2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Global Finite-time Stabilization for a Chain of Integrators in the Presence of Actuator Saturation by Homogeneous Feedback

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Abstract— **This paper considers the problem of globally stabilizing a chain of integrators with actuator saturation in finite time. The proposed method is adapted from an existing scheduled low gain approach by relaxing the range of the value of the low gain parameter. Due to the special structure of the considered system, the proposed feedback law not only results in finite-time stability but also is homogeneous in the whole state space. The homogeneity of the proposed feedback law allows its implementation through the discretization of a compact set of the state space and thus obviation from numerical difficulties in the existing scheduled low gain design. A numerical example is presented to validate our theoretical results and demonstrate that our approach achieves faster convergence than the existing methods.**

Index Terms— **Actuator saturation, low gain feedback, finite-time stabilization, global stabilization.**

I. INTRODUCTION

In this paper, we revisit the problem of globally stabilizing a chain of integrators in the presence of actuator saturation. The dynamics of the considered system is governed by

$$\dot{x} = Ax + B\mathrm{sat}\left(u\right), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, matrices A and B take the following form

$$A = \begin{bmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $\mathrm{sat} : \mathbb{R} \to \mathbb{R}$ is the standard decentralized saturation function, *i.e.*, $\mathrm{sat}(u) = \mathrm{sgn}(u)\min\{1, |u|\}$. This problem has been well studied and belongs to a much broader topic, global stabilization of linear systems with bounded controls. In what follows, we recollect some results concerning this problem from the literature.

The work of T. Hou and Y. Li is supported in part by the National Natural Science Foundation of China under Grant Nos. 62022055 and 61973215.

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It has been proven in [1] that, for a chain of integrators (1) with $n \geq 3$, there does exist a globally stabilizing linear feedback in the form of $u = -K^{\mathrm{T}}x$, where $K \in \mathbb{R}^n$. As a result, many efforts have been devoted to finding nonlinear feedback laws for global stabilization of (1). One of the most elegant control strategies was established in [2], which is referred to as the nested saturation method. This result was extended to general ANCBC systems in [3]. Here, ANCBC stands for asymptotically null controllable by bounded controls, which is a necessary and sufficient condition for global stabilization of linear systems subject to actuator saturation. The nested saturation method has also been studied by numerous works in the past decades to improve its convergence speed [4, 5], to incorporate with an event-trigger mechanism [6], or even to achieve finite-time stability [7].

Another important technique for solving the global stabilization problem of (1) is the scheduled low gain method [8], which was developed for general ANCBC systems. Utilizing this technique, the scheduled low-and-high gain strategy was developed [9, 10] for global practical stabilization in the presence of uncertainties and disturbances. Even though they yield appealing theoretical results, the scheduled low gain method and its variations are generally hard to implement in practical applications because they require solving an optimization problem online to determine the value of the state-dependent low gain parameter.

This paper aims to circumvent numerical difficulties encountered in global stabilization of (1) using the existing scheduled low gain feedback approach. In particular, by exploiting the special structure of (1), we find that solutions to a parameterized Riccati equation can be completely characterized by a solution to the same equation with a fixed parameter. This facilitates us to interpret the associated scheduled low gain feedback from a homogeneity perspective. Based on this observation, we further relax range of the value of the low gain parameter from $(0, 1]$ to $(0, \infty)$ to result in a homogeneous feedback law that renders the closed-loop system globally finite-time stable. Due to the homogeneity and the continuity of the resulting feedback law, we can implement it by discretizing a compact set of the state space, which reduces the online optimization in the existing scheduled low gain method to the offline computation of a finite number of feedback gains. A numerical example validates our theoretical results and shows that our method achieves faster convergence compared with the existing results.

Notation: In this paper, we use standard notions. For symmetric matrix $P \in \mathbb{R}^{n\times n}$, $P > 0$ means that P is positive definite. For a matrix $P > 0$ and a positive scalar $\rho > 0$, $\mathcal{L}_{\mathrm{V}}(P,\rho) := \{x \in \mathbb{R}^n : x^{\mathrm{T}}Px = \rho\}$. For a finite set $\mathcal{X}$, $|\mathcal{X}|$ denotes its cardinality. For a real number $c \in \mathbb{R}$, $\lceil c \rceil$ denotes the nearest integer greater than or equal to c. For a vector $a \in \mathbb{R}^n$, $\mathrm{diag}(a) \in \mathbb{R}^{n\times n}$ denotes the diagonal matrix with the ith diagonal entry being a_i. Λ_n denotes the unit simplex, *i.e.*, $\Lambda_n := \{\gamma = [\gamma_1\ \gamma_2\ \cdots\ \gamma_n]^{\mathrm{T}} : \sum_{i=1}^n \gamma_i = 1, \gamma_i \geq 0, i \in \{1, 2\cdots, n\}\}$

II. Preliminary: A New Interpretation of the Scheduled Low Gain Design Using a Parameterized Riccati Equation

In this section, we revisit the scheduled low gain design based on a parameterized Riccati equation and reveal some properties of the associated feedback law. These properties will help us to understand the scheduled low gain feedback from a homogeneity perspective and facilitate our subsequent development.

We first introduce the generalized homogeneity from [11].

Definition 1: A real function $u : \mathbb{R}^n \to \mathbb{R}$ is called homogeneous of order m with respect to the dilation matrix $D_r = \mathrm{diag}\left([r^n\ r^{n-1} \cdots r^1]^{\mathrm{T}}\right)$ if $u(x) = r^m u(D_r x)$.

