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Analysis of a Simple Neuromorphic Controller for Linear Systems: A Hybrid Systems Perspective

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Abstract— **In this paper we analyze a neuromorphic controller, inspired by the leaky integrate-and-fire neuronal model, in closed loop with a single-input single-output linear time-invariant system. The controller consists of two neuronal variables and generates a spiking control input. A spike is emitted by the controller whenever one of the neuronal states reaches a threshold. The control input is different from zero only at the spiking instants and, hence, between two spiking times the system evolves in open loop. Exploiting the hybrid nature of the integrate-and-fire neuronal dynamics, we present a hybrid modeling framework to design and analyze this new controller. In the particular case of single-state linear time-invariant plants, we prove a practical stability property for the closed-loop system, we ensure the existence of a strictly positive dwell-time between spikes, and we relate these properties to the parameters in the neurons. The results are illustrated in a numerical example.**

I. Introduction

The advantages of brain-inspired technology, such as energy efficiency, robustness, accuracy, and adaptability, have motivated many novel research topics in the last years. In particular, the pioneering work of Carver Mead [1], highlights the inefficiency of digital computers compared to biological systems. This work kick-started the field of neuromorphic engineering, where novel technologies for computing, controlling, sensing and actuating systems, mimicking the structure and function of neurons and the (human) brain have been developed, see, e.g., [2]–[5] and the references therein.

The key features of neuron activity and communication are related to the generation and processing of spikes, where the information is contained in the presence or absence of a spike, rather than in its magnitude. Different models were presented in the literature to model the neuron dynamics that generate these spikes, see, e.g., [6]–[9] for surveys. In particular, neuron models can be divided in two categories based on the approach used to describe the spiking behavior. The first one consists in describing the neuron dynamics using only continuous(-time) dynamical systems, where spikes are captured by specific (smooth) nonlinearities, as, for instance, in the Hodgkin–Huxley [10], the Fitzhugh-Nagumo [11] or the Hindmarsh-Rose [12] neuron models. The second category, which is the one we adopt in this paper, consists in describing the neuron dynamics by continuous-time dynamical systems with discrete spikes, see, e.g., [13]. In the latter approach, one of the simplest models to describe the neuron behavior is the (leaky) integrate-and-fire neuron model [13]–[15], [16, Chapter 4], which consists in combining a first-order differential equation, describing the dynamics of the membrane potential of the neuron between spikes, and a mechanism that resets the membrane potential whenever a certain threshold is reached. Spikes are thus events that are triggered at precise moments in time and the continuous-time dynamics in between spikes determines the times when these events are triggered, which naturally leads to the possibility of using hybrid systems frameworks, e.g., [17], [18], to model neuron dynamics.

In the field of control engineering, some of the first works illustrating the potential of brain-inspired control techniques are [19], [20], where the authors designed a proportional derivative neuromorphic controller for a DC motor. It is shown that the neuron servo controller works reliably when regulating the speed of the electric motor both at high and low reference speeds, unlike a conventional analog controller, which fails at low reference speeds because of the presence of friction. The interesting and promising results in [19], [20] are unfortunately missing a formal technical analysis of the properties guaranteed by the controller. More recently, a neuromorphic controller design has been proposed in [21], [22], where a positive feedback and a negative feedback are combined to obtain a mixed feedback controller that describes the mixed analog and digital nature of neurons. In particular, the positive feedback generates the spikes, while the negative feedback is used for regulation. Other research provides analysis tools for controllers whose input comes from neuromorphic (event-based) sensors (such as the event-based camera), see, e.g., [23], [24]. However, in this case, the control input is a continuous-time signal and not a spiking one and the controller is not designed by taking inspiration from the operation of the brain or from neuron dynamics.

As previously mentioned, the generation of spikes can be perceived as the triggering of an event. Therefore, neuromorphic control has a strong connection with event-based control, for which many works have been published in recent years, see, e.g., [25]–[29]. The main motivation of event-triggered control is to reduce the number of transmissions over a digital network compared to standard time-triggered periodic sampling, while still satisfying stability and performance properties. Thereto, the events are used to decide when a sample should be transmitted to obtain the desired stability and performance properties in an otherwise continuous control loop with continuously varying control signals taking any real value (not countable). In contrast,

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neurons, and consequently neuromorphic (spiky) controllers, operate through spikes, thereby controlling only through the time occurrences of events. The key information is therefore linked only to the triggering times. Interestingly, the original inspiration for event-based control design comes from the event-based nature of biological systems and the control through events using pulses [25].

