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Passivity-based Gradient-Play Dynamics for Distributed GNE Seeking via Parallel Feedforward Compensation

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Abstract— **We consider seeking generalized Nash equilibria for games with coupled nonlinear constraints over networks. We first revisit a well-known gradient-play dynamics to solve the problem from a passivity-based perspective, and address that the strict monotonicity on pseudo-gradients is a critical assumption to guarantee its convergence. Then we develop a novel passivity-based gradient-play dynamics by introducing parallel feedforward compensators. We prove that the dynamics achieves asymptotic convergence in merely monotone regimes. Moreover, in the absence of coupled constraints, we surprisingly find that the dynamics can handle hypomonotone games with inverse Lipschitz pseudo-gradients.**

I. INTRODUCTION

Recent years have witnessed a flurry of research on distributed generalized Nash equilibrium (GNE) seeking for noncooperative games with coupled constraints. This was motivated by their applications in different areas such as power allocation in communications, smart grids and social networks [1]–[3]. These are examples of multi-agent systems where each individual decision-maker (player) aims to minimize a local cost function which depends on its own action as well as on the actions of its opponents, and meanwhile, the shared constraints should be satisfied. A GNE is a reasonable solution to such a problem, whereby no player can decrease its local cost by unilaterally changing its own decision. Various distributed algorithms to seek GNEs have been explored, such as (projected) gradient-play dynamics, operator splitting approaches and pay-off-based dynamics [4]–[6], to name just a few.

One of the most studied methods is the gradient-play dynamics because it is easily implemented under full and partial-decision information settings. Convergence of the dynamics to an Nash equilibrium (NE) was addressed in [7], [8], while its extension to NE seeking over networks under partial-decision information was developed in [9]. A corresponding discrete-time algorithm was designed in [10] for strongly monotone games, which achieved faster convergence rates by incorporating a Nesterov's accelerated protocol. Based on a primal-dual framework, a fully distributed gradient-play dynamics was explored for games with separable nonlinear coupled constraints in [3]. Similar ideas were used to seek GNEs for multi-cluster games with nonsmooth payoff functions, coupled and set constraints in [11]. However, all algorithms mentioned above require pseudo-gradients of the local cost functions to be strictly or strongly monotone in order to ensure convergence. Unfortunately, the assumption fails in some applications, including zero-sum games, Cournot games and resource allocation problems [12]–[14].

It is well-known that passivity is a powerful tool for the analysis and design of control systems [15], [16]. More recently, the concept of passivity has been applied to NE computation [5]. On one hand, it provides us with a better understanding of existing algorithms by explaining why they can work under certain game settings. On the other hand, it guides us to design new algorithms. For instance, stable games and evolutionary dynamics were analyzed in [17] after they were modeled as passive dynamical systems. In [9], passivity was employed to prove the convergence of a gradient-play dynamics for distributed NE seeking, while in [18] equilibrium-independent passivity was applied to the analysis and design of reinforcement learning dynamics in multi-agent finite games. A heavy-anchor (HA) dynamics was proposed in [19], which allowed a relaxation of the strict monotonicity on pseudo-gradients, and could also handle a class of hypomonotone games under inverse-Lipschitz conditions. The authors of [20] designed a second-order mirror descent (MD2) dynamics, which converged to a variationally stable state without using techniques such as time-averaging or discounting. We note that both HA and MD2 were designed based on passivity-based modifications by introducing output-strictly passive systems, and both only dealt with NE seeking.

In this paper, we focus on GNE seeking. Our main contributions are summarized as follows.

a) We revisit a typical distributed gradient-play dynamics to seek GNEs for games with nonlinear coupled constraints from a passivity-based perspective, and conclude that the strictly monotone assumption on pseudo-gradients plays an important role on its convergence.
b) We develop a novel passivity-based gradient-play dynamics, by introducing parallel feedforward compensators (PFCs). We establish that the dynamics can achieve exact convergence to a GNE if the pseudo-gradient is merely monotone. Furthermore, we surprisingly find that in the absence of coupled constraints, it can also handle general hypomonotone games with inverse Lipschitz pseudo-gradients, unlike [19].

The rest of this paper is organized as follows. We introduce some preliminary background and formulate the problem in Section II. Then we revisit a well-known gradient-play dynamics in Section III. In Section IV, we propose a novel

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passivity-based gradient-play dynamics, whose convergence is addressed in Section V. Finally, we close this paper with some concluding remarks in Section VI.

Notation: Let $\mathbb{R}^m$, $\mathbb{R}^m_+$ and $\mathbb{R}^{m\times n}$ be the set of m-dimensional real column vectors, m-dimensional nonnegative real column vectors, and m-by-n dimensional real matrices. Denote 1_m (0_m) as the m-dimensional column vector with all entries of 1 (0), and I_n as the n-by-n identity matrix. We simply write $\mathbf{0}$ for vectors/matrices of zeros with appropriate dimensions when there is no confusion. Let $(\cdot)^{\mathrm{T}}$, $\otimes$ and $\|\cdot\|$ be the transpose, the Kronecker product and the Euclidean norm, respectively. The Euclidean inner product of x and y is $x^{\mathrm{T}}y$ or $\langle x, y\rangle$. For $x_i \in \mathbb{R}^{n_i}$, we define $\mathrm{col}\{x_i\}_{i\in\mathcal{I}} := [x_1^T, \ldots, x_N^T]^T \in \mathbb{R}^{\sum_{i\in\mathcal{I}} n_i}$, where $\mathcal{I} = \{1, \ldots N\}$. Given a differentiable function $J(x, y)$, $\nabla_x J(x, y)$ is the partial gradient of J with respect to x.

II. PRELIMINARY AND GAME SETUP

In this section, we introduce some necessary concepts, and then, formulate the distributed GNE seeking problem.

A. Mathematical Preliminary

Consider a multi-agent network modeled by an undirected graph $\mathcal{G}(\mathcal{I}, \mathcal{E}, \mathcal{A})$, where $\mathcal{I} = \{1, \ldots, N\}$ is the node set, $\mathcal{E} \subset \mathcal{I} \times \mathcal{I}$ is the edge set, and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$ is the adjacency matrix such that $a_{ij} = a_{ji} > 0$ if $(i, j) \in \mathcal{E}$, and otherwise, $a_{ij} = 0$. Suppose that there are no self-loops in $\mathcal{G}$. The Laplacian matrix $\mathcal{L}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A}$, where $\mathcal{D} = \mathrm{diag}\{d_i\} \in \mathbb{R}^{N\times N}$, and $d_i = \sum_{j\in\mathcal{I}} a_{ij}$. Node j is a neighbor of i if and only if $(i, j) \in \mathcal{E}$. Let $\mathcal{I}_i = \{j|(i, j) \in \mathcal{E}\}$ be the set of node i's neighbors. Graph $\mathcal{G}$ is connected if there exists a path between any pair of distinct nodes. If $\mathcal{G}$ is connected, then $\mathcal{L} = \mathcal{L}^{\mathrm{T}}$, $\mathrm{rank}(\mathcal{L}) = N - 1$, and $\ker(\mathcal{L}) = \{k1_N : k \in \mathbb{R}\}$.

Let $\Omega \subset \mathbb{R}^m$ be a convex set such that $\lambda x + (1 - \lambda)y \in \Omega, \forall x, y \in \Omega, \forall \lambda \in [0, 1]$. Its tangent cone at $x \in \Omega$ is

$$\mathcal{T}_\Omega(x) := \Big\{ \lim_{k\to\infty} \frac{x_k - x}{\tau_k} \mid x_k \in \Omega, x_k \to x, \tau_k > 0, \tau_k \to 0 \Big\},$$

while its normal cone is

$$\mathcal{N}_\Omega(x) := \{v \in \mathbb{R}^m | v^{\mathrm{T}}(y - x) \le 0, \forall y \in \Omega\}.$$

Take $\mathrm{proj}_\Omega(x) := \mathrm{argmin}_{y\in\Omega}\|y - x\|$ for $x \in \mathbb{R}^m$. The differentiated projection operator on $\mathcal{T}_\Omega(x)$ is defined by $\Pi_\Omega(x, v) := \mathrm{prox}_{\mathcal{T}_{\Omega(x)}}(v) = \lim_{h\to 0^+} \frac{\mathrm{proj}_\Omega(x+hv)-x}{h}$. It follows from [21, Theorem 6.30] that

$$v = \mathrm{proj}_{\mathcal{T}_{\Omega(x)}}(v) + \mathrm{proj}_{\mathcal{N}_{\Omega(x)}}(v), \ \forall v \in \mathbb{R}^m. \quad (1)$$

Given a differentiable function $f : \Omega \to \mathbb{R}$, $\nabla f(x)$ denotes its gradient at x. Function f is convex if $f(y) \ge f(x) + \langle \nabla f(x), y - x\rangle, \forall x, y \in \Omega, \forall \lambda \in [0, 1]$.