Let $P(r)$ be the solution of the following algebraic Riccati equation, parameterized in r,

$$P(r)A + A^{\mathrm{T}}P(r) - P(r)BB^{\mathrm{T}}P(r) + rP(r) = 0. \quad (2)$$

The scheduled low gain feedback law is constructed as

$$u(x) = -kB^{\mathrm{T}}P(\varepsilon(x))x, \quad (3)$$

where $\varepsilon(x)$ is the state-dependent low gain parameter determined by

$$\varepsilon(x) = \max\left\{r \in (0,1] : x^{\mathrm{T}}P(r)x \times \mathrm{Tr}(B^{\mathrm{T}}P(r)B) \leq 1\right\}, \quad (4)$$

and $k \geq 1$ is an arbitrary constant. The following theorem is adapted from [8], where a parameterized LQR Riccati equation, instead of the Riccati equation (2) is used.

Theorem 1: System (1) is globally asymptotically stabilized by the scheduled low gain feedback (3).

Due to the special structure of matrices A and B, solutions to the parameterized Riccati equation (2) can be completely characterized by its solution for a fixed value of r. This property has also been observed in [12] for a slightly different dilation matrix D_r.

Property 1: The solution to (2) takes the form of $P(r) = \frac{1}{r}D_rP(1)D_r$, where $P(1)$ is the solution of (2) with $r = 1$ and D_r is a dilation matrix defined as $D_r = \mathrm{diag}([r^n\ r^{n-1} \cdots\ r]^{\mathrm{T}})$.

Proof: Noting that $B^{\mathrm{T}}D_r = rB^{\mathrm{T}}, D_rA = rAD_r$, we verify that

$$\begin{aligned} &P(r)A + A^{\mathrm{T}}P(r) - P(r)BB^{\mathrm{T}}P(r) + rP \\ &= \frac{1}{r}D_rP(1)D_rA + \frac{1}{r}A^{\mathrm{T}}D_rP(1)D_r + D_rP(1)D_r \\ &- \frac{1}{r^2}D_rP(1)D_rBB^{\mathrm{T}}D_rP(1)D_r \\ &= D_r(P(1)A + A^{\mathrm{T}}P(1) - P(1)BB^{\mathrm{T}}P(1) + P(1))D_r \\ &= 0, \end{aligned}$$

where the last equality follows from the definition of $P(1)$. This completes the proof. ■

The property 1 enables us to write the scheduled low gain feedback in a more concise way.

Theorem 2: System (1) is globally asymptotically stabilized by the feedback law $u(x) = -kB^{\mathrm{T}}P(1)D_{\varepsilon(x)}x$, where $k \geq 1$ is any constant, $P(1)$ is the solution to (2) with $r = 1$, and $\varepsilon(x)$ is defined by

$$\varepsilon(x) = \max\left\{r \in (0,1] : x^{\mathrm{T}}D_rP(1)D_rx \times \mathrm{Tr}(B^{\mathrm{T}}P(1)B) \leq 1\right\}. \quad (5)$$

Proof: It follows from Property 1 that the feedback law (3) can be rewritten as $u = -kB^{\mathrm{T}}P(1)D_{\varepsilon(x)}x$, and (4) is equivalent to (5). ■

Theorem 2 reveals the quasi-homogeneity of the scheduled low gain feedback law, *i.e.*, (3) is homogeneous with respect to D_r in $\mathbb{R}^n$ excludes a compact set. To make this more obvious, we define $\mathcal{K} = \{x : \varepsilon(x) = 1\}$. Clearly, $\mathcal{K}$ is a compact set. In fact, $\mathcal{K}$ is exactly the set $\{x : x^{\mathrm{T}}P(1)x \times \mathrm{Tr}(B^{\mathrm{T}}P(1)B) \leq 1\}$. Moreover, for any $x \in \mathbb{R}^n \setminus \mathcal{K}$, (5) is equivalent to

$$\varepsilon(x) = \max\left\{r \in (0,1) : x^{\mathrm{T}}D_rP(1)D_rx \times \mathrm{Tr}(B^{\mathrm{T}}P(1)B) = 1\right\}. \quad (6)$$

This leads to the following property of the scheduled low gain feedback law.

Property 2: Let $x \in \mathbb{R}^n \setminus \mathcal{K}$, and suppose $y = D_\alpha x \in \mathbb{R}^n \setminus \mathcal{K}$, for some $\alpha \in (0,\infty)$. Then, we have $u(x) = u(y)$, where $u(x)$ has been defined in Theorem 2.

Proof: Since x and $y = D_\alpha x$ are within the set $\mathbb{R}^n \setminus \mathcal{K}$, it follows from (6) that $D_{\varepsilon(x)} = D_{\alpha\varepsilon(D_\alpha x)} = D_{\alpha\varepsilon(y)}$, *i.e.*, $\varepsilon(x) = \alpha\varepsilon(y)$. Therefore

$$\begin{aligned} u(y) = u(D_\alpha x) &= -kB^{\mathrm{T}}P(1)D_{\varepsilon(D_\alpha x)}D_\alpha x \\ &= -kB^{\mathrm{T}}P(1)D_{\varepsilon(x)}x = u(x). \end{aligned}$$

■

With these in hand, the scheduled low gain feedback law can be interpreted as a piecewise continuous feedback defined by

$$u(x) = \begin{cases} -kBP(1)x, & x \in \mathcal{K}, \\ -kBP(1)D_{\varepsilon(x)}x = -kBP(1)\bar{x}, & x \in \mathbb{R}^n \setminus \mathcal{K}, \end{cases} \quad (7)$$

where $\bar{x} := D_{\varepsilon(x)}x \in \mathcal{L}_{\mathrm{V}}(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))$. In other words, the scheduled low gain feedback law $u(x)$ is linear in $\mathcal{K}$ and is homogeneous in $\mathbb{R}^n \setminus \mathcal{K}$ with respect to D_r.

