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Local identification in dynamic networks using a multi-step least squares method

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***Abstract*— For identification of a single module in a linear dynamic network with correlated disturbances different methods are available in a prediction error setting. While indirect methods fully rely on the presence of a sufficient number of external excitation signals for achieving data-informativity, the local direct method with a MIMO predictor model can exploit also non-measured disturbance signals for data-informativity. However, a simple two-node example shows that this local direct method can also be conservative in terms of the number of external excitation signals that is required. Inspired by a recently introduced multi-step method for full network identification, we present a multi-step least squares method for single module identification. In a first indirect step a model is estimated that is used to reconstruct the innovation on a set of output signals, which in a second step is used to directly estimate the module dynamics with a MISO predictor model. The resulting path-based conditions for data-informativity show that the multi-step method requires a smaller number of excitation signals for data-informativity than the local direct method.**

I. INTRODUCTION

Modern day dynamic systems consist of interconnected subsystems that can be represented as large-scale networks. In the domain of system identification, data-driven modeling suitable for these dynamic networks has been the subject of many studies, in among others [1], [2], [3], [4], [5], [6].

In the problem of local identification the objective is to consistently identify a single module using local measured data within a network of which the interconnection structure is given. Currently available local identification methods in a prediction error setting are typically categorized under indirect methods and direct methods.

Indirect methods [7], [3], [8] typically estimate the tranfer functions from measured (user manipulated) external signals to the internal variables which are the nodes w. These methods therefore rely on the presences of a sufficient number of measured excitation signals to achieve data-informativity. Additionally, indirect methods require post-processing to obtain the estimates of the target module. Local direct methods [9], [10] directly estimate the transfer functions between the node signals, where the node signals can receive excitation from either measured (user manipulated) or unmeasured (disturbances) external signals to achieve data-informativity. The local direct method achieves a consistent estimate with maximum likelihood properties under the condition that an appropriate disturbance model is estimated. In the presence of correlated disturbances the local direct method might require the use of a multiple-input-multiple-output (MIMO) predictor model and thus a multivariate noise model. This is opposed to the indirect methods that achieve consistency, but do not require accurate modeling of the disturbances.

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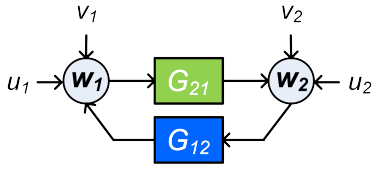

Fig. 1. 2-node example for local identification of target module G_{21}.

Consider the 2-node example shown in Figure 1, with target module G_{21} that represents the interconnecting dynamics from node signal w_1 to w_2. The node signals are additively driven by unmeasured disturbance signals v and external excitation signals u_1 and u_2. If v_1 and v_2 are uncorrelated, the example is a classic closed-loop problem for which we can use classic identification methods [11]. If v_1 and v_2 are correlated the single module identifiability check [12], which is a method independent check, indicates that G_{21} is identifiable when either u_1 or u_2 is present. This is confirmed by the indirect method that can estimate G_{21} by identifying the mappings from $u_1 \to w_1$ and $u_1 \to w_2$ and taking the quotient of these estimates, while relying on the presence of u_1 only, and irrespective of any correlation between v_1 and v_2. The local direct method [9] requires a MIMO predictor model with both w_1 and w_2 as outputs to appropriately model the disturbances. However, as shown in [13], [14], [15] this predictor model requires both u_1 and u_2 to be present to achieve data-informativity. Apparently in this example the data-informativity conditions for the local direct method are conservative when compared to the single module identifiability results and the indirect method.

In this paper, our aim is to develop a local identification method that has the key advantages of the local direct method but requires less external excitation signals u compared to the local direct method.

The multi-step least squares method [16], developed for full network identification, relies on accurate modeling of the disturbances. When applied to the 2-node example it shows that both G_{21} and G_{12} can be estimated while requiring the presence of either u_1 or u_2 to achieve data-informativity.

However, there is no local identification version of this multi-step method available. In this paper we will develop such a local identification method, and show that it indeed allows for a smaller number of excitation signals compared to the local direct method.

Section 2 presents the network setup and the results of [9], [15] and [16]. Section 3 describes the layout of the algorithm for local identification, where the 3 main steps of the algorithm are elaborated in Sections 4 to 6. Section 7 provides the data-informativity conditions of the local identification algorithm. A more extensive example is then worked out in Section 8, and the results are concluded in Section 9. Proofs of results are collected in an Appendix.

II. DYNAMIC NETWORK SETUP AND CURRENT METHODS

A. Dynamic network setup

We follow the module framework setup in [2], and consider dynamic networks represented by

$$w(t) = G(q)w(t)+v(t)+u(t), \text{ with } v(t) = H(q)e(t), \quad (1)$$

where q^{-1} is the delay operator, i.e. $q^{-1}w(t) = w(t-1)$, w is an L-dimensional vector, $G(q)$ is a hollow and rational transfer function matrix and $H(q)$ is a monic and rational disturbance model. Moreover, $e(t)$ is an L-dimensional vector containing white noise processes. The inputs signals $u(t) = R(q)r(t)$ are driven by the measured external excitation signals $r(t)$, with $R(q)$ a known rational transfer function matrix. We assume that the network is well-posed, and that all elements in $G(q)$ are strictly proper transfer functions. The set of node indexes $\{1 \cdots L\}$ is denoted by $\mathcal{L}$.