An operator (or mapping) $F : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is monotone if $\langle F(x) - F(y), x - y\rangle \ge 0, \forall x, y \in \Omega$, strictly monotone if the strict inequality holds for all $x \ne y$, and ν-hypomonotone if there exists $\nu > 0$ such that $\langle F(x) - F(y), x - y\rangle \ge -\nu\|x - y\|^2, \forall x, y \in \Omega$. F is θ-Lipschitz continuous if there is $\theta > 0$ such that $\|F(x) - F(y)\| \le \theta\|x - y\|, \forall x, y \in \Omega$, and R-inverse Lipschitz if there exists $R > 0$ such that $\|x - y\| \le R\|F(x) - F(y)\|, \forall x, y \in \Omega$ (see [19] for more details). Note that for a convex $f : \Omega \to \mathbb{R}^n$, ∇f is monotone.

Consider a system Σ given by

$$\Sigma : \dot{x} = f(x, u), \ y = g(x, u)$$

where $x \in \mathbb{R}^n$, $u, y \in \mathbb{R}^m$, f is locally Lipschitz, g is continuous, $f(0, 0) = 0$, and $h(0, 0) = 0$. If there exists a continuous differentiable positive semi-definite storage function V such that $\dot{V} = \nabla V(x)^T f(x, u) \le u^T y, \ \forall (x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, then Σ is said to be passive. Moreover, Σ is input feedforward passive if $\dot{V} \le u^T y - \delta u^T u$ for some $\delta \in \mathbb{R}$, where the sign of δ denotes an excess or shortage of passivity. Specifically, Σ is input feedforward passive-excess if $\delta > 0$, and it is input feedforward passive-short if $\delta < 0$.

B. Game Setup

Consider a set $\mathcal{I} = \{1, \ldots, N\}$ of N players (agents) involved in a game. Player i controls its decision (action) $x^i \in \mathbb{R}^{n_i}$, where $\sum_{i=1}^N n_i = n$. Let $x = (x^i, x^{-i}) \in \mathbb{R}^n$ be the N-tuple of all agents' actions, where x^{-i} is the $(N-1)$-tuple of all agents' actions except agent i's decision. Alternatively, $x = \mathrm{col}\{x^i\}_{i\in\mathcal{I}} \in \mathbb{R}^n$. Player i aims to minimize its local cost function $J_i(x^i, x^{-i}) : \mathbb{R}^n \to \mathbb{R}$, which depends on both its local decision x^i as well on those of its opponents x^{-i}. Furthermore, the following separable coupled constraints should be satisfied $X = \{x \in \mathbb{R}^n \mid g(x) \le 0_m\}$, where $g(x) := \sum_{i\in\mathcal{I}} g_i(x^i)$, and $g_i : \mathbb{R}^{n_i} \to \mathbb{R}^m$ is a private function only known by agent i. Given x^{-i}, the feasible decision set of agent i is $X_i(x^{-i}) = \{x^i \in \mathbb{R}^{n_i} : (x^i, x^{-i}) \in X\}$. Player i aims to solve

$$\min_{x^i\in\mathbb{R}^{n_i}} J_i(x^i, x^{-i}), \ \text{s.t. } x^i \in X_i(x^{-i}). \quad (2)$$

A collective decision profile $x^* = (x^{i,*}, x^{-i,*})$ is called a generalized Nash equilibrium (GNE) if

$$x^{i,*} \in \mathrm{argmin}_{x^i} J_i(x^i, x^{-i,*}), \ \text{s.t. } (x^i, x^{-i,*}) \in X, \ \forall i \in \mathcal{I}. \quad (3)$$

Particularly, if there are no coupled constraints X, i.e., $g_i(x^i) \equiv \mathbf{0}$, $\forall i \in \mathcal{I}$, then x^* satisfying (3) is a Nash equilibrium (NE).

To ensure the well-posedness of (2), we make the following well-known assumption.

Assumption 1: For every $i \in \mathcal{I}$, J_i is continuously differentiable and convex in x^i, given x^{-i}, and g_i is continuous, differentiable and convex. Furthermore, X is non-empty and satisfies the Slater's constraint qualification.

Under Assumption 1, x^* is a GNE of (2) if and only if the following Karush-Kuhn-Tucker (KKT) conditions hold [22]:

$$\begin{aligned} 0_{n_i} &= \nabla_{x^i} J_i(x^{i,*}, x^{-i,*}) + \nabla g_i(x^{i,*})^T \lambda^{i,*}, \\ 0_m &\in -g(x^*) + \mathcal{N}_{\mathbb{R}^m_+}(\lambda^{i,*}), \end{aligned} \quad (4)$$

where $\lambda^{i,*} \in \mathbb{R}^m$ is the Lagrangian multiplier of agent i.

Given x^* as a GNE of (2), the corresponding Lagrangian multipliers may be different for the players, i.e., $\lambda^{i,*} \ne \lambda^{j,*}$ for $i \ne j$. In this work, we focus on seeking a GNE with the same Lagrangian multiplier, named variational GNE (v-GNE), i.e., $\lambda^{i,*} = \lambda_c^*$, $\forall i \in \mathcal{I}$ [22], [23], and we simply call it a GNE.

III. Gradient-Play Dynamics

In this section, we revisit a typical distributed gradient-play dynamics for (2) from a passivity-based perspective.

Consider that agent i only knows J_i and g_i, but has the knowledge of all its opponents' decisions x^{-i}. As discussed in [1], [4], [25], this is a full-decision information setting, and agent i can compute the partial gradient $\nabla_{x^i} J_i(x^i, x^{-i})$ at each step. Let λ^i be the estimation of the consensus multiplier $\boldsymbol{\lambda}^*$. To handle the coupled constraint $\sum_{i\in\mathcal{I}} g_i(x^i) \leq \mathbf{0}$ and ensure all local multipliers λ^i reaching consensus, we introduce a auxiliary variable $z^i \in \mathbb{R}^m$ for agent i. Suppose that all agents exchange their local data through a graph $\mathcal{G}_c(\mathcal{I}, \mathcal{E}, \mathcal{A})$. Specifically, agent i can receive $\{\lambda^j, z^j\}$ from agent j if and only if $j \in \mathcal{I}_i$, where $\mathcal{I}_i$ is the neighbor set of agent i. We make the following assumption on $\mathcal{G}_c$.

Assumption 2: Graph $\mathcal{G}_c$ is undirected and connected.

Referring to [3], [11], a well-known gradient-play dynamics to solve (2) is given by

$$\begin{cases} \dot{x}^i = -\nabla_{x^i} J_i(x^i, x^{-i}) - \nabla g_i(x^i)^T \lambda^i, \; x^i(0) \in \mathbb{R}^{n_i} \\ \dot{z}^i = \sum_{j\in\mathcal{I}_i} a_{ij}(\lambda^i - \lambda^j), \; z^i(0) \in \mathbb{R}^m \\ \dot{\lambda}^i = \Pi_{\mathbb{R}_+^m}\big[\lambda^i, g_i(x^i) - \sum_{j\in\mathcal{I}_i} a_{ij}(z^i - z^j) - \sum_{j\in\mathcal{I}_i} a_{ij}(\lambda^i - \lambda^j)\big], \\ \qquad \lambda^i(0) \in \mathbb{R}_+^m, \end{cases} \tag{5}$$

where a_{ij} is the (i,j)-th entry of $\mathcal{A}$.