The focus of this work is to propose a formal framework to analyze spiky controllers, that are based on the leaky integrate-and-fire neuron model [13]–[15], [16, Chapter 4], for linear time-invariant systems, which exploits the hybrid nature of neuron dynamics. In particular, we consider a neuromorphic controller consisting of two neuronal states in closed loop with a single-input single-output linear plant. As explained in [13], the leaky integrate-and-fire neuron model is only a threshold model and lacks the capability to generate spikes. To overcome this drawback and to better approximate the neuron behavior, in this work we add a mechanism that fires a spike whenever one of these variables reaches the threshold and is reset. The system is therefore controlled through spiking control inputs (such as the DC motor example mentioned above [19], [20]) with fixed amplitude different from zero only at some discrete-time (triggering) instants.

The first main contribution of this work is the formulation of the structure of the neuron-inspired controller for a single-input single-output plant. We will exploit the hybrid nature of the neuronal dynamics and develop a suitable hybrid modeling setup for the closed-loop system. Specifically, we model the overall system as a hybrid system using the formalism of [18], where a jump corresponds to the occurrence of a spiking control action.

Our second contribution is that, for the particular case of single-state linear time-invariant plants, we prove a practical stability property of the closed-loop system, which is not trivial in view of the structure of the proposed neuron-inspired controller and several technical difficulties have to be overcome. Indeed, the system evolves in open loop except for some discrete instants and the control action has a fixed amplitude. Therefore, the system is controlled only through the time occurrence of events. As such, the design of the leaky integrate-and-fire neuronal parameters, which essentially determine the times when spiking control actions occur, is instrumental to guarantee a stability property of the system state, especially when the open-loop plant is not stable. In addition, we prove the existence of a strictly positive dwell-time and thus we guarantee that the time between two consecutive spiking control actions is always greater than or equal to a strictly positive constant, thereby ruling out the Zeno phenomenon. Finally, we provide a quantitative relationship between the controller parameters (threshold, amplitude of the spiking control action and leaking parameter) and the guaranteed ultimate bound, region of attraction and dwell-time. Compared to the existing works on neuromorphic control in the literature, this work proposes a novel modeling perspective and different tools to formally analyze a neuromorphic spiky controller with

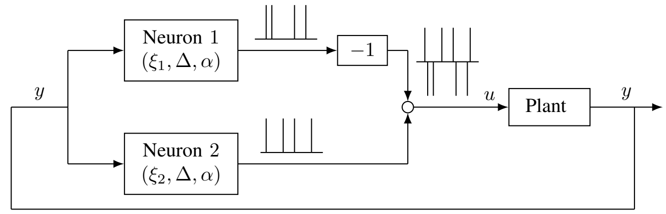

Fig. 1. Block diagram representing the system architecture

stability guarantees based on a simple hybrid model of neuronal dynamics. We illustrate these results also using a numerical case study. As the analysis is already technically challenging, we consider a simple class of systems, but our ambition for future work is to extend this framework to more general classes of systems, which may require the use of multiple pairs of neurons.

The remainder of the paper is organized as follows. Preliminaries are reported in Section II. The neuromorphic controller and the hybrid modeling framework are given in Section III, while Section IV presents the main analysis and design. A numerical example is given in Section V. Finally, Section VI concludes the paper and presents possible future research directions.