B. Local direct method

The local direct method for single module identification [9] is a prediction error method that is built on a representation of a subnetwork through a predictor model

$$w_{\mathcal{Y}} = \bar{G}(q)w_{\mathcal{D}} + \bar{H}(q)\xi_{\mathcal{Y}} + u_{\mathcal{Y}} \quad (2)$$

focusing on identifying the target module G_{ji}, with $i \in \mathcal{D}$ and $j \in \mathcal{Y}$. The sets $\mathcal{Y}, \mathcal{D} \subset \mathcal{L}$ are chosen in such a way that the target module remains invariant in the predictor model, i.e. $\bar{G}_{ji}(q) = G_{ji}(q)$, while confounding variables[1] are treated in such a way that $\bar{G}_{ji}(q)$ can be estimated consistently. For a proper treatment of confounding variables, a possibly multivariate noise model $\bar{H}(q)$ is exploited through which correlations between disturbances on nodes in $w_{\mathcal{Y}}$ and $w_{\mathcal{D}}$ can be explicitly modelled, with $\xi_{\mathcal{Y}}$ being a vector white noise process. Note that it is allowed that $\mathcal{Y} \cap \mathcal{D} \neq \emptyset$.
While $u_{\mathcal{Y}}$ can be decomposed as $u_{\mathcal{Y}} = \bar{J}(q)u_{\mathcal{K}} + \bar{S}u_{\mathcal{P}}$ with $\bar{J}(q)$ dynamic and $\bar{S}$ a binary matrix [15], data informativity is achieved if the vector signal $\kappa := [w_{\mathcal{D}}^T \ \xi_{\mathcal{Y}}^T \ u_{\mathcal{K}}^T]^T$ is persistently exciting, typically achieved by requiring that the spectral density $\Phi_\kappa(\omega) \succ 0$ for almost all ω. The data-informativity condition can be satisfied generically by satisfying path-based conditions on the graph of the network [15].

[1] A confounding variable is an unmeasured variable that has paths to both the input and the output of an estimation problem [17].

C. Multi-step least squares

The full network identification method in [16] inspired by among others the Sequential Linear Regressions method [18], first models the mapping $u \to w$ on the basis of a high-order Auto Regressive eXogenous (ARX) model. This ARX model is then used to reconstruct the innovation signals in vector e. We manipulate (1) according to

$$w = Gw + (H - I)e + Ie + u. \quad (3)$$

Next the reconstructed innovation $\hat{e}$ can be treated as a measured input in the predictor

$$\hat{w}(t|t-1) = Gw + (H - I)\hat{e} + u, \quad (4)$$

that now represents a network model with a noise model equal to I, implying that there are no correlated disturbances present. Identification of the full network can therefore be done using multiple-input-single-output (MISO) predictors for each row $j \in \mathcal{L}$ of (3). These MISO predictor models directly estimate the inneighboring modules of output node w_j and the disturbances that have a direct path to w_j. In contrast to the local direct method, which models the disturbance using unmeasured innovation signals, the multi-step method treats the reconstructed innovation signals as measured signals in the modeling procedure.

III. LAYOUT OF THE ALGORITHM

We will introduce the following stepwise algorithm that is actually a local version of the network identification algorithm presented in [16].

Step 1:

a. Select a subset of nodes $\mathcal{S} \subset \mathcal{L}$, with $i, j \in \mathcal{S}$, on the basis of which we are going to estimate the dynamics of the single module G_{ji};
b. Remove all non-selected nodes from the network by immersion [19], i.e. by adjusting the network representation such that the node signals $w_{\mathcal{S}}$ remain invariant.
c. Decompose the set $\mathcal{S}$ in $\mathcal{Y}$ and $\mathcal{U}$, with $j \in \mathcal{Y}$, in such a way that there are no confounding variables for the estimation problem $u_{\mathcal{U}} \to w_{\mathcal{Y}}$ [20], [9]. This can always be done, if necessary by choosing $\mathcal{U} = \emptyset$.

Step 2:

- In the resulting subnetwork description

$$\underbrace{\begin{bmatrix} w_{\mathcal{Y}} \\ u_{\mathcal{U}} \end{bmatrix}}_{w_{\mathcal{S}}} = \bar{G}_{\mathcal{S}}(q) \begin{bmatrix} w_{\mathcal{Y}} \\ u_{\mathcal{U}} \end{bmatrix} + \begin{bmatrix} \bar{H}(q) & 0 \\ 0 & \bar{H}_{\mathcal{U}}(q) \end{bmatrix} \begin{bmatrix} \xi_{\mathcal{Y}} \\ \xi_{\mathcal{U}} \end{bmatrix} + u_{\mathcal{S}}$$

estimate a high order predictor model for the node signals $w_{\mathcal{Y}}$ in the mapping $u_{\mathcal{S}} \to w_{\mathcal{S}}$, and use that model to estimate the innovation signal $\xi_{\mathcal{Y}}$.