For ease of notation, we define a pseudo-gradient mapping F as $F(x) := \mathrm{col}\{\nabla_{x^i} J_i(x^i, x^{-i})\}_{i\in\mathcal{I}} \in \mathbb{R}^n$. Take $L := \mathcal{L} \otimes I_m \in \mathbb{R}^{Nm\times Nm}$, where $\mathcal{L}$ is the Laplacian matrix of $\mathcal{G}_c$. Define $G(x) := \mathrm{col}\{g_i(x^i)\}_{i\in\mathcal{I}} \in \mathbb{R}^{Nm}$, $\nabla G(x) := \mathrm{blkdiag}\{\nabla g_i(x^i)\}_{i\in\mathcal{I}} \in \mathbb{R}^{Nm\times n}$, $z := \mathrm{col}\{z^i\}_{i\in\mathcal{I}} \in \mathbb{R}^{Nm}$, and $\lambda := \mathrm{col}\{\lambda^i\}_{i\in\mathcal{I}} \in \mathbb{R}^{Nm}$. Then (5) can be rewritten as

$$\begin{cases} \dot{x} = -F(x) - \nabla G(x)^T \lambda \\ \dot{z} = L\lambda, \\ \dot{\lambda} = \Pi_{\mathbb{R}_+^{Nm}}\big[\lambda, G(x) - Lz - L\lambda\big], \end{cases} \tag{6}$$

where $x(0) \in \mathbb{R}^n$, $z(0) \in \mathbb{R}^{Nm}$, and $\lambda(0) \in \mathbb{R}_+^{Nm}$.

Remark 1: By the viability theorem in [26], $\lambda(t) \in \mathbb{R}_+^{Nm}$ for all $t \geq 0$ because of the projection operator $\Pi_{\mathbb{R}_+^{Nm}}[\lambda, \cdot]$. If $g_i \equiv \mathbf{0}$, (6) degenerates into the dynamics discussed in [8], [9]. If J_i only depends on x^i, then (6) is consistent with the primal-dual method for constrained optimization in [27].

By Theorem 4.1 in [11] or Lemma 2 in [3], we obtain the following lemma.

Lemma 1: Let Assumptions 1 and 2 hold, and F be monotone. Then x^* is a GNE of (2) if and only if there exists (λ^*, z^*) such that (x^*, λ^*, z^*) is an equilibrium point of (6).

In the following, we put (6) in a block-diagram representation. Note that x in the first equation of (6) can be represented via a bank of integrators, $x(s) = [(1/s)I_n]v_x(s)$, where we defined $v_x := \mathrm{col}\{v_x^i\}_{i\in\mathcal{I}}$. Similarly, $z(s) = [(1/s)I_{Nm}]v_z(s)$, where $v_z := \mathrm{col}\{v_z^i\}_{i\in\mathcal{I}}$.

To handle λ obtained as in (6) via the projection operator $\Pi_{\mathbb{R}_+^{Nm}}[\lambda, \cdot]$ in cascade with a bank of integrators, we introduce the notation, $\lambda(s) = [(1/s)I_{Nm}]^+ v_\lambda(s)$, where $v_\lambda := \mathrm{col}\{v_\lambda^i\}_{i\in\mathcal{I}}$.

With these notations, (6) can be represented by the block-diagram in Fig. 1.

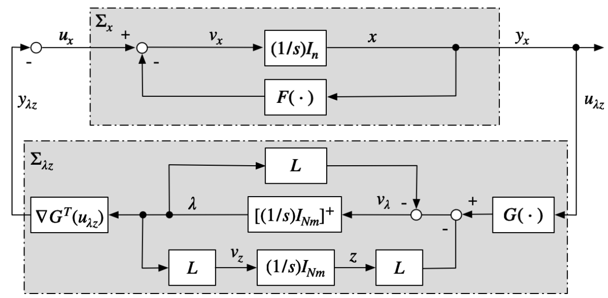

Fig. 1. Block diagram of dynamics (6).

Note that we can decompose (6) into two interconnected subsystems Σ_x and $\Sigma_{\lambda z}$, where

$$\Sigma_x : \begin{cases} \dot{x} = -F(x) + u_x, \\ y_x = x \end{cases} \tag{7}$$

and

$$\Sigma_{\lambda z} : \begin{cases} \dot{\lambda} = \Pi_{\mathbb{R}_+^{Nm}}\big[\lambda, G(u_{\lambda z}) - Lz - L\lambda\big], \\ \dot{z} = L\lambda, \\ y_{\lambda z} = \nabla G(u_{\lambda z})^T \lambda. \end{cases} \tag{8}$$

Let (x^*, λ^*, z^*) be an equilibrium point of (6). Consequently, $F(x^*) + \nabla G(x^*)^T \lambda^* = \mathbf{0}$, $L\lambda^* = \mathbf{0}$, and $\Pi_{\mathbb{R}_+^{Nm}}\big[\lambda^*, G(x^*) - Lz^* - L\lambda^*\big] = \mathbf{0}$. Take

$$\begin{aligned} \tilde{u}_x &:= u_x - u_x^*, \quad \tilde{u}_{\lambda z} := u_{\lambda z} - u_{\lambda z}^*, \\ \tilde{y}_x &:= y_x - y_x^*, \quad \tilde{y}_{\lambda z} := y_{\lambda z} - y_{\lambda z}^*, \end{aligned} \tag{9}$$

where $u_{\lambda z}^* = y_x^* = x^*$ and $u_x^* = -y_{\lambda z}^* = -\nabla G(x^*)^T \lambda^*$.

The next theorem addresses the convergence of (6).

Theorem 1: Consider dynamics (6). Let Assumptions 1 and 2 hold, and F be monotone. Then

a) The subsystem Σ_x (7) is passive from $\tilde{u}_x$ to $\tilde{y}_x$ with respect to the storage function $V_x = \frac{1}{2}\|x - x^*\|^2$.
b) The subsystem $\Sigma_{\lambda z}$ (8) is passive from $\tilde{u}_{\lambda z}$ to $\tilde{y}_{\lambda z}$ with respect to the storage function $V_{\lambda,z} = \frac{1}{2}\|\lambda - \lambda^*\|^2 + \frac{1}{2}\|z - z^*\|^2$.
c) If F is strictly monotone, then every trajectory $\big(x(t), \lambda(t), z(t)\big)$ converges to an equilibrium point (x^*, λ^*, z^*), where x^* is a GNE of (2).

Proof: Here we provide an overview for the proof, and the details are given in the appendix.

a) We show that along trajectories of (7),

$$\dot{V}_x = -\langle x - x^*, F(x) - F(x^*)\rangle + \langle y_x - y_x^*, u_x - u_x^*\rangle. \tag{10}$$

b) We prove that $\dot{V}_{\lambda,z} \leq \langle u_{\lambda z} - u_{\lambda z}^*, y_{\lambda z} - y_{\lambda z}^*\rangle - \lambda^T L\lambda$.

c) Resorting to passivity analysis for feedback systems, we show that every trajectory $\big(x(t), \lambda(t), z(t)\big)$ converges to an equilibrium point of (6). $\square$

Remark 2: Theorem 1 is an alternative result to those presented in [3], [11], [27], in that the convergence of (6) is addressed from a passivity-based perspective. Note that the assumption of strict monotonicity on F plays a critical role on the asymptotic convergence of (6). If the assumption fails, by (10), (6) may be passive lossless and not convergent. An illustrative example is given as follows.

Example 1: Consider a two-players zero-sum game problem. Take $J_1(x^1, x^2) = x^1x^2$, $J_2(x^2, x^1) = -x^1x^2$, and $g_i(x^i) \equiv \mathbf{0}$, where $x^1, x^2 \in \mathbb{R}$. Then (6) is given by

$$\dot{x}^1 = -x^2,\ \dot{x}^2 = x^1. \tag{11}$$

Clearly, the NE is $\mathbf{0}$, but $x(t)$ will cycle around the NE and never converge if $x(0) \neq \mathbf{0}$.

Dynamics (11) can be viewed as a feedback interconnection of the two passive integrators. Referring to [16, Th. 6.3], its asymptotic stability cannot be guaranteed because neither strictly passive nor output strictly passive terms exist, and it is a passive lossless system.

IV. PASSIVITY-BASED GRADIENT-PLAY DYNAMICS

As mentioned above, to ensure the convergence of (6), F is assumed to be strictly monotone. In this section, we develop a novel passivity-based gradient-play dynamics that converges under relaxed assumptions.

Referring to [15, Th. 2.10], the passivity of a system will be preserved if a passive compensator is added in parallel, and the parallel feedforward compensator (PFC) may enhance the stability of the original system. Recent applications of PFCs could be found in [28]–[30]. Motivated by the observations, we focus on enhancing the convergence of (6) by introducing PFCs for Σ_x (7) and $\Sigma_{\lambda z}$ (8).