In the next section, we will show that the quasi-homogeneity of (5) becomes homogeneity if we relax the range of the value of the low gain parameter r in (5) from $(0,1]$ to $(0,+\infty)$. This will also render the closed-loop system finite-time stable.

III. GLOBAL FINITE-TIME STABILIZATION BY HOMOGENEOUS FEEDBACK

To obtain a homogeneous feedback law, we relax the range of the value of the parameter r in (5) from $(0, 1]$ to $(0, \infty)$. The resulting feedback law is given as

$$\tilde{u}(x) = \begin{cases} 0, & x = 0, \\ -kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x)}x, & x \neq 0, \end{cases} \tag{8}$$

where $k \geq 1$ is any constant and $\tilde{\varepsilon}(x)$ is defined by

$$\tilde{\varepsilon}(x) = \max\{r \in (0, \infty) : x^{\mathrm{T}}D_rP(1)D_rx \times \mathrm{Tr}(B^{\mathrm{T}}P(1)B) \leq 1\}. \tag{9}$$

For completeness, we let $\tilde{\varepsilon}(0) = +\infty$ and $\tilde{u}(0) = 0$.

Property 3: The scheduled feedback (8) is homogeneous of degree 0 with respect to the dilation matrix D_r in $\mathbb{R}^n \setminus \{0\}$, *i.e.*, $\tilde{u}(x) = \tilde{u}(D_rx)$ for any $x \in \mathbb{R}^n \setminus \{0\}$ and $r \in (0, \infty)$.

Proof: It is clear that, for a given $x \in \mathbb{R}^n \setminus \{0\}$, Eq. (9) is equivalently to

$$\tilde{\varepsilon}(x) = \max\{r \in (0, \infty) : x^{\mathrm{T}}D_rP(1)D_rx \times \mathrm{Tr}(B^{\mathrm{T}}P(1)B) = 1\}. \tag{10}$$

Hence, following the same steps as those in the proof of Property 2, we deduce that, for any $x \in \mathbb{R}^n \setminus \{0\}$ and any $r \in (0, \infty)$, $u(x) = u(D_rx)$. ■

The difference between (5) and (9) is that we allow the parameter r to tend to $+\infty$ as x approaches the origin. Nevertheless, in view of the fact that

$$\begin{aligned} |\tilde{u}(x)| &\leq k|B^{\mathrm{T}}P(1)^{\frac{1}{2}}||P(1)^{\frac{1}{2}}D_{\tilde{\varepsilon}(x)}x| \\ &\leq k|B^{\mathrm{T}}P(1)B| \times |x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)D_{\tilde{\varepsilon}(x)}x| \leq k, \end{aligned}$$

the homogeneous feedback (8) is globally bounded. Next, we will show that this homogeneous feedback law results in global finite-time stabilization. To this end, we first recall the definition of finite-time stability from [13] and then establish some technical lemmas for the proof.

Definition 2: System (1) is said to be globally finite-time stable at the origin if it is globally asymptotically stable and there exists a locally bounded function $T : \mathbb{R}^n \to [0, \infty)$, such that $x(t, x_0) = 0$ for any $t \geq T(x_0)$, where $x(t, x_0)$ is the trajectory of (1) originating from x_0. In particular, the function T is called the settling-time function.

Next, we adapt a useful lemma from [14] concerning the monotonicity of the solution to (2) with respect to r.

Lemma 1: For any $r \in (0, \infty)$, the solution $P(r)$ to (2) satisfies $\frac{\mathrm{d}P(r)}{\mathrm{d}r} > 0$.

Lemma 1 enables us to derive the following property.

Lemma 2: The matrix $D_r(P(1)\Psi + \Psi P(1))D_r$ is positive definite for any $r \in (0, \infty)$, where $\Psi = \mathrm{diag}([n\; n{-}1 \cdots 1]^{\mathrm{T}})$

Proof: It follows from Lemma 1 and Property 1 that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}r}P(r) = &\frac{1}{r}\left(\left(\frac{\mathrm{d}}{\mathrm{d}r}D_r\right)P(1)D_r + D_rP(1)\left(\frac{\mathrm{d}}{\mathrm{d}r}D_r\right)\right) \\ &- \frac{1}{r^2}D_rP(1)D_r > 0. \end{aligned}$$

Since $r \in (0, \infty)$, we have

$$\left(\frac{\mathrm{d}}{\mathrm{d}r}D_r\right)P(1)D_r + D_rP(1)\left(\frac{\mathrm{d}}{\mathrm{d}r}D_r\right) > 0. \tag{11}$$

Noting that

$$\frac{\mathrm{d}D_r}{\mathrm{d}r} = \begin{bmatrix} nr^{n-1} & 0 & \cdots & 0 \\ 0 & (n-1)r^{n-2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = \frac{1}{r}\Psi D_r = \frac{1}{r}D_r\Psi,$$

we observe from (11) that $D_r\left(P(1)\Psi + \Psi P(1)\right)D_r > 0$. This completes the proof. ■

Now, we are ready to establish that, under the homogeneous feedback law (8), the closed-loop system is globally finite-time stable.