Step 3:

- Estimate a MISO model for the node signal w_j, using the estimated innovation signal $\hat{\xi}_{\mathcal{Y}}$ as a measured input, next to the relevant input signals that are present in $w_{\mathcal{S}}$.

In the following Sections we will present and analyse the separate steps in the Algorithm in more detail, and pay particular attention to the data-informativity conditions.

IV. STEP 1: NETWORK EQUATIONS FOR A SUBNETWORK

A. Subnetwork selection and immersion

In the problem of local identification we can use the topology information to decide which node signals $\mathcal{S} \subset \mathcal{L}$ to select for the identification procedure. The subnetwork with selected nodes is then obtained by a dynamic network operation that appropriately removes the unselected nodes while keeping the selected node signals invariant, referred to as immersion [19]. The subnetwork is then represented by the immersed network according to

$$w_{\mathcal{S}} = \bar{G}_{\mathcal{S}} w_{\mathcal{S}} + \bar{H}_{\mathcal{S}} \xi_{\mathcal{S}} + u_{\mathcal{S}}, \tag{5}$$

where $w_{\mathcal{S}}$ is invariant, $\bar{G}_{\mathcal{S}}$ represents the immersed module dynamics, $u_{\mathcal{S}}$ contains the immersed external excitation signals according $u_{\mathcal{S}} = \bar{R}(q) u_{\mathcal{L}}$ with $\bar{R}(q) \in \mathbb{R}(z)^{dim(\mathcal{S}) \times L}$, and where the disturbance process $\bar{H}_{\mathcal{S}} \xi_{\mathcal{S}}$ is the result of a spectral factorization applied to the immersed disturbance model, with $\xi_{\mathcal{S}}$ a vector of white noise processes. It is assumed that the disturbance process $\bar{H}_{\mathcal{S}} \xi_{\mathcal{S}}$ is full rank.

The use of an immersed network allows local identification methods to use local network measurements, where the selected node signals should satisfy the following condition.

Condition 1. Consider a target module G_{ji}. A subset of node signals $\mathcal{S} \subset \mathcal{L}$ with $\{w_i, w_j\} \in w_{\mathcal{S}}$ is said to satisfy the Parallel Path and Loop (PPL) condition, if every path from w_i to w_j that does not pass through G_{ji}, and every loop around w_j, pass through a node in $\mathcal{S}$.

The condition stems from [19], where it has been shown that it guarantees that after immersion of the nodes in $\mathcal{L} \backslash \mathcal{S}$, leading to the system representation (5), the module dynamics of G_{ji} remains invariant in the representation of the immersed network.

Any subnetwork $\mathcal{S}$ that satisfies Condition 1 will allow for the identification of a consistent estimate of the target module. The particular choice of $\mathcal{S}$ will however affect the conditions on data-informativity, that are intrinsically present in the consistency results. For the continuation of the algorithm we assume that the subset $\mathcal{S}$ has been selected by the user.

B. Decomposition of the subnetwork

Inspired by the work in [20] and [9] we decompose set $\mathcal{S}$ in subsets $\mathcal{Y}$ and $\mathcal{U}$, for which there is no innovation signal $\xi_k, k \in \mathcal{S}$ that has direct paths to both $\mathcal{Y}$ and $\mathcal{U}$, meaning there are no confounding variable for the estimation problem $u_{\mathcal{U}} \to w_{\mathcal{Y}}$. Then the disturbance model of the subnetwork adheres to a block diagonal structure written as

$$\underbrace{\begin{bmatrix} w_{\mathcal{Y}} \\ w_{\mathcal{U}} \end{bmatrix}}_{w_{\mathcal{S}}} = \underbrace{\begin{bmatrix} \bar{G}_{\mathcal{Y}\mathcal{Y}} & \bar{G}_{\mathcal{Y}\mathcal{U}} \\ \bar{G}_{\mathcal{U}\mathcal{Y}} & \bar{G}_{\mathcal{U}\mathcal{U}} \end{bmatrix}}_{\bar{G}_{\mathcal{S}}} \underbrace{\begin{bmatrix} w_{\mathcal{Y}} \\ w_{\mathcal{U}} \end{bmatrix}}_{w_{\mathcal{S}}} + \underbrace{\begin{bmatrix} \bar{H} & 0 \\ 0 & \bar{H}_{\mathcal{U}} \end{bmatrix}}_{\bar{H}_{\mathcal{S}}} \underbrace{\begin{bmatrix} \xi_{\mathcal{Y}} \\ \xi_{\mathcal{U}} \end{bmatrix}}_{\xi_{\mathcal{S}}} + \underbrace{\begin{bmatrix} u_{\mathcal{Y}} \\ u_{\mathcal{U}} \end{bmatrix}}_{u_{\mathcal{S}}}, \tag{6}$$

with $\xi_{\mathcal{S}}$ a vector of white noise processes, $\bar{H}$ is monic, stable and stably invertible, and $\bar{G}_{\mathcal{Y}\mathcal{Y}}$ is hollow. The target module G_{ji} remains invariant during the decomposition, as formulated in the following result.