For Σ_x in (7), only passive integrators $\frac{1}{s}I_n$ are involved in the evolution of x as shown in Fig. 1. Here we introduce static and dynamical PFCs for each integrator $\frac{1}{s}$. Then, instead of $\frac{1}{s}I_n$, we employ a diagonal matrix given by $M(s) = \operatorname{diag}\{M_1(s), \ldots, M_n(s)\}$, where

$$M_r(s) = \frac{\alpha^x_{r1}}{s} + \sum_{\rho=2}^{\kappa^x_r} \frac{\alpha^x_{r\rho}}{s + \beta^x_{r\rho}} + \gamma^x_r,\ r \in \{1, \ldots, n\} \tag{12}$$

with $\beta^x_{r\kappa^x_r} > \cdots > \beta^x_{r2} > 0$, $\alpha^x_{r\rho} > 0$, $\forall \rho \in \{1, \ldots, \kappa^x_r\}$ and $\gamma^x_r \geq 0$. As a result, we have

$$x_r(s) = M_r(s)v_{xr}(s), \tag{13}$$

where v_{xr} and x_r be the r-th entry of v_x and x for $r \in \{1, \ldots, n\}$. Clearly, the proposed method is distributed because $M(s)$ is a diagonal matrix. For $i \in \mathcal{I}$, the evolution of x^i is given by

$$x^i(s) = M^i(s)v^i_x(s)$$

where $M^i(s) = \mathcal{R}^iM(s)$, $\mathcal{R}^i = [0_{n_i\times n_{<i}}, I_{n_i}, 0_{n_i\times n_{>i}}]$, $n_{<i} = \sum_{j<i, j\in\mathcal{I}} n_j$, and $n_{>i} = \sum_{j>i, j\in\mathcal{I}} n_j$.

Remark 3: To ensure that the PFCs are strictly passive, we assume that $\alpha^x_{r\rho} > 0$, $\beta_{r\rho} > 0$ and $\gamma^x_r \geq 0$. Thus, $M_r(s)$ has at least one nonnegative (stable) zero. As will be discussed in Remark 6, the PFCs will preserve the passivity of Σ_x, and meanwhile, enhance the convergence of (6) by preventing (6) to be passive lossless.

A state-space representation of (13) is

$$\begin{cases} \dot{\xi}_{r1} = \alpha^x_{r1}v_{xr}, \\ \dot{\xi}_{r\rho} = -\beta^x_{r\rho}\xi_{r\rho} + \alpha^x_{r\rho}v_{xr},\ \rho \in \{2, \ldots, \kappa^x_r\} \\ x_r = 1^T_{\kappa^x_r}\xi_r + \gamma^x_r v_{xr},\ r \in \{1, \ldots, n\} \end{cases} \tag{14}$$

where $\xi_r = [\xi_{r1}, \ldots, \xi_{r\kappa^x_r}]^T$.

As a simple example, we take $\kappa^x_r = 2$, $\alpha^x_{r1} = \alpha^x_{r2} = \beta^x_{r\rho} = 1$ for $r \in \{1, \ldots, n\}$. Then (14) is cast into

$$\dot{\xi}_1 = v_x,\ \dot{\xi}_2 = -\xi_2 + v_x,\ x = \xi_1 + \xi_2 + v_x. \tag{15}$$

The evolution of ξ_1 is the same as that of x in (6). However, a strictly passive system $\dot{\xi}_2 = -\xi_2 + v_x$ is added, and moreover, the output x aggregates the states ξ_1, ξ_2 and the input v_x.

Remark 4: In [19], without the coupled constraints X, a heavy-anchor (HA) dynamics is proposed for (2) as

$$\dot{r} = \alpha(x - r),\quad \dot{x} = -F(x) - \beta(x - r) \tag{16}$$

where $\alpha, \beta > 0$, and $r \in \mathbb{R}^n$ are auxiliary variables. In fact, (16) can be viewed as adding an output feedback compensator (OFC) to Σ_x in (7). It was shown that HA could achieve exact convergence to NE for games in merely monotone regimes. Following the same idea, a second-order mirror descent (MD2) dynamics was developed in [20]. Different from HA and MD2, we introduce PFCs rather than OFCs and deal with GNE seeking. Both PFCs and OFCs can deal with general monotone games, but PFCs are more powerful to handle hypomonotone games as will be discussed in Remark 8.

Returning to $\Sigma_{\lambda z}$, (8), Fig. 1, by a similar procedure, we substitute the integrator $\frac{1}{s}I_{Nm}$ for z in Fig. 1 by a diagonal matrix $K(s) = \operatorname{diag}\{K_1(s), \ldots, K_{Nm}(s)\}$, where

$$K_p(s) = \frac{\alpha^z_{p1}}{s} + \sum_{\tau=2}^{\kappa^z_p} \frac{\alpha^z_{p\tau}}{s + \beta^z_{p\tau}} + \gamma^z_p,\ p \in \{1, \ldots, Nm\},$$

with $\beta^z_{p\kappa^z_p} > \cdots > \beta^z_{p2} > 0$, $\alpha^z_{p\tau} > 0$, $\forall \tau \in \{1, \ldots, \kappa^z_p\}$ and $\gamma^z_p \geq 0$. Then a state-space realization for z is

$$\begin{cases} \dot{\zeta}_{p1} = \alpha^z_{p1}v_{zp}, \\ \dot{\zeta}_{p\tau} = -\beta^z_{p\tau}\zeta_{p\tau} + \alpha^z_{p\tau}v_{zp},\ \tau \in \{2, \ldots, \kappa^z_p\} \\ z_p = 1^T_{\kappa^z_p}\zeta_p + \gamma^z_p v_{zp},\ p \in \{1, \ldots, Nm\} \end{cases} \tag{17}$$

where $\zeta_p = [\zeta_{p1}, \ldots, \zeta_{p\kappa^z_p}]^T$, v_{zp} and z_p are the p-th entry of v_z and z, respectively.

Recall that for the evolution of λ, from Fig. 1, $[(1/s)I_{Nm}]^+$ is used as a short-hand notation for projection in cascade with a bank of integrators. We introduce a similar notation $y(s) = [1/(\alpha s + \beta)]^+u(s)$, to denote a short-hand notation for projection in cascade with $1/(\alpha s + \beta)$, where $\alpha \geq 0$ and $\beta > 0$. To be specific, in time-domain, $\dot{y} =$

$\Pi_{\mathbb{R}_+}\big[y,-(\beta/\alpha)y+(1/\alpha)u\big]$ if $\alpha>0$, and otherwise, $y=(1/\beta)\max\{0,u\}$. Consequently, $\lambda(s)=[(1/s)I_{Nm}]^+v_\lambda(s)$.

Then, with these notations, instead of $[(1/s)I_{Nm}]^+$ for the evolution of λ in Fig. 1, we consider a diagonal matrix given by $[H(s)]^+=\mathrm{diag}\{[H_1(s)]^+,\dots,[H_{Nm}(s)]^+\}$, where

$$[H_q(s)]^+=\Big[\frac{\alpha^\lambda_{q1}}{s}\Big]^+ + \sum_{\eta=2}^{\kappa^\lambda_q}\Big[\frac{\alpha^\lambda_{q\eta}}{s+\beta^\lambda_{q\eta}}\Big]^+ + \big[\gamma^\lambda_q\big]^+, q\in\{1,\dots,Nm\}$$

with $\beta^\lambda_{q\kappa^\lambda_q}>\cdots>\beta^\lambda_{q2}>0$, $\alpha^\lambda_{q\eta}>0$, $\forall\eta\in\{1,\dots,\kappa^\lambda_q\}$, and $\gamma^\lambda_q\geq 0$. Consequently, a state-space realization for λ is

$$\begin{cases}\dot\omega_{q1}=\Pi_{\mathbb{R}_+}\big[\omega_{q1},\alpha^\lambda_{q1}v_{\lambda q}\big],\\ \dot\omega_{q\eta}=\Pi_{\mathbb{R}_+}\big[\omega_{q\eta},-\beta^\lambda_{q\eta}\omega_{q\eta}+\alpha^\lambda_{q\eta}v_{\lambda q}\big],\eta\in\{2,\dots,\kappa^\lambda_q\}\\ \lambda_q=1^T_{\kappa^\lambda_q}\omega_q+\gamma^\lambda_q\max\{0,v_{\lambda q}\},\ q\in\{1,\dots,Nm\}\end{cases} \tag{18}$$

where $\omega_q=[\omega_{q1},\dots,\omega_{q\kappa^\lambda_q}]^T$, $v_{\lambda q}$ and λ_q be the q-th entry of v_λ and λ.