Theorem 3: System (1) under the homogeneous feedback law (8) is globally finite-time stable at the origin, and, for any initial state $x_0 \in \mathbb{R}^n$, the settling time is upper bounded by $1/(\kappa\tilde{\varepsilon}(x_0))$, where κ is constant independent of x_0.

Proof: Choose the Lyapunov function as

$$V(x) = x^{\mathrm{T}}P(\tilde{\varepsilon}(x))x = \frac{1}{\tilde{\varepsilon}(x)}x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)D_{\tilde{\varepsilon}(x)}x. \tag{12}$$

Then, it follows from (10) that $V(x) = \frac{1}{\mathrm{Tr}(B^{\mathrm{T}}P(1)B)\tilde{\varepsilon}(x)}$. Obviously, $V(x) > 0$ if $x \in \mathbb{R}^n \setminus \{0\}$, $V(x) \to \infty$ as $|x| \to \infty$, and, by our definition of $\tilde{\varepsilon}(0)$, $V(0) = 0$.

Furthermore, we have

$$\dot{V}(x) = -\frac{1}{\mathrm{Tr}(B^{\mathrm{T}}P(1)B)}\frac{\dot{\tilde{\varepsilon}}(x)}{\tilde{\varepsilon}^2(x)}. \tag{13}$$

Note that (10) implies that

$$x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)D_{\tilde{\varepsilon}(x)}x \times \mathrm{Tr}(B^{\mathrm{T}}P(1)B) = 1, \quad x \in \mathbb{R}^n \setminus \{0\}.$$

Taking differentiation of both sides of the above equation, we obtain, for any $x \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} &2x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)\frac{\mathrm{d}D_{\tilde{\varepsilon}(x)}}{\mathrm{d}\tilde{\varepsilon}(x)}x\dot{\tilde{\varepsilon}}(x) \\ &= -2x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)D_{\tilde{\varepsilon}(x)}(Ax + B\mathrm{sat}(-kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x)}x)) \\ &= -\tilde{\varepsilon}(x)x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}(P(1)A + A^{\mathrm{T}}P(1))D_{\tilde{\varepsilon}(x)}x \\ &\quad + 2\tilde{\varepsilon}(x)x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)B\mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x)}x) \\ &= \tilde{\varepsilon}(x)x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}(P(1) + P(1)BB^{\mathrm{T}}P(1))D_{\tilde{\varepsilon}(x)}x \\ &\quad + 2\tilde{\varepsilon}(x)x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)B(\mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x)}x) \\ &\quad - B^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x)}x) \\ &\geq \tilde{\varepsilon}(x)x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}(P(1) + P(1)BB^{\mathrm{T}}P(1))D_{\tilde{\varepsilon}(x)}x. \end{aligned} \tag{14}$$

By the definition of $\tilde{\varepsilon}(x)$, we have

$$\begin{aligned} &2x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)\frac{\mathrm{d}D_{\tilde{\varepsilon}(x)}}{\mathrm{d}\tilde{\varepsilon}(x)}x\dot{\tilde{\varepsilon}}(x) \\ &= \frac{\dot{\tilde{\varepsilon}}(x)}{\tilde{\varepsilon}(x)}x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}(P(1)\Psi + \Psi P(1))D_{\tilde{\varepsilon}(x)}x \\ &> 0, \quad x \in \mathbb{R}^n \setminus \{0\}, \end{aligned} \tag{15}$$

where the last inequality follows from Lemma 2 and the fact that $\tilde{\varepsilon}(x) > 0$ for any $x \in \mathbb{R}^n \setminus \{0\}$. Consequently, from (14) and (15), we obtain

$$\dot{\tilde{\varepsilon}}(x) \geq \frac{\tilde{\varepsilon}^2(x) x^{\mathrm{T}} D_{\tilde{\varepsilon}(x)} (P(1) + P(1)BB^{\mathrm{T}}P(1)) D_{\tilde{\varepsilon}(x)} x}{x^{\mathrm{T}} D_{\tilde{\varepsilon}(x)} (P(1)\Psi + \Psi P(1)) D_{\tilde{\varepsilon}(x)} x}. \tag{16}$$

Substitution of (16) into (13) yields

$$\begin{aligned} \dot{V}(x) &\leq -\frac{x^{\mathrm{T}} D_{\tilde{\varepsilon}(x)} (P(1) + P(1)BB^{\mathrm{T}}P(1)) D_{\tilde{\varepsilon}(x)} x}{x^{\mathrm{T}} D_{\tilde{\varepsilon}(x)} (P(1)\Psi + \Psi P(1)) D_{\tilde{\varepsilon}(x)} x} \\ &\quad \times \frac{1}{\mathrm{Tr}(B^{\mathrm{T}}P(1)B)}, \\ &< 0, \quad x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{17}$$

Clearly, $V(x)$ always decreases along nontrivial trajectories of the closed-loop system. Next, we will show that $\dot{V}(x)$ is upper bounded by a negative number, which implies that the closed-loop system reaches the origin in finite time. Note that $x^{\mathrm{T}} D_{\tilde{\varepsilon}(x)} P(1) D_{\tilde{\varepsilon}(x)} x \times \mathrm{Tr}(B^{\mathrm{T}}P(1)B) = 1$, *i.e.*, for any $x \in \mathbb{R}^n \setminus \{0\}$, $D_{\tilde{\varepsilon}(x)} x \in \mathcal{L}_V(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))$. In view of this observation, we define

$$g(z) = \frac{z^{\mathrm{T}}(P(1) + P(1)BB^{\mathrm{T}}P(1))z}{z^{\mathrm{T}}(P(1)\Psi + \Psi P(1))z},$$

where $z \in \mathcal{L}_V(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))$. Since the level set $\mathcal{L}_V(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))$ is compact and $g(z)$ is a continuous function of z, $g(z)$ attains its maximum and minimum at some points in $\mathcal{L}_V(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))$. Let $\gamma = \min_{z \in \mathcal{L}_V(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))} g(z)$. Note that $\gamma > 0$ is a constant independent of z. Then, it follows from (17) that

$$\dot{V}(x) \leq -\frac{\gamma}{\mathrm{Tr}(B^{\mathrm{T}}P(1)B)} := -\kappa, \quad x \in \mathbb{R}^n \setminus \{0\}. \tag{18}$$