Proposition 1. For every set $\mathcal{S}$ that satisfies the PPL condition in Condition 1 there exists a decomposition in sets $\mathcal{Y}$ and $\mathcal{U}$, with $j \in \mathcal{Y}$, for which there exists a structured form (6) that has no confounding variables for the estimation problem $u_{\mathcal{U}} \to w_{\mathcal{Y}}$ and that satisfies $\bar{G}_{ji} = G_{ji}$.

Proof: The Proposition is a special case of Theorem 1 in [9], for the situation $\mathcal{Q} = \mathcal{Y}$, $\mathcal{O} = \mathcal{B} = \emptyset$, where the work in [9] shows that a decomposition in a structured form (6) can always be found for a selected subnetwork. □

As an immediate result of Proposition 1 the expression for nodes $w_{\mathcal{Y}}$ equals the first block row of (6), which includes the target output w_j. The external excitation signals $u_{\mathcal{Y}}$ that enter nodes $w_{\mathcal{Y}}$ require further specification.

Depending on the selected nodes in the subnetwork $w_{\mathcal{S}}$, the immersion operation leads to unknown dynamics appearing, for example for the path from $u_k, k \in \mathcal{L} \backslash S$ to $w_{\mathcal{Y}}$. Then the external excitation signals enter the nodes $w_{\mathcal{Y}}$ with either an unknown dynamic term or a known constant term, as formulated next.

Proposition 2. The signals $u_{\mathcal{Y}}$ in (6), can be written as

$$u_{\mathcal{Y}} = \bar{J}(q) u_{\mathcal{K}} + \bar{S} u_{\mathcal{P}}, \tag{7}$$

with $\bar{J}(q)$ a dynamic transfer function and $\bar{S}$ a binary constant matrix and where $u_{\mathcal{K}}$ and $u_{\mathcal{P}}$ are defined as follows.

1) $\mathcal{P} \subset \mathcal{Y}$ and for $\ell \in \mathcal{Y}$, $u_\ell \in u_{\mathcal{P}}$ if all loops around w_ℓ pass through a node in $\mathcal{Y} \cup \mathcal{U}$.
2) The set of excitation signals $u_{\mathcal{K}}$ is composed of
   - all signals $u_{\mathcal{Y}}$ that are not in $u_{\mathcal{P}}$, and
   - all signals $u_{\mathcal{L}\backslash\mathcal{S}}$ of which the effect on $u_{\mathcal{Y}}$ is not covered by the inputs $u_{\mathcal{U}}$.

Proof: The Proposition is a special case of Proposition 3 in [15], for the situation $\mathcal{Q} = \mathcal{Y}$, $\mathcal{O} = \mathcal{B} = \emptyset$. □

Using the results in Proposition 1 and 2 we arrive at an expression for the nodes $w_{\mathcal{Y}}$

$$w_{\mathcal{Y}} = \bar{G}(q) \begin{bmatrix} w_{\mathcal{Y}} \\ u_{\mathcal{U}} \end{bmatrix} + \bar{H}(q) \xi_{\mathcal{Y}} + \bar{J}(q) u_{\mathcal{K}} + \bar{S} u_{\mathcal{P}}, \tag{8}$$

where in the algorithm's next step we show how we use predictor models of node signals $w_{\mathcal{Y}}$ to reconstruct the innovation signals $\xi_{\mathcal{Y}}$.

V. STEP 2: HIGH ORDER ARX MODEL TO RECONSTRUCT THE INNOVATION SIGNAL $\xi_{\mathcal{Y}}$

For obtaining a reconstruction of the innovation signals we model the dynamics present in the mapping $u_{\mathcal{S}} \to w_{\mathcal{S}}$ and use this model to reconstruct the innovation signal $\xi_{\mathcal{Y}}$.

Analogous to the full network identification method in Section 2, we first consider the closed-loop network equations that represent the mapping from $u_{\mathcal{S}} \to w_{\mathcal{S}}$ based on the subnetwork equation (5)

$$w_{\mathcal{S}} = \Big(I - \bar{H}_{\mathcal{S}}^{-1}(q)\big(I - \bar{G}_{\mathcal{S}}(q)\big)\Big)w_{\mathcal{S}} + \bar{H}_{\mathcal{S}}^{-1}(q)u_{\mathcal{S}} + \xi_{\mathcal{S}}. \quad (9)$$

Based on the mapping (9) we construct an expression for the one step ahead predictor

$$\hat{w}_{\mathcal{S}}(t|t-1) = \Big(I - \bar{H}_{\mathcal{S}}^{-1}(q)\big(I - \bar{G}_{\mathcal{S}}(q)\big)\Big)w_{\mathcal{S}} + \bar{H}_{\mathcal{S}}^{-1}(q)u_{\mathcal{S}}. \quad (10)$$