Combining (14), (17) with (18), our proposed passivity-based gradient-play dynamics is given as

$$\begin{cases}\dot\xi_{r1}=-\alpha^x_{r1}[F(x)+\nabla G(x)^T\lambda]_r,\\ \dot\xi_{r\rho}=-\beta^x_{r\rho}\xi_{r\rho}-\alpha^x_{r\rho}[F(x)+\nabla G(x)^T\lambda]_r,\\ x_r=1^T_{\kappa^x_r}\xi_r-\gamma^x_r[F(x)+\nabla G(x)^T\lambda]_r,\\ \dot\zeta_{p1}=\alpha^z_{p1}[L\lambda]_p,\\ \dot\zeta_{p\tau}=-\beta^z_{p\tau}\zeta_{p\tau}+\alpha^z_{p\tau}[L\lambda]_p,\\ z_p=1^T_{\kappa^z_p}\zeta_p+\gamma^z_p[L\lambda]_p,\\ \dot\omega_{q1}=\Pi_{\mathbb{R}_+}\big[\omega_{q1},\alpha^\lambda_{q1}[G(x)-Lz-L\lambda]_q\big],\\ \dot\omega_{q\eta}=\Pi_{\mathbb{R}_+}\big[\omega_{q\eta},-\beta^\lambda_{q\eta}\omega_{q\eta}+\alpha^\lambda_{q\eta}[G(x)-Lz-L\lambda]_q\big],\\ \lambda_q=1^T_{\kappa^\lambda_q}\omega_q+\gamma^\lambda_q\max\big\{0,[G(x)-Lz-L\lambda]_q\big\},\end{cases} \tag{19}$$

where $[\cdot]_j$ denotes the j-th entry of a vector.

Remark 5: From the perspective of agent i, in frequency-domain, dynamics (19) is given as $x^i(s)=M^i(s)v^i_x(s)$, $\lambda^i(s)=[H^i(s)]^+v^i_\lambda(s)$, and $z^i(s)=N^i(s)v^i_z(s)$, where $[H^i(s)]^+=\tilde{\mathcal{R}}^i[H(s)]^+$, $N^i(s)=\tilde{\mathcal{R}}^iN(s)$, and $\tilde{\mathcal{R}}^i=[0_{m\times(i-1)m},I_m,0_{m\times(N-i)m}]$. We should mention that similar to (5), only the first-order (pseudo-)gradient information, such as F and ∇G, is used in the proposed dynamics. Fig. 2 shows the block diagram of (19). We generalize (6) by substituting $\frac{1}{s}I_n$, $\frac{1}{s}I_{Nm}$ and $[\frac{1}{s}I_{Nm}]^+$ in Fig. 1 as $M(s)$, $K(s)$ and $[H(s)]^+$ in Fig. 2.

Let $(\xi^*,\omega^*,\zeta^*)=\big(\mathrm{col}\{\xi^*_r\},\mathrm{col}\{\omega^*_q\},\mathrm{col}\{\zeta^*_p\}\big)$ be an equilibrium point of (19), and (x^*,λ^*,z^*) be the corresponding output. Then $F(x^*)+\nabla G(x^*)^T\lambda^*=\mathbf{0}$, $L\lambda^*=\mathbf{0}$, and $\Pi_{\mathbb{R}^{Nm}_+}\big[\lambda^*,G(x^*)-Lz^*-L\lambda^*\big]=\mathbf{0}$. Moreover,

$$\begin{aligned}\xi^*_r&=[\xi^*_{r1},\xi^*_{r2},\dots,\xi^*_{r\kappa^x_r}]^T&&=[x^*_r,0,\dots,0]^T,\\ \omega^*_q&=[\omega^*_{q1},\omega^*_{q2},\dots,\omega^*_{q\kappa^\lambda_q}]^T&&=[\lambda^*_q,0,\dots,0]^T,\\ \zeta^*_p&=[\zeta^*_{p1},\zeta^*_{p2},\dots,\zeta^*_{p\kappa^z_p}]^T&&=[z^*_p,0,\dots,0]^T.\end{aligned} \tag{20}$$

The next lemma addresses the relationship between equilibria of (19) and GNEs of (2). Its proof is similar to that of Lemma 1, and omitted here.

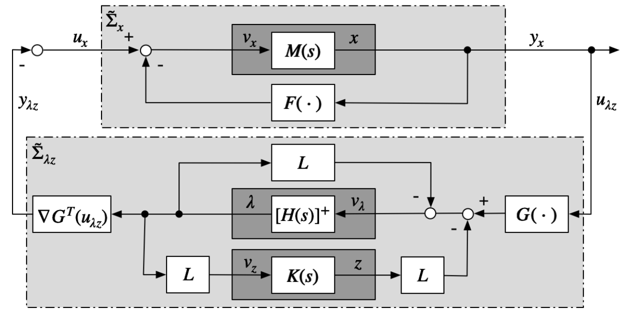

Fig. 2. Block diagram of dynamics (19).

Lemma 2: Consider dynamics (19). Let Assumptions 1 and 2 hold, and F be monotone. If (ξ^*,ω^*,ζ^*) is an equilibrium point of (19), then $x^*=[\xi^*_{11},\xi^*_{21},\dots,\xi^*_{n1}]^T$ (called the x component) is a GNE of (2). Conversely, if x^* is a GNE of (2), then there exists (ξ^*,ω^*,ζ^*) such that it is an equilibrium point of (19) with output (x^*,λ^*,z^*) satisfying (20).

V. MAIN RESULTS

In this section, we analyze the convergence of (19). Similar to (6), we decompose (19) into two interconnected subsystems $\widetilde\Sigma_x$ and $\widetilde\Sigma_{\lambda z}$ as shown in Fig. 2, where

$$\widetilde\Sigma_x:\begin{cases}\dot\xi_{r1}=\alpha^x_{r1}[-F(x)+u_x]_r,\\ \dot\xi_{r\rho}=-\beta^x_{r\rho}\xi_{r\rho}+\alpha^x_{r\rho}[-F(x)+u_x]_r,\\ x_r=1^T_{\kappa^x_r}\xi_r+\gamma^x_r[-F(x)+u_x]_r,\\ y_x=x\end{cases} \tag{21}$$

and

$$\widetilde\Sigma_{\lambda z}:\begin{cases}\dot\omega_{q1}=\Pi_{\mathbb{R}_+}\big[\omega_{q1},\alpha^\lambda_{q1}[G(u_{\lambda z})-Lz-L\lambda]_q\big],\\ \dot\omega_{q\eta}=\Pi_{\mathbb{R}_+}\big[\omega_{q\eta},-\beta^\lambda_{q\eta}\omega_{q\eta}+\alpha^\lambda_{q\eta}[G(u_{\lambda z})-Lz-L\lambda]_q\big],\\ \lambda_q=1^T_{\kappa^\lambda_q}\omega_q+\gamma^\lambda_q\max\big\{0,[G(u_{\lambda z})-Lz-L\lambda]_q\big\},\\ \dot\zeta_{p1}=\alpha^z_{p1}[L\lambda]_p,\\ \dot\zeta_{p\tau}=-\beta^z_{p\tau}\zeta_{p\tau}+\alpha^z_{p\tau}[L\lambda]_p,\\ z_p=1^T_{\kappa^z_p}\zeta_p+\gamma^z_p[L\lambda]_p,\\ y_{\lambda z}=\nabla G(u_{\lambda z})^T\lambda.\end{cases} \tag{22}$$

Let (ξ^*,ω^*,ζ^*) be an equilibrium point of (19) with output (x^*,λ^*,z^*) satisfying (20). For $\widetilde\Sigma_x$ and $\widetilde\Sigma_{\lambda z}$, we also define $\tilde u_x,\tilde u_{\lambda z},\tilde y_x$ and $\tilde y_{\lambda z}$ by (9). Then the following result holds.