In other words, for any initial state $x_0 \in \mathbb{R}^n \setminus \{0\}$, the Lyapunov function $V(x)$ decreases to zero, and thus the closed-loop system reaches the origin in a finite time no bigger than $1/(\kappa \tilde{\varepsilon}(x_0))$. Moreover, by our definition of $\tilde{\varepsilon}(0)$, we have $\tilde{u}(0) = 0$. Hence, the closed-loop system will remain at the origin afterward. On the other hand, if $x_0 = 0$, then the corresponding settling time is $1/(\kappa \tilde{\varepsilon}(x_0)) = 0$. This completes the proof. ∎

Remark 1: The results obtained in this paper can be easily extended to more general Riccati equations in the form

$$P(r)A + A^{\mathrm{T}}P(r) - P(r)BB^{\mathrm{T}}P(r) + Q(r) = 0, \tag{19}$$

where $Q(r) = D_r Q D_r$ with $Q > 0$ and $\frac{\mathrm{d}Q(r)}{\mathrm{d}r} > 0$. Particularly, Q can be chosen as any diagonal matrices with positive entries. Note that, in essence, Eq. (2) is a parameterized Lyapunov (linear) equation in $P^{-1}(r)$, for which an analytic solution may be obtained through algebraic manipulation. On the other hand, Eq. (19) is a parameterized Riccati (nonlinear) equation, which is harder to solve analytically compared with (2).

IV. Practical Implementation

The homogeneous design method is advantageous over the existing scheduled low gain feedback in practical implementation. To demonstrate this, we first introduce the dilated norm from [15].

Definition 3: The dilated norm $\mathcal{N} : \mathcal{R}^n \to [0, \infty)$ with respect to the dilation matrix D_r is defined as

$$\mathcal{N}(x) := \left(\sum_{i=1}^{n} x_i^{\frac{d}{n+1-i}} \right)^{\frac{1}{d}} \tag{20}$$

where d is chosen as $d = 2n!$.

By the definition of the dilated norm (20), we have $\mathcal{N}(0) = 0$, $\mathcal{N}(x) > 0$ if $x \neq 0$, and $\mathcal{N}(D_r^a x) = rN(x)$.

With the dilated norm defined in (20), we can construct a projection $\mathcal{P}(x)$ from $\mathbb{R}^n \setminus \{0\}$ onto the dilated unit sphere $\mathcal{N}^{-1}(1)$. Particularly, $\mathcal{P}(x)$ is given by

$$\mathcal{P}(x) = D_{1/\mathcal{N}(x)} x. \tag{21}$$

Then, it follows immediately from the definition of $\mathcal{P}(x)$ and $\mathcal{N}(x)$ that $\mathcal{P}(D_r x) = \mathcal{P}(x)$, $r \in (0, \infty)$. With these in hand, the homogeneous feedback law given in Theorem 3 can be written in a more computationally friendly way for practical implementation.

Theorem 4: The homogeneous feedback law $\tilde{u}(x)$, as given in (22), can be rewritten as

$$\tilde{u}(x) = -kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(\mathcal{P}(x))}\mathcal{P}(x), \tag{22}$$

under which system (1) is globally finite-time stable at the origin.

Proof: The proof follows from the definition of the map $\mathcal{P}(x)$ and Theorem 3. ∎

When the dimension of system (1) is low, we may be able to construct an analytic expression of $\tilde{\varepsilon}(x)$. If this is not the case, the homogeneous feedback in the form (22) is helpful for practical implementation because the feedback law (22) can be executed in two steps: first project x onto the dilated unit sphere $\mathcal{N}^{-1}(1)$, then calculate the corresponding $\tilde{\varepsilon}(\mathcal{P}(x))$. Since $\mathcal{N}^{-1}(1)$ is a compact set, we could discretize it into sufficiently dense grid points x_d, compute the corresponding $\tilde{\varepsilon}(x_d)$ for each point offline, and store it in memory. When executing the homogeneous feedback law (22) in real-time, we will project x onto $\mathcal{N}^{-1}(1)$, select the grid point x_d nearest to $\mathcal{P}(x)$, and use the corresponding $\tilde{\varepsilon}(x_d)$ for $\tilde{\varepsilon}(\mathcal{P}(x))$. In particular, this practical homogeneous feedback can be written as

$$\tilde{u}_{\mathcal{X}}(x) = -kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x_d)}\mathcal{P}(x), \tag{23}$$

where $k \geq 1$, x_d is selected through

$$x_d \in \mathcal{S}(x) := \left\{ x_d \in \mathcal{X} : \arg\min_{x_d \in \mathcal{X}} |x_d - \mathcal{P}(x)| \right\}, \tag{24}$$

and $\mathcal{X}$ is the set of points distributed on $\mathcal{N}^{-1}(1)$. By implementing the homogeneous feedback law in this way, we can circumvent numerical problems, for example, solving the optimization problem (9) online, as in the existing scheduled low gain design. It is also noteworthy to mention that, under the practical homogeneous feedback law (23), the closed-loop system is a dynamic system with a discontinuous right-hand side. In this paper, the solution of the closed-loop system is understood in the sense of Filippov [16]. Particularly, in view of (24), the closed-loop system takes the following form

$$\dot{x} = Ax - \sum_{i\in|\mathcal{S}(x)|} \alpha_i(x) B\mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x_i)}\mathcal{P}(x)), \quad (25)$$

where $x_i \in \mathcal{S}(x)$ and $\alpha(x) \in \Lambda_{|\mathcal{S}(x)|}$.