Then due to the block-diagonal structure in $\bar{H}_{\mathcal{S}}$ (6), we can isolate the $\hat{w}_{\mathcal{Y}}(t|t-1)$ part of the predictor (10), omitting q for convenience

$$\hat{w}_{\mathcal{Y}}(t|t-1) = \Big(\big[I \quad 0\big] - \underbrace{\big[\bar{H}^{-1} \quad 0\big]\,(I - \bar{G}_{\mathcal{S}})}_{A(q)}\Big)w_{\mathcal{S}} + \underbrace{\bar{H}^{-1}\,\big[\bar{S} \quad \bar{J}\big]}_{B(q)}\begin{bmatrix} u_{\mathcal{P}} \\ u_{\mathcal{K}} \end{bmatrix}, \quad (11)$$

where the decomposition of $u_{\mathcal{Y}}$ according to (7) has been substituted. We use a high order ARX model, where the parametrization of the filters $A(q)$ and $B(q)$ is chosen according

$$\begin{aligned} A(q,\zeta) &= I + A_1 q^{-1} + \cdots + A_n q^{-n}, \\ B(q,\zeta) &= B_0 + B_1 q^{-1} + \cdots + B_{n-1} q^{-(n-1)}, \end{aligned} \quad (12)$$

where the model order n is chosen sufficiently high [21] so as to accurately approximate the rational filters $A(q)$ and $B(q)$ with polynomial functions. The ARX model is estimated according to $\hat{\zeta}_N^n = \text{argmin}_\zeta \frac{1}{N}\sum_{t-1}^{N} \varepsilon_{\mathcal{Y}}^\top(t,\zeta)\varepsilon_{\mathcal{Y}}(t,\zeta)$, with data length N and

$$\varepsilon_{\mathcal{Y}}(t,\zeta) = w_{\mathcal{Y}}(t) - \hat{w}_{\mathcal{Y}}(t|t-1,\zeta) = w_{\mathcal{Y}}(t) - \varphi^\top(t)\zeta \quad (13)$$

and $\varphi(t)$ composed of the appropriate terms in $w_{\mathcal{S}}, u_{\mathcal{P}}$ and $u_{\mathcal{K}}$, which leads to the analytical solution

$$\hat{\zeta}_N^n = \Big[\frac{1}{N}\sum_{t=1}^{N}\varphi(t)\varphi^\top(t)\Big]^{-1}\frac{1}{N}\sum_{t=1}^{N}\varphi(t)w_{\mathcal{Y}}(t). \quad (14)$$

We will show that estimates $\hat{\zeta}_N^n$ are consistent.

Proposition 3. Consider a subnetwork (6) that satisfies Condition 1, and consider predictor model (11). Then, with analytical solution (14), the estimates $A(\hat{\zeta}_N^n)$ and $B(\hat{\zeta}_N^n)$ are consistent if the following conditions hold

1) The external excitation $u_{\mathcal{P}\cup\mathcal{K}}$ is uncorrelated to the noise $\xi_{\mathcal{Y}}$
2) The spectral density $\Phi_\kappa(\omega) \succ 0$ for almost all ω, with $\kappa = \begin{bmatrix} w_{\mathcal{S}} \\ u_{\mathcal{P}\cup\mathcal{K}} \end{bmatrix}$ (data-informativity condition)
3) $A(q,\zeta)$ and $B(q,\zeta)$ are of high-order n, with $n \to \infty$
4) There exists a parameter ζ^0 such that $\big(A(\zeta^0), B(\zeta^0)\big)$ represents the dynamics in the closed-loop mapping $u_{\mathcal{S}} \to w_{\mathcal{S}}$ in (9).

Proof: The Proposition is a special case of Proposition 1 in [16], for the situation that $w_{\mathcal{S}}$ has full rank disturbances. □

The consistency property of estimate $\hat{\zeta}_N^n$ implies that

$$\varepsilon_{\mathcal{Y}}(t,\hat{\zeta}_N^n) \to \xi_{\mathcal{Y}}(t) \quad \text{w.p. 1 as } N \to \infty \ \forall\, t. \quad (15)$$

Therefore, the reconstructed innovation $\varepsilon_{\mathcal{Y}}(t,\hat{\zeta}_N^n) = w_{\mathcal{Y}} - \hat{w}_{\mathcal{Y}}(t|t-1,\hat{\zeta}_N^n)$ is a consistent estimate of $\xi_{\mathcal{Y}}(t)$. Note that the consistency of estimates $\hat{\zeta}_N^n$ is a sufficient condition for a consistent estimate of $\xi_{\mathcal{Y}}$ and might be conservative, as shown in [16] for the reduced rank noise situation.