II. Preliminaries

The notation $\mathbb{R}$ stands for the set of real numbers, $\mathbb{R}_{\geq 0} := [0, +\infty)$ and $\mathbb{R}_{>0} := (0, +\infty)$. We use $\mathbb{Z}$ to denote the set of integers, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, ...\}$ and $\mathbb{Z}_{>0} := \{1, 2, ...\}$. We consider hybrid systems in the formalism of [18], namely

$$\mathcal{H} : \begin{cases} \dot{x} &= F(x), \qquad x \in \mathcal{C}, \\ x^+ &\in G(x), \qquad x \in \mathcal{D}, \end{cases} \tag{1}$$

where $\mathcal{C} \subseteq \mathbb{R}^{n_x}$ is the flow set, $\mathcal{D} \subseteq \mathbb{R}^{n_x}$ is the jump set, $F : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is the flow map and $G : \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ is the jump map. We consider hybrid time domains as defined in [18, Definition 2.3] and we use the notion of solution for system (1) as advocated in [18, Definition 2.6]. Given a solution x for system (1) and its hybrid time domain $\operatorname{dom} x$, $\sup_j \operatorname{dom} x := \sup\{j \in \mathbb{Z}_{\geq 0} : \exists\, t \in \mathbb{R}_{\geq 0} \text{ such that } (t, j) \in \operatorname{dom} x\}$.

III. Neuromorphic controller and hybrid modeling framework

In this section we present the neuromorphic controller and we propose a new hybrid modeling framework for the closed-loop system.

Consider the linear time-invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{2}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $y \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ the spiking control input and $n_x \in \mathbb{Z}_{>0}$. In particular, we consider the setting shown in Fig. 1, where the input u is generated by a neuromorphic controller consisting of two variables, $\xi_\ell \in \mathbb{R}_{\geq 0}$, $\ell \in \{1, 2\}$, whose dynamics is inspired by the leaky integrate-and-fire neuron model [13]–[15], [16, Chapter 4]. In particular, $\xi_\ell \in \mathbb{R}_{\geq 0}$, $\ell \in \{1,2\}$, represents the membrane potential of the neuron, whose continuous-time dynamics is a first-order differential equation affected by a non-negative current input, and it is then reset whenever it reaches a positive threshold. Note that, as explained in [16], the integrate-and-fire neuron model generates spiking times only when fed with non-negative inputs and it is insensitive to negative inputs. The emitted spike, considered in the form of a Dirac delta pulse $\alpha\delta(t-t_j)$, with $\alpha \in \mathbb{R}_{>0}$ a given and constant spike amplitude, $t \in \mathbb{R}_{\geq 0}$ and t_j $j \in \mathbb{Z}_{\geq 0}$ the considered spiking time, is absorbed by the actuator and leads to the reset of the plant state $x \in \mathbb{R}^{n_x}$, which we will make more precise below, see (5). In our neuromorphic controller design, the input to the neurons is the continuous-time measured output y of system (2), which, depending on its sign, influences either ξ_1 or ξ_2. Indeed, since the measured output $y \in \mathbb{R}$ and the neuron dynamics is affected only by non-negative inputs, in the controller design we need one neuron sensitive to y and one to $-y$.

Inspired by the leaky integrate-and-fire neuron model [13]–[15], [16, Chapter 4], we thus consider the following dynamics for the neuronal variables ξ_ℓ, $\ell \in \{1,2\}$, between two control actions,

$$\dot{\xi}_1 = -\mu\xi_1 + \max(0,y), \quad \dot{\xi}_2 = -\mu\xi_2 + \max(0,-y), \tag{3}$$

with $\mu \in \mathbb{R}_{\geq 0}$ a "leaking" parameter of the neurons. Moreover, the neuronal variables ξ_ℓ, $\ell \in \{1,2\}$, trigger spikes when there exists $\ell \in \{1,2\}$ such that the condition

$$\xi_\ell \geq \Delta \tag{4}$$

is satisfied, where $\Delta \in \mathbb{R}_{>0}$ is, again, a neuron parameter. For ease of exposition, we take μ and Δ the same for both neurons, but they can also be taken differently, see Remark 4 below. Inspired by the leaky integrate-and-fire neuron model, where the membrane potential is reset whenever it reaches a threshold, we reset the variable ξ_ℓ, $\ell \in \{1,2\}$, when it triggers a spiking control action, while it is not modified when the other neuronal variable reaches the threshold, i.e., when neuron $\ell \in \{1,2\}$ satisfies (4), we have, in terms of the hybrid system notation in Section II,