In what follows, we will show that if we discretize $\mathcal{N}^{-1}(1)$ densely enough, the practical feedback law (23) will retain the finite-time stability property. Let $d_{\mathcal{X}} = \max_{z\in\mathcal{N}^{-1}(1),\tilde{x}\in\mathcal{X}} |\tilde{x} - z|$. Clearly, $d_{\mathcal{X}}$ could be viewed as a measure of the density of $\mathcal{X}$ on $\mathcal{N}^{-1}(1)$.

Theorem 5: For any given η satisfying $0 < \eta < \kappa$, where κ has been defined in (18), and for any $k \geq 1$, there exists a positive number ϱ, such that system (1) is globally finite-time stabilized by (23), with a settling time $T(x_0) \leq 1/(\eta\tilde{\varepsilon}(x_0))$, provided that $d_{\mathcal{X}} < \varrho$.

Proof: The argument of the proof follows from the continuity of the homogeneous feedback law and that of the derivative of the Lyapunov function. Adopting the same Lyapunov function $V(x)$ as defined in (12) and continuing from (14), (15), (25), we obtain

$$\nu(x)\dot{V}(x) \leq -\upsilon(x) + 2\sum\nolimits_{i\in|\mathcal{S}(x)|} \alpha_i(x)\big(x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)B \times \mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x_i)}\mathcal{P}(x))\big), \quad (26)$$

where $x_i \in \mathcal{S}(x)$, and, for notational simplicity, we define

$$\nu(x) = \mathrm{Tr}(B^{\mathrm{T}}P(1)B) \times x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}(P(1)\Psi + \Psi P(1))D_{\tilde{\varepsilon}(x)}x,$$
$$\upsilon(x) = x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}(P(1) - P(1)BB^{\mathrm{T}}P(1))D_{\tilde{\varepsilon}(x)}x.$$

Notice that $\tilde{\varepsilon}(x) = \tilde{\varepsilon}(\mathcal{P}(x))/\mathcal{N}(x)$. Then, by (21), we obtain

$$\mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x_i)}\mathcal{P}(x)) = \mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x_i)/\tilde{\varepsilon}(\mathcal{P}(x))}D_{\tilde{\varepsilon}(x)}x).$$

It then follows from (26) that

$$\dot{V}(x) \leq -\frac{\upsilon(x) - \sum_{i\in|\mathcal{S}(x)|} \alpha_i(x)\psi(x, x_i)}{\nu(x)},$$

where

$$\psi(x, x_i) = 2x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)B \times \mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\tilde{\varepsilon}(x_i)/\tilde{\varepsilon}(\mathcal{P}(x))}D_{\tilde{\varepsilon}(x)}x).$$

Since $\alpha_i(x) \geq 0$, $i \in \{1, 2, \cdots, n\}$, and $\sum_{i=1}^{|\mathcal{S}(x)|} \alpha_i(x) = 1$, to guarantee global finite-time stability of (25) with settling time $T(x_0) \leq 1/(\tilde{\varepsilon}(x_0)\eta)$, it is sufficient to have

$$\psi(x, x_i) \geq \eta\nu(x) - \upsilon(x), \forall x_i \in \mathcal{S}(x), x \in \mathbb{R}^n. \quad (27)$$

Next, to get an estimate of $d_{\mathcal{X}}$, we define a new function $g : [0, \infty) \times \mathcal{L}_{\mathrm{V}}(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B)) \to \mathbb{R}$ as

$$g(\epsilon, z) = \bar{\psi}(\epsilon, z) - \eta\bar{\nu}(z) + \bar{\upsilon}(z), \quad (28)$$

where $\bar{\psi}(\epsilon, z) = 2z^{\mathrm{T}}P(1)B\mathrm{sat}(kB^{\mathrm{T}}P(1)D_{\epsilon}z)$, $\bar{\nu}(x) = \mathrm{Tr}(B^{\mathrm{T}}P(1)B) \times z^{\mathrm{T}}(P(1)\Psi + \Psi P(1))z$ and $\upsilon(x) = z^{\mathrm{T}}(P(1) - P(1)BB^{\mathrm{T}}P(1))z$. Since $0 < \eta < \kappa$, we have $g(1, z) > 0$ for any $z \in \mathcal{L}_{\mathrm{V}}(P(1),\ 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))$. Let $\bar{\epsilon}(z) := \{\sup_{\epsilon>1} \epsilon : g(\epsilon, z) \geq 0\}$, $\underline{\epsilon}(z) := \{\min_{0\leq\epsilon<1} \epsilon : g(\epsilon, z) \geq 0\}$, $\mathcal{E}(z) := \min\{\bar{\epsilon}(z) - 1, 1 - \underline{\epsilon}(z)\}$. Because $g(1, z) > 0$ and $g(\epsilon, z)$ is continuous in ϵ, $\mathcal{E}(z)$ is well defined and bounded for any given z. Moreover, $\mathcal{E}(z)$ is a continuous function of z by noting that both $\bar{\epsilon}(z)$ and $\underline{\epsilon}(z)$ are continuous in z. Since $\mathcal{L}_V(P(1), 1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))$ is compact, we let $L_1 = \min_{z\in\mathcal{L}_V(P(1),1/\mathrm{Tr}(B^{\mathrm{T}}P(1)B))} \mathcal{E}(z)$. Taking into consideration of the relation between (27) and (28), the following condition guarantees (27) holds

$$\left|\frac{\tilde{\varepsilon}(x_i)}{\tilde{\varepsilon}(\mathcal{P}(x))} - 1\right| \leq L_1, \forall x_i \in \mathcal{S}(x), \forall x \in \mathbb{R}^n. \quad (29)$$