VI. Step 3: Parametric target module estimate

Similar to the full network situation in Section II.C, we manipulate the term $\bar{H}(q)\xi_{\mathcal{Y}}$ in (8) as

$$\bar{H}(q)\xi_{\mathcal{Y}} = (\bar{H}(q) - I)\xi_{\mathcal{Y}} + I\xi_{\mathcal{Y}},$$

and use $(\bar{H}(q) - I)\hat{\varepsilon}_{\mathcal{Y}}$ in the network model, with $\hat{\varepsilon}_{\mathcal{Y}} := \varepsilon_{\mathcal{Y}}(t,\hat{\zeta}_N^n)$ being the reconstructed innovation signal that acts as a measured input. Therefore, for the identification of the target module we only consider the MISO predictor model for the $j-th$ row of (8)

$$\begin{aligned} \hat{w}_j(t|t-1) = &\sum_{k\in\mathcal{N}_j^-}\bar{G}_{jk}(\eta)w_k + \sum_{\ell\in\mathcal{Y}}\check{H}_{j\ell}(\eta)\hat{\varepsilon}_\ell \\ &+ \sum_{\gamma\in\mathcal{K}_j}\bar{J}_{j\gamma}(\eta)u_\gamma + \sum_{\beta\in\mathcal{P}_j}\bar{S}_{j\beta}u_\beta, \end{aligned} \quad (16)$$

with $\check{H}_{j\ell}$ an element of $\check{H} = \bar{H} - I$, $\mathcal{N}_j^-$ are the inneighbors of w_j where $\mathcal{N}_j^-$ is a subset of $\mathcal{U}\cup\mathcal{Y}\backslash j$, and $u_{\mathcal{P}_j} = u_j$ if present and $u_{\mathcal{K}_j}$ are the u signals on unselected nodes $\mathcal{L}\backslash\mathcal{S}$ that have a direct path or a path through unselected nodes to the target output w_j.

The model is estimated according to

$$\hat{\eta}_N^n = \text{argmin}_\eta \frac{1}{N}\sum_{t-1}^{N}\varepsilon_j^\top(t,\eta)\varepsilon_j(t,\eta), \quad (17)$$

with $\varepsilon_j(t,\eta) = w_j(t) - \hat{w}_j(t|t-1,\eta)$. The conditions for consistency of estimates $\hat{\eta}_N^n$ are formulated next.

Proposition 4. Consider a dynamic network that satisfies Condition 1 and the one-step ahead predictor (16), for which a consistent estimate $\hat{\varepsilon}_{\mathcal{Y}}$ is available and the external excitation signals are defined below equation (16). Then, with criterion (17) the estimates $\bar{G}_{jk}(\hat{\eta}_N^n)$, $\bar{H}_{j\ell}(\hat{\eta}_N^n)$ and $\bar{J}_{j\gamma}(\hat{\eta}_N^n)$ are consistent if the following conditions hold

1) The spectral density $\Phi_{\bar{\kappa}}(\omega) \succ 0$ for almost all ω, with $\bar{\kappa} = \begin{bmatrix} w_{\mathcal{N}_j^-} \\ u_{\mathcal{K}_j} \\ \xi_{\mathcal{Y}} \end{bmatrix}$ (data-informativity condition).
2) The data generating system is in the model set, i.e. there exists a η^0 such that $\bar{G}_{jk}(q,\eta^0) = \bar{G}_{jk}(q)$, $\check{H}_{j\ell}(q,\eta^0) = \bar{H}_{j\ell}(q) - I_{jj}$ and $\bar{J}_{j\gamma}(q,\eta^0) = \bar{J}_{j\gamma}(q)$.

Proof: The Proposition is a special case of Proposition 2 in [16], for the situation that $w_{\mathcal{S}}$ has full rank disturbances. The proof is structured similarly to the proof in [16] with some slight adaptations as given in the Appendix. □

Note that Proposition 4 holds for all model structures that satisfy the system in the model set condition.

It appears that the local multi-step least squares method results in a MISO predictor model of the $j-th$ row to consistently estimate the target module, given that the required reconstructed innovation signals are consistently estimated in the previous step.

Solution (17) leads to a convex optimization problem if we first model the rational components in (16) as polynomial functions and continue with the Weighted Null Space Fitting method [22], [23] to estimate the rational transfer functions in (16) as was done in [16]. Alternatively one can use a kernel-based identification method [24] to identify the dynamics in the predictor model (16). In the next Section we will further analyse the spectral data-informativity conditions in Propositions 3 and 4.

VII. PATH-BASED CONDITIONS FOR DATA-INFORMATIVITY

The spectral conditions in Proposition 3 and 4 that reflect data-informativity, can generically be satisfied by verifying path-based conditions on the graph of the network. This has been shown in [14], [15], based on the graph-based results in [3]. The resulting path-based conditions for the two Propositions are formulated next, for the mapping from persistently excited external signals to vectors κ and $\bar{\kappa}$.

Proposition 5. The spectrum condition in Proposition 3 is generically satisfied if there are $dim(w_{\mathcal{S}})$ vertex-disjoint paths from $(\xi_{\mathcal{S}}, u_{\mathcal{L}\setminus\mathcal{P}\cup\mathcal{K}}) \rightarrow w_{\mathcal{S}}$, and the external signals on the starting nodes of the vertex disjoint paths are persistently exciting.

Proposition 6. Let $\bar{e}_{\mathcal{U}\setminus\mathcal{N}_j^-}$ be defined as all signals e that have a direct or unmeasured path to $w_{\mathcal{U}\setminus\mathcal{N}_j^-}$. Then the spectrum condition in Proposition 4 is generically satisfied if there are $dim(w_{\mathcal{N}_j^-\setminus\mathcal{U}})$ vertex disjoint paths from $(u_{\mathcal{L}\setminus\mathcal{K}_j}, \bar{e}_{\mathcal{U}\setminus\mathcal{N}_j^-}) \rightarrow w_{\mathcal{N}_j^-\setminus\mathcal{U}}$ that do not pass through $w_{\mathcal{N}_j^-\cap\mathcal{U}}$, and the external signals on the starting nodes of the vertex disjoint paths are persistently exciting.