$$\xi_\ell^+ = 0 \ \text{ and } \ \xi_{3-\ell}^+ = \xi_{3-\ell}. \tag{5}$$

It is important to notice that the leaky integrate-and-fire neuron model does not include a spike generation mechanism, but, as explained in [13], it is only a threshold mechanism and therefore does not intrinsically produce a spike. To represent the neuron behavior more accurately, in our design we add a spike generation mechanism to the leaky integrate-and-fire neuron model. Indeed, whenever ξ_ℓ, $\ell \in \{1,2\}$, reaches the threshold Δ, a spiking control action, which can be interpreted by the release of charge of the membrane potential affecting the connected system, is fired, whose sign depends on which neuronal variable reaches Δ. In particular, when ξ_1 reaches the threshold Δ, the spiking control action at the spiking time t_j, $j \in \mathbb{Z}_{\geq 0}$, is given by, with some abuse of notation, $-\alpha\delta(t-t_j)$, with $t \in \mathbb{R}_{\geq 0}$, which results, in terms of the hybrid system notation in Section II,

$$x^+ = x - B\alpha. \tag{6}$$

Similarly, when ξ_2 reaches the threshold Δ, the spiking control action at the spiking time t_j, $j \in \mathbb{Z}_{\geq 0}$, is given by, again with some abuse of notation, $\alpha\delta(t-t_j)$, with $t \in \mathbb{R}_{\geq 0}$, and thus

$$x^+ = x + B\alpha. \tag{7}$$

Note that, equations (6)-(7) assume that the actuator can cope with spiking control signals, which is reasonable in some applications, see, e.g., [3], [4]. Interestingly, the number and type of neuromorphic devices, such as actuators and sensors, which are important for the real-life implementation of true neuromorphic controllers, such as (3), (4), leading to (6), (7), are growing in the recent years due the engineering community's interests in developing neuromorphic closed-loop systems. A prominent example is the event-based camera [23], [24].

Between two spiking control instants, i.e., when for all $\ell \in \{1,2\}$, $\xi_\ell \leq \Delta$, $u = 0$ and thus the system state evolves in open loop according to

$$\dot{x} = Ax. \tag{8}$$

To capture all of the above, we define the overall state as $q := (x, \xi_1, \xi_2) \in \mathbb{R}^{n_x+2}$ and we obtain the hybrid system

$$\begin{cases} \dot{q} = F(q), & q \in \mathcal{C} \\ q^+ \in G(q), & q \in \mathcal{D}, \end{cases} \tag{9}$$

where the flow map F is defined, for any $q \in \mathcal{C}$, from (3) and (8),

$$F(q) := (Ax, -\mu\xi_1 + \max(0,y), -\mu\xi_2 + \max(0,-y)). \tag{10}$$

The flow set $\mathcal{C}$ in (9) is defined as

$$\mathcal{C} := \mathcal{C}_1 \cap \mathcal{C}_2 \tag{11}$$

with

$$\mathcal{C}_\ell := \left\{ q \in \mathbb{R}^{n_x+2} : \xi_\ell \leq \Delta \right\}, \quad \ell \in \{1,2\}. \tag{12}$$

The jump set $\mathcal{D}$ in (9) is defined as, from (4),

$$\mathcal{D} := \mathcal{D}_1 \cup \mathcal{D}_2 \tag{13}$$

with

$$\mathcal{D}_\ell := \left\{ q \in \mathbb{R}^{n_x+2} : \xi_\ell \geq \Delta \right\}, \quad \ell \in \{1,2\}. \tag{14}$$

The jump map G in (9) is defined as, for any $q \in \mathcal{D}$, from (6)-(5),

$$G(q) := G_1(q) \cup G_2(q), \tag{15}$$

with

$$G_1(q) := \begin{cases} \left\{ \begin{pmatrix} x - B\alpha \\ 0 \\ \xi_2 \end{pmatrix} \right\} & q \in \mathcal{D}_1 \\ \emptyset & q \notin \mathcal{D}_1, \end{cases} \tag{16}$$

and

$$G_2(q) := \begin{cases} \left\{ \begin{pmatrix} x + B\alpha \\ \xi_1 \\ 0 \end{pmatrix} \right\} & q \in \mathcal{D}_2 \\ \emptyset & q \notin \mathcal{D}_2. \end{cases} \quad (17)$$

Remark 1. *The current formulation includes a pair of neuronal variables. We can envision a model with multiple pairs of neuronal variables, with possibly a control input u with dimension greater than 1. This is the subject of future research.*

Now that we have proposed the neuromorphic controller and provided a suitable modeling framework, it is our objective to derive conditions on the neuron parameters α, μ, Δ to guarantee a stability property of system (9). In the next section we present results for the case when (2) is a scalar state linear time-invariant system.