Theorem 2: Consider dynamics (19). Let Assumptions 1 and 2 hold, and F be monotone. Then

a) The subsystem $\widetilde\Sigma_x$ (21) is passive from $\tilde u_x$ to $\tilde y_x$ with respect to the storage function $S_x=\sum_{r=1}^n S_{xr}$, where

$$S_{xr}:=\frac{1}{2\alpha^x_{r1}}(\xi_{r1}-x^*_r)^2+\sum_{\rho=2}^{\kappa^x_r}\frac{1}{2\alpha^x_{r\rho}}\xi^2_{r\rho}.$$

b) The subsystem $\widetilde\Sigma_{\lambda z}$ (22) is passive from $\tilde u_{\lambda z}$ to $\tilde y_{\lambda z}$ with respect to the storage function $S_{\lambda,z}=S_\lambda+S_z$, where

$S_\lambda = \sum_{q=1}^{Nm} S_{\lambda q}$, $S_z = \sum_{p=1}^{Nm} S_{zp}$,

$$S_{\lambda q} := \frac{1}{2\alpha_{q1}^\lambda}(\omega_{q1} - \lambda_q^*)^2 + \sum_{\eta=2}^{\kappa_q^\lambda} \frac{1}{2\alpha_{q\eta}^\lambda}\omega_{q\eta}^2,$$

and moreover,

$$S_{zp} := \frac{1}{2\alpha_{p1}^z}(\zeta_{p1} - z_p^*)^2 + \sum_{\tau=2}^{\kappa_p^z} \frac{1}{2\alpha_{p\tau}^z}\zeta_{p\tau}^2.$$

c) If $\gamma_r^x > 0$ or $\kappa_r^x \geq 2$ for all $M_r(s)$ in (12), then every trajectory $\big(\xi(t), \omega(t), \zeta(t)\big)$ converges to an equilibrium point of (19), where the x component is a GNE of (2).

Proof. a) Recalling (21) gives

$$\begin{aligned}\dot{S}_x =& \sum_{r=1}^{n} \langle 1_{\kappa_r^x}^T \xi_r - x_r^*, [-F(x) + u_x]_r\rangle - \sum_{r=1}^{n}\sum_{\rho=2}^{\kappa_r^x} \frac{\beta_{r\rho}^x}{\alpha_{r\rho}^x}\xi_{r\rho}^2 \\ =& -\sum_{r=1}^{n} \gamma_r^x \|[-F(x)+u_x]_r\|^2 - \sum_{r=1}^{n}\sum_{\rho=2}^{\kappa_r^x} \frac{\beta_{r\rho}^x}{\alpha_{r\rho}^x}\xi_{r\rho}^2 \\ & - \langle x - x^*, F(x) - F(x^*)\rangle + \langle y_x - y_x^*, u_x - u_x^*\rangle. \end{aligned} \tag{23}$$

The monotonicity of F indicates $\dot{S}_x \leq \langle \tilde{u}_x, \tilde{y}_x\rangle$, and then, part a) holds.

b) It follows from (22) that

$$\begin{aligned}\dot{S}_{\lambda,z} =& \sum_{q=1}^{Nm} \frac{1}{\alpha_{q1}^\lambda}\langle \omega_{q1} - \lambda_q^*, \dot{\omega}_{q1}\rangle + \sum_{q=1}^{Nm}\sum_{\eta=2}^{\kappa_q^\lambda} \frac{1}{\alpha_{q\eta}^\lambda}\langle \omega_{q\eta}, \dot{\omega}_{q\eta}\rangle \\ & + \langle z - z^*, L\lambda\rangle - \sum_{p=1}^{Nm} \gamma_p^z \|[L\lambda]_p\|^2 - \sum_{p=1}^{Nm}\sum_{\tau=2}^{\kappa_p^z} \frac{\beta_{p\tau}^z}{\alpha_{p\tau}^z}\zeta_{p\tau}^2. \end{aligned} \tag{24}$$

According to (1), we obtain

$$\begin{aligned}\big\langle \omega_{q1} - \lambda_q^*, \dot{\omega}_{q1}\big\rangle =& \big\langle \omega_{q1} - \lambda_q^*, \alpha_{q1}^\lambda [G(u_{\lambda z}) - Lz - L\lambda]_q \\ & - \mathrm{proj}_{\mathcal{N}_{\mathbb{R}_+}(\omega_{q1})}\big[\alpha_{q1}^\lambda[G(u_{\lambda z}) - Lz - L\lambda]_q\big]\big\rangle \\ \leq & \big\langle \omega_{q1} - \lambda_q^*, \alpha_{q1}^\lambda [G(u_{\lambda z}) - Lz - L\lambda]_q\big\rangle. \end{aligned}$$

Similarly, we have

$$\langle \omega_{q\eta}, \dot{\omega}_{q\eta}\rangle \leq \langle \omega_{q\eta}, -\beta_{q\eta}^\lambda \omega_{q\eta} + \alpha_{q\eta}^\lambda [G(u_{\lambda z}) - Lz - L\lambda]_q]\rangle.$$

Substituting the above two inequalities to (24), we obtain

$$\begin{aligned}\dot{S}_{\lambda,z} \leq & \langle \lambda - \lambda^*, G(u_{\lambda z}) - Lz - L\lambda\rangle + \langle z - z^*, L\lambda\rangle \\ & - \sum_{p=1}^{Nm}\sum_{\tau=2}^{\kappa_p^z} \frac{\beta_{p\tau}^z}{\alpha_{p\tau}^z}\zeta_{p\tau}^2 - \sum_{q=1}^{Nm}\sum_{\eta=2}^{\kappa_q^\lambda} \frac{\beta_{q\eta}^\lambda}{\alpha_{q\eta}^\lambda}\omega_{q\eta}^2 \\ =& \langle \lambda - \lambda^*, G(u_{\lambda z})\rangle - \lambda^T L z^* - \lambda^T L\lambda \\ & - \sum_{p=1}^{Nm}\sum_{\tau=2}^{\kappa_p^z} \frac{\beta_{p\tau}^z}{\alpha_{p\tau}^z}\zeta_{p\tau}^2 - \sum_{q=1}^{Nm}\sum_{\eta=2}^{\kappa_q^\lambda} \frac{\beta_{q\eta}^\lambda}{\alpha_{q\eta}^\lambda}\omega_{q\eta}^2 \end{aligned}$$

Note that $\lambda(t) \in \mathcal{T}_{\mathbb{R}_+^{Nm}}(\lambda^*)$, $L\lambda^* = 0$, and $G(u_{\lambda z}^*) - L\lambda^* - Lz^* \in \mathcal{N}_{\mathbb{R}_+^{Nm}}(\lambda^*)$. Consequently, $\langle \lambda - \lambda^*, G(u_{\lambda z}^*) - Lz^*\rangle \leq 0$, and moreover,

$$\begin{aligned}\dot{S}_{\lambda,z} \leq & \big\langle \lambda - \lambda^*, G(u_{\lambda z}) - G(u_{\lambda z}^*)\big\rangle - \lambda^T L\lambda \\ & - \sum_{p=1}^{Nm}\sum_{\tau=2}^{\kappa_p^z} \frac{\beta_{p\tau}^z}{\alpha_{p\tau}^z}\zeta_{p\tau}^2 - \sum_{q=1}^{Nm}\sum_{\eta=2}^{\kappa_q^\lambda} \frac{\beta_{q\eta}^\lambda}{\alpha_{q\eta}^\lambda}\omega_{q\eta}^2. \end{aligned} \tag{25}$$

Because of the convexity of G and $\lambda \in \mathbb{R}_+^{Nm}$,

$$\begin{aligned}\big\langle \lambda, G(u_{\lambda z}) - G(u_{\lambda z}^*)\big\rangle \leq & \langle \lambda, \nabla G(u_{\lambda z})(u_{\lambda z} - u_{\lambda z}^*)\rangle \\ =& \langle y_{\lambda z}, u_{\lambda z} - u_{\lambda z}^*\rangle, \end{aligned}$$

and moreover,

$$-\big\langle \lambda^*, G(u_{\lambda z}) - G(u_{\lambda z}^*)\big\rangle \leq -\langle y_{\lambda z}^*, u_{\lambda z} - u_{\lambda z}^*\rangle.$$

Combining the above inequalities with (25), we have

$$\begin{aligned}\dot{S}_{\lambda,z} \leq & \langle y_{\lambda z} - y_{\lambda z}^*, u_{\lambda z} - u_{\lambda z}^*\rangle - \lambda^T L\lambda \\ & - \sum_{p=1}^{Nm}\sum_{\tau=2}^{\kappa_p^z} \frac{\beta_{p\tau}^z}{\alpha_{p\tau}^z}\zeta_{p\tau}^2 - \sum_{q=1}^{Nm}\sum_{\eta=2}^{\kappa_q^\lambda} \frac{\beta_{q\eta}^\lambda}{\alpha_{q\eta}^\lambda}\omega_{q\eta}^2. \end{aligned} \tag{26}$$

Thus, $\dot{S}_{\lambda,z} \leq \langle \tilde{u}_{\lambda z}, \tilde{y}_{\lambda z}\rangle$, and part b) holds.