The results in Propositions 3 and 5 show that due to full rank disturbances on $w_{\mathcal{S}}$, there will always be $dim(w_{\mathcal{S}})$ vertex disjoint paths from $\xi_{\mathcal{S}}$ to $w_{\mathcal{S}}$, and so the data-informativity conditions for Step 2 of the algorithm are always generically satisfied. The question whether we need additional external excitation signals $u_{\mathcal{L}\setminus\mathcal{K}_j}$ therefore depends on whether these are needed to satisfy the data-informativity conditions in Proposition 4 and 6.

When comparing the spectral conditions for generic data-informativity: $\Phi_{\kappa}(\omega) \succ 0$ for the local direct method [15] and $\Phi_{\bar{\kappa}}(\omega) \succ 0$ for the multistep method (Proposition 4) with

$$\kappa = \begin{bmatrix} w_{\mathcal{D}} \\ u_{\mathcal{K}} \\ \xi_{\mathcal{Y}} \end{bmatrix}, \qquad \bar{\kappa} = \begin{bmatrix} w_{\mathcal{N}_j^-} \\ u_{\mathcal{K}_j} \\ \xi_{\mathcal{Y}} \end{bmatrix} \tag{18}$$

it can be observed that $\bar{\kappa}$ has fewer elements than κ, since $w_{\mathcal{D}}$ typically contains all in-neighbors (after immersion) of the outputs in the possibly MIMO predictor model, while $w_{\mathcal{N}_j^-}$ just contains the in-neighbors (after immersion) of the scalar target module output. At the same time, since $dim(u_{\mathcal{K}_j})$ will typically be smaller than $dim(u_{\mathcal{K}})$ this shows that we can expect less restrictive generic data-informativity conditions for the multi-step method when compared to the local direct method. This will be illustrated in a more extensive Example in the next Section.

VIII. EXAMPLE

We consider the 4-node example in Figure 2 with target module G_{12}, that has correlated disturbances v_1 and v_3. We select $\mathcal{S} = \{1, 2, 3\}$ as our selected subnetwork for the local identification method, where node w_4 is immersed to obtain the network equation (5). Through this choice the PPL condition (Condition 1) is satisfied. Due to the correlated disturbances on w_1 and w_3, with $j = 1$, we select $\mathcal{Y} = \{1, 3\}$ and $\mathcal{U} = \{2\}$. We reconstruct the innovation signals ξ_1 and ξ_3, and use them as measured inputs in the MISO predictor model (16) which results in

$$\hat{w}_1(t|t-1,\eta) = \sum_{k\in\{2,3\}} \bar{G}_{1k}(\eta) w_k + \sum_{\ell\in\{1,3\}} \check{H}_{1\ell}(\eta)\hat{\varepsilon}_\ell + u_1 \tag{19}$$

where $\check{H}_{11} = \bar{H}_{11} - 1$, and $\hat{\varepsilon} = \varepsilon(t, \hat{\zeta}_N^n)$. Since u_1 has a known path to the output node w_j, $u_1 \in u_{\mathcal{P}_j}$. The effect of external excitation signal u_3 is incorporated in input w_3 and therefore does not appear in (19), which implies that $\mathcal{K}_j = \emptyset$. Note that if u_4 would have been present on unselected node w_4 then $u_4 \in u_{\mathcal{K}_j}$. Considering the data-informativity condition in Proposition 6, we have that $\mathcal{N}_j^-\setminus\mathcal{U} = \{w_3\}$, $\mathcal{U}\setminus\mathcal{N}_j^- = \{\emptyset\}$ and $\mathcal{N}_j^- \cap \mathcal{U} = \{w_2\}$. This implies that there should be one path from external signals u to w_3, and this is true for u_1 and u_3 since these have paths that do not pass through $w_2 \in w_{\mathcal{N}_j^-\cap\mathcal{U}}$. As a result the example can be solved by having either u_1 or u_3 present to obtain a consistent estimate of G_{12}.

The 2-node example in Figure 1 with correlated disturbances v_1 and v_2 follows a similar reasoning, where we have sets $\mathcal{Y} = \mathcal{S} = \mathcal{L} = \{1, 2\}$ and $\mathcal{U} = \emptyset$ and external excitation signal $u_2 \in u_{\mathcal{P}_j}$ and $\mathcal{K}_j = \emptyset$. As a result the 2-node example can be solved by having either u_1 or u_2 present, whereas the local direct method [9], [14], [15] requires both u_1 and u_2 to be present to obtain a consistent estimate of the target module G_{21}.