On the other hand, on account of (10), $\tilde{\varepsilon}(x)$ is a continuous differentiable function of $x \in \mathbb{R}^n \setminus \{0\}$ according to the implicit function theorem. Moreover, we have

$$\frac{\partial\tilde{\varepsilon}(x)}{\partial x} = -\frac{2\tilde{\varepsilon}(x)x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}P(1)D_{\tilde{\varepsilon}(x)}}{x^{\mathrm{T}}D_{\tilde{\varepsilon}(x)}(P(1)\Psi + \Psi P(1))D_{\tilde{\varepsilon}(x)}x},$$

Hence, for any $z_1,\ z_2 \in \mathcal{N}^{-1}(1)$, we deduce $|\tilde{\varepsilon}(z_1) - \tilde{\varepsilon}(z_2)| \leq L_2|z_1 - z_2|$, where $L_2 = \max_{x\in\mathcal{N}^{-1}(1)} \left|\frac{\partial\tilde{\varepsilon}(x)}{\partial x}\right|$. This further implies

$$\left|\frac{\tilde{\varepsilon}(x_i)}{\tilde{\varepsilon}(\mathcal{P}(x))} - 1\right| \leq \frac{L_2}{\varsigma}|x_i - \mathcal{P}(x)| \leq \frac{L_2}{\varsigma}d_{\mathcal{X}}, \quad \forall x_i \in \mathcal{S}(x), \forall x \in \mathbb{R}^n, \quad (30)$$

where $\varsigma = \min_{x\in\mathcal{N}^{-1}(1)} \tilde{\varepsilon}(x)$. Therefore, in view of (29) and (30), to guarantee the finite-time stability of (25), it suffices to have $d_{\mathcal{X}} \leq L_1\varsigma/L_2 := \varrho$. ■

In light of Theorem 5, we get an upper bound of $|\mathcal{X}|$ as $\lceil\sqrt{n}/\varrho\rceil^n$ after some tedious computation. However, this estimate is highly conservative. A more accurate estimate may be found through detailed analysis of $\dot{V}$ and $\mathcal{N}^{-1}(1)$, which is beyond the scope of this paper. Developing an algorithm to generate $\mathcal{X}$ with $d_{\mathcal{X}} \leq \varrho$ is also an important issue. One possible way is to first generate a certain number of equally distributed points on $\mathbb{B}^{n-1}$, project these points onto the dilated sphere $\mathcal{N}^{-1}(1)$, and then examine whether $d_{\mathcal{X}} < \varrho$. If not, we increase the number of points on $\mathbb{B}^{n-1}$ and repeat this procedure until $d_{\mathcal{X}} < \varrho$. The problem of generating equidistributed points on $\mathbb{B}^{n-1}$ has been well studied[17].

Remark 2: Since $\tilde{u}(x) = -\tilde{u}(-x)$, we can restrict ourselves on the upper-half of $\mathcal{N}^{-1}(1)$ in practical implementation, which further reduces the size of $\mathcal{X}$.

Remark 3: The authors are aware that the homogeneous feedback established in this paper is similar to the implicit Lyapunov design method proposed in [13]. However, our method is different in two aspects. First, we establish the connections between the implicit Lyapunov function and the solution to a parameterized Ricatti equation (see Remark 1). Second, thanks to the homogeneity and the continuity of the feedback, we propose a practical implementation approach

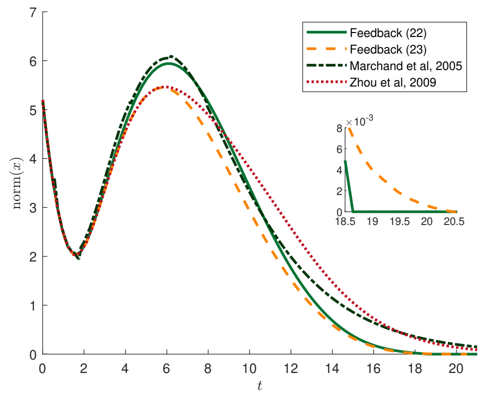

Fig. 1. Trajectories of $|x|$ originating from x_0 under (22), (23) and control strategies proposed in [5] and [18]

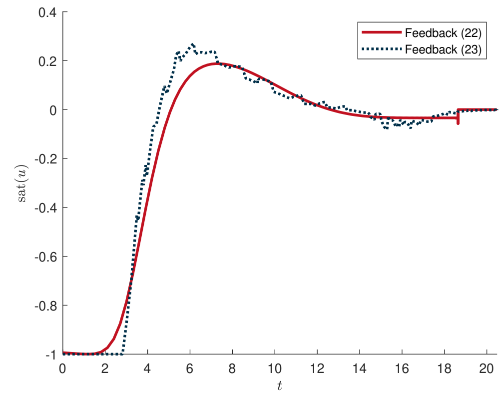

Fig. 2. Trajectories of $\mathrm{sat}(u)$ associated with (22) and (23)

that retains finite-time stability through a sufficiently dense partition of the dilated unit sphere, whereas [13] settles with asymptotic stability by adopting a sample-and-hold strategy for practical implementation.