IV. Main analysis and design

The goal of this section is to prove that the proposed neuromorphic controller guarantees a stability property for system (9) in the particular case of single-state linear time-invariant systems. In particular, in the next theorem we show that system (9) with $A = a \in \mathbb{R}_{>0}$, $B = C = 1$, satisfies a semi-global practical stability property, and we ensure the existence of a strictly positive dwell-time, thereby ruling out the Zeno phenomenon. We will also show how the control parameters (leaking rate μ, firing threshold Δ and spike amplitude α) and plant parameter a relate to the region of attraction, the ultimate bound and the dwell-time guarantees. Note that, the case $B = C = 1$ is considered in Theorem 1 without loss of generality and the results hold *mutatis mutandi* for any $B, C \in \mathbb{R}_{>0}$. The case $B = C = 1$ is considered just to simplify notations.

Theorem 1. *Consider system* (9) *with $n_x = 1$ and $A = a \in \mathbb{R}_{>0}$, $B = C = 1$. Select $\rho \in (0,1)$. Define $\Psi := \dfrac{\rho+1}{(\rho+1)^2-1}\alpha$. Then for all $\Delta \in \left(0, \dfrac{\rho\alpha}{\mu+a}\right]$, and all $\sigma \in \left[\dfrac{(\rho+1)^2-1}{(\rho+1)^2}, 1\right)$, any solution q of the controlled system* (9) *such that $|x(0,0)| \leq \sigma\Psi$ and $\xi_1(0,0) = \xi_2(0,0) = 0$ satisfies the following properties:*

(i) For all $(t,j) \in \operatorname{dom} q$

$$|x(t,j)| \leq \gamma^j |x(0,0)| + 2\alpha, \quad (18)$$

where $\gamma := \left(1 - (1-\sigma)(\rho+1)^2\right)^{\frac{1}{2}} \in (0,1)$.

(ii) The solution q has a minimum dwell-time $\tau := \dfrac{\Delta}{\Upsilon} \in \mathbb{R}_{>0}$, with $\Upsilon := \Psi + 2\alpha \in \mathbb{R}_{>0}$, i.e., for all $(s,i),(t,j) \in \operatorname{dom} q$ with $s + i \leq t + j$ we have $j - i \leq \dfrac{t-s}{\tau} + 1$.

The proof of Theorem 1 is omitted for space reasons and can be found in [30]. Theorem 1 provides conditions on the control parameters α, Δ, μ, to ensure that the proposed neuromorphic controller guarantees a semi-global practical stability property for system (9). Indeed, item *(i)* guarantees that, when the system initial state is such that $|x(0,0)| \leq \sigma\Psi$, with Ψ that can be arbitrarily large (by taking ρ close to zero), it converges to a neighborhood of the origin, whose size is given by 2α, as the time grows. Since the controller operates via fixed amplitude spikes, it is not possible to obtain an asymptotic stability property for the closed-loop system, and, thus, proving a practical one as in Theorem 1, is the best we can achieve. Note that, since the solution to system (2) is diverging between spikes, because of $a > 0$, and the magnitude α of the proposed controller is given and fixed, the stability result provided in this theorem is not trivial. Indeed, the time occurrence of the spiking control actions needs to be sufficiently fast to guarantee that each continuous-time divergent behavior between two consecutive control actions is compensated by the discrete spike. This is reflected by the a, α and μ dependence of the upperbound on the firing threshold Δ. Consequently, for a given spike amplitude, the further the state x is from the origin, the faster the spikes must be generated, which provides a trade-off between region of attraction and the dwell-time, as reflected by the conditions on the parameters in Theorem 1. Note that, it is possible to select the design parameters so that the region of attraction can be arbitrarily large. Indeed, as can be seen from the conditions in Theorem 1, the smaller $\rho \in (0,1)$ is chosen, the larger Ψ will be. However, the smaller ρ is selected, the smaller is the upperbound on the firing threshold Δ that guarantees the stability property. Thus, in turn, as stated in item *(ii)*, decreases the guaranteed dwell-time. Similarly, also the parameter $\mu \in \mathbb{R}_{\geq 0}$ has an impact on the threshold Δ. Indeed, the larger the μ (the "leakier" the neuronal dynamics are), the smaller the upperbound on Δ is, which decreases the guaranteed dwell-time. However, for any choice of the design parameters satisfying the conditions in Theorem 1, item *(ii)* guarantees the existence of a strictly positive dwell-time, which implies that the Zeno phenomenon cannot occur. In addition, item *(i)* provides an upperbound on the guaranteed convergence speed. In particular, the smaller the σ, that is, the farther the initial state is from the border of the region of attraction, the larger γ is, which affects the convergence speed. Finally, the result holds for any spike amplitude $\alpha \in \mathbb{R}_{>0}$. From Theorem 1 we have that the smaller the α, the smaller the ultimate bound towards which the state converges is. However, the smaller the α, the smaller the region of attraction and the upperbound on the threshold Δ are.