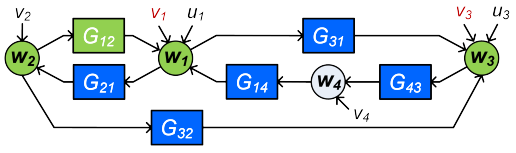

Fig. 2. 4-node example for local identification of target module G_{12}, where node w_4 is not selected and disturbances v_1 and v_3 are correlated

IX. CONCLUSIONS

We have introduced a multi-step least squares method for single module identification that uses a MISO predictor model to directly and consistently estimate the target module,

in the presence of correlated disturbances. We have shown that, when compared to current local direct methods, the data-informativity conditions for the local multi-step least squares method are relaxed while keeping key advantages of current local direct methods. A follow up question involves investigating the role of the particular choice of $\mathcal{S}$, i.e. the set of selected nodes, on the performance of the method. Construction of predictor models with the new multi-step method has been implemented in the MATLAB app and toolbox SYSDYNET [25].

APPENDIX

Proof of Proposition 4. From Proposition 3 we know that $\hat{\zeta}_N^n$ is consistent and therefore $\varepsilon_{\mathcal{Y}}(t, \hat{\zeta}_N^n)$ is a consistent estimate of $\xi_{\mathcal{Y}}$. The prediction error is then written as

$$\begin{aligned}\varepsilon_j(t,\eta) = w_j - \hat{w}_j(t|t-1,\eta) = \sum_{k\in\mathcal{N}_j^-} \Delta\bar{G}_{jk}(\eta)w_k + \\ \sum_{\ell\in\mathcal{Y}} \Delta\check{H}_{j\ell}(\eta)\xi_\ell + \xi_j + \sum_{\gamma\in\mathcal{K}_j} \Delta\bar{J}_{j\gamma}(\eta)u_\gamma,\end{aligned} \tag{20}$$

with $\Delta\bar{G}_{jk} = \bar{G}^0_{jk} - \bar{G}_{jk}(\eta)$, $\Delta\check{H}_{j\ell}(\eta) = (\bar{H}^0_{jk} - I_{jj}) - \check{H}_{j\ell}(\eta)$, $\Delta\bar{J}_{j\gamma} = \bar{J}^0_{j\gamma} - \bar{J}_{j\gamma}(\eta)$. From here on we can follow the proof of Proposition 2 in [16]. □

Proof of Proposition 5. The generic data-informativity conditions are satisfied if there are $dim(\kappa)$ vertex disjoint paths in the mapping $(\xi_{\mathcal{S}}, u_{\mathcal{L}}) \rightarrow (w_{\mathcal{S}}, u_{\mathcal{P}\cup\mathcal{K}})$. We can remove the external excitation signals that have the same components on the right and left hand side, which results in $(\xi_{\mathcal{S}}, u_{\mathcal{L}\setminus\mathcal{P}\cup\mathcal{K}}) \rightarrow w_{\mathcal{S}}$. □

Proof of Proposition 6. The generic data-informativity conditions are satisfied if there are $dim(\bar{\kappa})$ vertex disjoint paths $(u_{\mathcal{L}}, \xi_{\mathcal{S}}) \rightarrow (w_{\mathcal{N}_j^-}, \xi_{\mathcal{Y}}, u_{\mathcal{K}_j})$, or equivalently $(u_{\mathcal{L}}, \xi_{\mathcal{Y}}, \xi_{\mathcal{U}}) \rightarrow (w_{\mathcal{N}_j^-}, \xi_{\mathcal{Y}}, u_{\mathcal{K}_j})$. The external signals that have similar components on both the left hand side and right hand side of the mapping can be removed, leading to $(u_{\mathcal{L}\setminus\mathcal{K}_j}, \xi_{\mathcal{U}}) \rightarrow w_{\mathcal{N}_j^-}$, or equivalently $(u_{\mathcal{L}\setminus\mathcal{K}_j}, \xi_{\mathcal{U}\setminus\mathcal{N}_j^-}, \xi_{\mathcal{N}_j^-\cap\mathcal{U}}) \rightarrow (w_{\mathcal{N}_j^-\setminus\mathcal{U}}, w_{\mathcal{N}_j^-\cap\mathcal{U}})$. Under the full rank noise assumption we know that $\xi_{\mathcal{N}_j^-\cap\mathcal{U}}$ has $dim(w_{\mathcal{N}_j^-\cap\mathcal{U}})$ vertex disjoint paths to $w_{\mathcal{N}_j^-\cap\mathcal{U}}$. Therefore we can remove $w_{\mathcal{N}_j^-\cap\mathcal{U}}$ and $\xi_{\mathcal{N}_j^-\cap\mathcal{U}}$ under the condition that the paths do not pass through $w_{\mathcal{N}_j^-\cap\mathcal{U}}$. This results in $(u_{\mathcal{L}\setminus\mathcal{K}_j}, \xi_{\mathcal{U}\setminus\mathcal{N}_j^-}) \rightarrow w_{\mathcal{N}_j^-\setminus\mathcal{U}}$, where the vertex disjoint paths are not allowed to pass through $w_{\mathcal{N}_j^-\cap\mathcal{U}}$. Relating the signals $\xi_{\mathcal{U}\setminus\mathcal{N}_j^-}$ to the noise signals in the original network then leads to the result. □

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