c) Construct a Lyapunov function candidate as $S = S_x + S_{\lambda,z}$. Note that $u_{\lambda z} = y_x$ and $u_x = -y_{\lambda z}$. Recalling (23) and (26) yields

$$\begin{aligned}\dot{S} \leq & -\sum_{r=1}^{n} \gamma_r^x \|[F(x) + \nabla G(x)^T \lambda]_r\|^2 - \sum_{r=1}^{n}\sum_{\rho=2}^{\kappa_r^x} \frac{\beta_{r\rho}^x}{\alpha_{r\rho}^x}\xi_{r\rho}^2 \\ & - \langle x - x^*, F(x) - F(x^*)\rangle - \lambda^T L\lambda, \\ & - \sum_{p=1}^{Nm}\sum_{\tau=2}^{\kappa_p^z} \frac{\beta_{p\tau}^z}{\alpha_{p\tau}^z}\zeta_{p\tau}^2 - \sum_{q=1}^{Nm}\sum_{\eta=2}^{\kappa_q^\lambda} \frac{\beta_{q\eta}^\lambda}{\alpha_{q\eta}^\lambda}\omega_{q\eta}^2. \end{aligned} \tag{27}$$

The monotonicity of F implies $\dot{S} \leq 0$. Then every trajectory $\big(\xi(t), \omega(t), \zeta(t)\big)$ is bounded since S is radially unbounded.

Let $\mathcal{R} = \{(\xi, \omega, \zeta) \mid \dot{S} = 0\}$, and $\mathcal{M}$ be the largest invariant subset of $\mathcal{R}$. By the LaSalle's invariance principle [16, Th. 4.4], $\big(\xi(t), \omega(t), \zeta(t)\big) \to \mathcal{M}$ as $t \to \infty$. Suppose that $\big(\bar{\xi}(t), \bar{\omega}(t), \bar{\zeta}(t)\big)$ is a trajectory of (19), and $(\bar{x}(t), \bar{\lambda}(t), \bar{z}(t))$ is the corresponding output. Since $\mathcal{M}$ is a positive invariant set, $\big(\bar{\xi}(t), \bar{\omega}(t), \bar{\zeta}(t)\big) \in \mathcal{M}$ for all $t \geq 0$ if $\big(\bar{\xi}(0), \bar{\omega}(0), \bar{\zeta}(0)\big) \in \mathcal{M}$. Next we characterize $\mathcal{M}$. Recall that $\gamma_r^x > 0$ or $\kappa_r^x \geq 2$ for all $M_r(s)$. From $\dot{S} = 0$ it follows that either $[F(\bar{x}(t)) + \nabla G(\bar{x}(t))^T \bar{\lambda}(t)]_r = 0$, if $\gamma_r^x > 0$, or $\bar{\xi}_{r\rho}(t) = 0$, $\dot{\bar{\xi}}_{r\rho}(t) = 0$, and $[F(\bar{x}(t)) + \nabla G(\bar{x}(t))^T \bar{\lambda}(t)]_r = 0$, if $\kappa_r^x \geq 2$, $\forall r \in \{1, \ldots, n\}$, $\forall \rho \in \{2, \ldots, \kappa_r^x\}$. In summary, it holds that

$$F(\bar{x}(t)) + \nabla G(\bar{x}(t))^T \bar{\lambda}(t) = \mathbf{0},$$

$\dot{\bar{\xi}}(t) = \mathbf{0}$, and $\bar{x}(t) = \bar{x}(0)$. Furthermore, $\dot{S} = 0$ implies $L\bar{\lambda}(t) = \mathbf{0}$, $\dot{\bar{\zeta}}(t) = \mathbf{0}$, and $\bar{z}(t) = \bar{z}(0)$. By (19), we have

$$\dot{\bar{\omega}}_{q1} = \Pi_{\mathbb{R}_+}\big[\bar{\omega}_{q1}, \alpha_{q1}^\lambda [G(\bar{x}(0)) - L\bar{z}(0)\big].$$

If $\dot{\bar{\omega}}_{q1} \neq \mathbf{0}$, then $\lim_{t\to\infty} \bar{\omega}_{q1}(t) = \infty$, which contradicts the boundness of the trajectory, hence $\dot{\bar{\omega}}_{q1} = \mathbf{0}$. Thus, by $\dot{S} = 0$, $\dot{\bar{\omega}}(t) = \mathbf{0}$, and $\bar{\lambda}(t) = \bar{\lambda}(0)$. Therefore, any $\big(\xi, \omega, \zeta\big) \in \mathcal{M}$ is an equilibrium point of (19).

In conclusion, every trajectory $(\xi(t), \omega(t), \zeta(t))$ converges to an equilibrium point of (19), where by Lemma 2, the x component is a GNE of (2). □

Remark 6: Theorem 2 indicates that the passivity of Σ_x and $\Sigma_{\lambda z}$ is preserved after introducing the PFCs. On the other hand, dynamics (19) can deal with a broader class of games than (6) since it does not require F to be strictly monotone. Intuitively speaking, as a result of the compensators, $\tilde{\Sigma}_x$ and $\tilde{\Sigma}_{\lambda z}$ cannot be both passive lossless as shown in (23) and (26), due to the additional negative terms introduced by the compensators. In Theorem 2, we take $\kappa_r^x \geq 2$ or $\gamma_r^x > 0$ as the general case for the analysis, and we can simply take $\kappa_r^x = 2$ or $\gamma_r^x > 0$ in practice. However, we should mention that different PFCs will affect the performance including the convergence rate of (19).

Remark 7: The concept of passivity has been adopted in the analysis and design of optimization algorithms [29], [31]. However, in [31], the cost functions are strictly monotone. In [29], PFCs were designed for a primal-dual dynamics to solve constrained optimization problems with general convex cost functions. To the best of our knowledge, this paper introduces PFCs to seek a GNE for the first time. Note that dynamics (6) is different from that of [29] because J_i depends on x^i as well as x^{-i}, and the coupled constraints $g(x) \leq \mathbf{0}$ are considered. Furthermore, we solve (2) distributedly rather than via a centralized method as [29].

In the following, we discuss a special case of (2), where $g_i(x^i) \equiv \mathbf{0}$. Then the formulation is the same as that of [19], and dynamics (19) degenerates into

$$\begin{cases} \dot{\xi}_{r1} = -\alpha_{r1}^x [F(x)]_r, \\ \dot{\xi}_{r\rho} = -\beta_{r\rho}^x \xi_{r\rho} - \alpha_{r\rho}^x [F(x)]_r, \ \rho \in \{2, \ldots, \kappa_r^x\} \\ x_r = 1_{\kappa_r^x}^T \xi_r - \gamma_r^x [F(x)]_r, \ r \in \{1, \ldots, n\}. \end{cases} \tag{28}$$

By (10), Σ_x in (7) is a passive-short system if F is hypomonotone. Referring [15, Chapter 2], a passive-short system can be passivated by adding passive-excess compensators. In light of (23), $\tilde{\Sigma}_x$ in (21) is a passive-excess system by taking $\gamma_r^x > 0$. Thus, this is potential to deal with hypomonotone games as shown in the next theorem.

Theorem 3: Consider dynamics (28). Let Assumptions 1 hold, and moreover, F be ν-hypomonotone and R-inverse Lipschitz. If $\gamma_r^x > \nu R^2, \forall r \in \{1, \ldots, n\}$, then the trajectory of $x(t)$ converges to an NE x^* of (2).

Proof. Let $\xi^* = \mathrm{col}\{\xi_r^*\}$ be an equilibrium point of (28), and x^* be the corresponding output, where $\xi_r^* = [\xi_{r1}^*, \xi_{r2}^*, \ldots, \xi_{r\kappa_r^x}^*]^T$. Then $[\xi_{r1}^*, \xi_{r2}^*, \ldots, \xi_{r\kappa_r^x}^*]^T = [x_r^*, 0, \ldots, 0]^T$, $F(x^*) = \mathbf{0}$, and x^* is an NE of (2).