V. Numerical Examples

Example 1: Consider (1) with $n = 3$ and let the initial condition be $x_0 = [3 \;\; -3 \;\; 3]$. For both the homogeneous feedback law (22) and the practical homogeneous feedback law (23), we set $k = 1$ and $P(1)$ as the solution to (2) with $r = 1$. In this case, we find $\kappa = 0.14$ and $\tilde{\varepsilon}(x_0) = 0.12$. Hence, Theorem 3 gives an estimate of settling time as $T(x_0) = 59.52$s for the homogeneous feedback law (22). Simulation results in Fig. 1 show that the trajectory reaches the origin at 18.62s under (22). For the practical homogeneous feedback (23), we set $|\mathcal{X}| = 1200$. These points are generated by projecting 1200 points equidistributed on the dilated unit sphere onto $\mathcal{N}^{-1}(1)$. The norm of trajectory originating from x_0 under (23) is also plotted in Fig. 1. It reaches zero at 20.36s. The corresponding saturated inputs associated with (22) and (23) are plotted in Fig. 2. For comparison, we also depict the results obtained in [5] and [18] in Fig. 1, from where it can be seen that our control laws achieve faster convergence.

VI. Conclusions

In this paper, we considered the problem of globally stabilizing a chain of integrators subject to actuator saturation in finite time. Based on an existing scheduled low gain approach, we proposed a homogeneous feedback law by relaxing the low gain parameter from taking values in $(0, 1]$ to $(0, \infty)$. The homogeneity of the proposed feedback enables us to practically implement the feedback by discretizing a compact set in the state space and computing a finite number of optimization problems. This reduces the online optimization in the existing scheduled low gain method into offline computation and thus circumvents numerical issues in the existing scheduled low gain method. A numerical result validated the effectiveness of our proposed approach.

References

[1] H. J. Sussmann and Y. Yang, "On the stabilizability of multiple integrators by means of bounded feedback controls," in *Proc. 30th CDC*. IEEE Publications, 1991, Conference Proceedings, pp. 70–72.

[2] A. R. Teel, "Global stabilization and restricted tracking for multiple integrators with bounded controls," *Systems & Control Letters*, vol. 18, no. 3, pp. 165–171, 1992.

[3] H. J. Sussmann, E. D. Sontag, and Y. Yang, "A general result on the stabilization of linear systems using bounded controls," *IEEE Transactions on Automatic Control*, vol. 39, no. 12, pp. 2411–2425, 1994.

[4] N. Marchand and A. Hably, "Global stabilization of multiple integrators with bounded controls," *Automatica*, vol. 41, no. 12, pp. 2147–2152, 2005.

[5] B. Zhou and G.-R. Duan, "Global stabilization of linear systems via bounded controls," *Systems & Control Letters*, vol. 58, no. 1, pp. 54–61, 2009.

[6] Y. Xie and Z. Lin, "Event-triggered global stabilization of general linear systems with bounded controls," *Automatica*, vol. 107, pp. 241–254, 2019.

[7] K. Mei, L. Ma, R. He, and S. Ding, "Finite-time controller design of multiple integrator nonlinear systems with input saturation," *Applied Mathematics and Computation*, vol. 372, p. 124986, 2020.

[8] A. Megretski, "L_2 BIBO output feedback stabilization with saturated control," in *IFAC Proceedings*, vol. 29, 1996, Conference Proceedings, pp. 1872–1877.

[9] Z. Lin, "Global control of linear systems with saturating actuators," *Automatica*, vol. 34, no. 7, pp. 897–905, 1998.

[10] ——, *Low Gain Feedback*. Springer, 1999, vol. 240.

[11] H. Hermes, "Nilpotent and high-order approximations of vector field systems," *SIAM Review*, vol. 33, no. 2, pp. 238–264, 1991.

[12] Y. Chitour, M. Harmouche, and S. Laghrouche, "L_p-stabilization of integrator chains subject to input saturation using Lyapunov-based homogeneous design," *SIAM Journal on Control and Optimization*, vol. 53, no. 4, pp. 2406–2423, 2015.

[13] A. Polyakov, D. Efimov, and W. Perruquetti, "Finite-time and fixed-time stabilization: Implicit lyapunov function approach," *Automatica*, vol. 51, pp. 332–340, 2015.

[14] B. Zhou, G. Duan, and Z. Lin, "A parametric Lyapunov equation approach to the design of low gain feedback," *IEEE Transactions on Automatic Control*, vol. 53, no. 6, pp. 1548–1554, 2008.

[15] L. Grüne, "Homogeneous state feedback stabilization of homogenous systems," *SIAM Journal on Control and Optimization*, vol. 38, no. 4, pp. 1288–1308, 2000.

[16] A. F. Filippov, *Differential Equations with Discontinuous Right-hand Sides*, ser. Mathematics and Its Applications. Springer, Dordrecht, 1988.

[17] M. Deserno, "How to generate equidistributed points on the surface of a sphere," *If Polymerforshung (Ed.)*, vol. 99, no. 2, 2004.

[18] S. Gayaka, L. Lu, and B. Yao, "Global stabilization of a chain of integrators with input saturation and disturbances: A new approach," *Automatica*, vol. 48, no. 7, pp. 1389–1396, 2012.