Construct a Lyapunov function candidate as

$$\tilde{V} = \sum_{r=1}^n (\xi_{r1} - x_r^*)^2 + \sum_{r=1}^n \sum_{\rho=2}^{\kappa_r^x} \xi_{r\rho}^2.$$

By a similar procedure as the proof of (23), we obtain

$$\begin{aligned} \dot{\tilde{V}} = &- \sum_{r=1}^n \sum_{\rho=2}^{\kappa_r^x} \frac{\beta_{r\rho}^x}{\alpha_{r\rho}^x} \xi_{r\rho}^2 - \sum_{r=1}^n \gamma_r^x \| [F(x) - F(x^*)]_r \|^2 \\ &- \langle x - x^*, F(x) - F(x^*) \rangle. \end{aligned} \tag{29}$$

Since F is ν-hypomonotone and R-inverse Lipschitz,

$$\langle x - x^*, F(x) - F(x^*) \rangle \geq -\nu \|x - x^*\|^2,$$

and moreover,

$$\sum_{r=1}^n \gamma_r^x \| [F(x) - F(x^*)]_r^2 \| \geq \min_r \{\gamma_r^x / R^2\} \|x - x^*\|^2.$$

Substituting the above two inequalities to (29), we obtain $\dot{\tilde{V}} \leq -\min_r\{\gamma_r^x/R^2 - \nu\}\|x - x^*\|^2$. If $\gamma_r^x > \nu R^2$, then there exists $\delta > 0$ such that $\dot{\tilde{V}} \leq -\delta \|x - x^*\|^2$. Invoking the LaSalle invariance principle [16, Th. 4.4], $x(t)$ converges to x^*. This completes the proof. □

Remark 8: To the best of our knowledge, only NE seeking for hypomonotone games has been discussed in existing literatures such as [19], and thus, Theorem 3 also considers the unconstrained case. The inverse Lipschitz condition can hold in many practical applications (see [19] for more details). It was shown that HA in [19] could achieve exact convergence for hypomonotone games satisfying $\nu R < 1$. Theorem 3 indicates that dynamics (28) can handle a broader class of hypomonotone games than HA, since the restriction on νR is removed. The main reason is that $\tilde{\Sigma}_x$ in (21) is a passive-excess system, and moreover, its passivity index can be adjusted by selecting suitable γ_r^x.

VI. CONCLUSION

This paper focused on the distributed generalized Nash equilibrium seeking for noncooperative games with nonlinear coupled constraints. Inspired by the concept of passivity, a novel gradient-play dynamics was proposed by introducing parallel feedforward compensators. The dynamics allowed a relaxation of the strictly monotone assumption on pseudo-gradients, which was important to ensure the exact convergence of the standard gradient-play dynamics. Furthermore, it could also compute Nash equilibria for hypomonotone games without coupled constraints. Possible directions for future work include analyzing the convergence rate of (19), and extending the dynamics to a discrete-time setup for its numerical implementation.

APPENDIX

Proof of Theorem 1: a) Due to $u_x^* = F(x^*)$,

$$\begin{aligned} \dot{V}_x =& \langle x - x^*, -F(x) + u_x \rangle \\ =& -\langle x - x^*, F(x) - F(x^*) \rangle + \langle y_x - y_x^*, u_x - u_x^* \rangle. \end{aligned} \tag{30}$$

By the monotonicity of F, $\dot{V}_x \leq \langle \tilde{u}_x, \tilde{y}_x \rangle$, and part a) holds.

b) It follows from (8) that

$$\begin{aligned} \dot{V}_{\lambda,z} = & \langle \lambda - \lambda^*, \Pi_{\mathbb{R}_+^{Nm}} [\lambda, G(u_{\lambda z}) - Lz - L\lambda] \\ & - \Pi_{\mathbb{R}_+^{Nm}} [\lambda^*, G(u_{\lambda z}^*) - Lz^* - L\lambda^*] + \langle z - z^*, L\lambda - L\lambda^* \rangle. \end{aligned}$$

Recalling (1) gives $[G(u_{\lambda z}) - Lz - L\lambda] - \Pi_{\mathbb{R}_+^{Nm}} [\lambda, G(u_{\lambda z}) - Lz - L\lambda] \in \mathcal{N}_{\mathbb{R}_+^{Nm}}(\lambda)$. Due to $\lambda^* - \lambda \in \mathcal{T}_{\mathbb{R}_+^{Nm}}(\lambda)$, we obtain

$$\begin{aligned} & \langle \lambda - \lambda^*, \Pi_{\mathbb{R}_+^{Nm}} [\lambda, G(u_{\lambda z}) - Lz - L\lambda] \rangle \\ & \leq \langle \lambda - \lambda^*, G(u_{\lambda z}) - Lz - L\lambda \rangle. \end{aligned}$$

By a similar procedure, we have

$$-\big\langle \lambda-\lambda^*, \Pi_{\mathbb{R}_+^{Nm}}\big[\lambda^*, G(u_{\lambda z}^*)-Lz^*-L\lambda^*\big]\big\rangle \\ \leq -\big\langle \lambda-\lambda^*, G(u_{\lambda z}^*)-Lz^*-L\lambda^*\big\rangle.$$

Consequently,

$$\dot{V}_{\lambda,z} \leq \big\langle \lambda-\lambda^*, G(u_{\lambda z})-G(u_{\lambda z}^*)\big\rangle - \lambda^T L\lambda.$$

Because of the convexity of G and $\lambda \in \mathbb{R}_+^{Nm}$, $\langle \lambda, G(u_{\lambda z})-G(u_{\lambda z}^*)\rangle \leq \langle \lambda, \nabla G(u_{\lambda z})(u_{\lambda z}-u_{\lambda z}^*)\rangle = \langle y_{\lambda z}, u_{\lambda z}-u_{\lambda z}^*\rangle$, and moreover, $-\langle \lambda^*, G(u_{\lambda z})-G(u_{\lambda z}^*)\rangle \leq -\langle y_{\lambda z}^*, u_{\lambda z}-u_{\lambda z}^*\rangle$. As a result,

$$\dot{V}_{\lambda,z} \leq \langle u_{\lambda z}-u_{\lambda z}^*, y_{\lambda z}-y_{\lambda z}^*\rangle - \lambda^T L\lambda. \tag{31}$$

Therefore, $\dot{V}_{\lambda,z} \leq \langle \tilde{u}_{\lambda z}, \tilde{y}_{\lambda z}\rangle$, and part b) is proved.

c) Construct a Lyapunov function candidate as $V = V_x + V_{\lambda,z}$. Combining (30) with (31), we obtain

$$\dot{V} \leq -\langle x-x^*, F(x)-F(x^*)\rangle - \lambda^T L\lambda \\ + \langle u_x-u_x^*, y_x-y_x^*\rangle + \langle u_{\lambda z}-u_{\lambda z}^*, y_{\lambda z}-y_{\lambda z}^*\rangle.$$

According to $u_x = -y_{\lambda z}$ and $y_x = u_{\lambda z}$,

$$\dot{V} \leq -\langle x-x^*, F(x)-F(x^*)\rangle - \lambda^T L\lambda \leq 0. \tag{32}$$

Therefore, (x^*, λ^*, z^*) is a Lyapunov stable equilibrium point. Furthermore, since V is radially unbounded, every trajectory $\big(x(t), \lambda(t), z(t)\big)$ is bounded.

The strict monotonicity of F implies that x^* is unique. Let $\mathcal{R} = \{(x,\lambda,z) : \dot{V} = 0\} \subset \{(x,\lambda,z) : x = x^*, L\lambda = \mathbf{0}\}$, and $\mathcal{M}$ be the largest invariant subset of $\mathcal{R}$. From LaSalle's invariance principle [16, Theorem 4.4], every trajectory $(x(t),\lambda(t),z(t)) \to \mathcal{M}$ as $t \to \infty$. Moreover, if $(x(0),\lambda(0),z(0)) \in \mathcal{M}$, then $(x(t),\lambda(t),z(t)) \in \mathcal{M}$ for all $t \geq 0$, i.e., $x(t) = x^*$, $L\lambda(t) = \mathbf{0}$. To characterize $\mathcal{M}$, from $x(t) = x^*$ and $L\lambda(t) = \mathbf{0}$, it follows that $z(t) = z(0)$, and $\dot{\lambda} = \Pi_{\mathbb{R}_+^{Nm}}\big[\lambda, G(x^*)-Lz(0)\big]$. If $\dot{\lambda} \neq \mathbf{0}$, then $\lim_{t\to\infty}\lambda(t) = \infty$, which contradicts the boundness of $\lambda(t)$. Hence, $\dot{\lambda} = \mathbf{0}$. To sum up, any $(x,\lambda,z) \in \mathcal{M}$ is an equilibrium point of (6), and every trajectory $(x(t),\lambda(t),z(t))$ converges to an equilibrium point in $\mathcal{M}$. By Lemma 1, x^* is a GNE of (2). This completes the proof. $\square$

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