Conversely, from a controller design perspective, Theorem 1 guarantees that it is possible to obtain any desirable region of attraction $\Psi \in \mathbb{R}_{>0}$ and ultimate bound $2\alpha \in \mathbb{R}_{>0}$ for the closed-loop system for any given μ, if there is freedom in the selection of the parameters α and Δ of neuronal variables ξ_ℓ, $\ell \in \{1, 2\}$. Indeed, the ultimate bound is directly related to the amplitude of the spikes α. Moreover, given α and the desired Ψ, it is always possible to find $\rho \in (0,1)$ such that $\Psi = \frac{\rho+1}{(\rho+1)^2-1}\alpha$ is satisfied. Then, for any given μ, by selecting Δ such that $\Delta \in \left(0, \frac{\rho\alpha}{\mu+a}\right]$ holds, the practical stability property with the desired ultimate bound 2α and region of attraction Ψ is guaranteed.

From (3)-(4), (5), and since $\xi_\ell(0,0) = 0$, $\ell \in \{1,2\}$, from the conditions in Theorem 1, we have that the neuronal variables are always bounded, namely $\xi_\ell(t,j) \leq \Delta$, $\ell \in \{1,2\}$, for all $(t,j) \in \operatorname{dom} q$.

Remark 2. *In Theorem 1 the neuronal variables ξ_1 and ξ_2 are initialized at $\xi_1(0,0) = \xi_2(0,0) = 0$. This is just a design choice that simplifies the proof of Theorem 1, but any other choice for the initial condition would not compromise the stability result in item (i). However, the dwell-time property in item (ii) would hold only after the first jump of each neuronal variable.*

Remark 3. *Theorem 1 provides a stability results for system* (9) *with $A = a > 0$, $B = C = 1$. Similar results hold* mutatis mutandis *for system* (9) *with $A = a \leq 0$, $B, C \in \mathbb{R}_{>0}$. Moreover, when $a < 0$ an asymptotic stability property can be proved under some conditions.*

Remark 4. *The neuron parameters α, μ, Δ are assumed to be the same in the two neuronal variables ξ_ℓ, $\ell \in \{1,2\}$ in* (9). *The proposed model and results hold* mutatis mutandis *when considering different neuron parameters for the two variables. In particular, we can consider asymmetric spike amplitude, namely $u \in \{0, +\alpha_1, -\alpha_2\}$, with $\alpha_1, \alpha_2 \in \mathbb{R}_{>0}$, different leaking parameters $\mu_1, \mu_2 \in \mathbb{R}_{\geq 0}$ and thresholds $\Delta_1, \Delta_2 \in \mathbb{R}_{>0}$. More details will be provided in future work.*

V. Numerical Example

We consider system (2) with $A = 1$, $B = 1$, $C = 1$ and we apply the proposed neuromorphic spiking controller, where the neuron parameters are $\alpha = 0.5$, $\mu = 0.5$, $\Delta = 0.1$, which satisfy the conditions of Theorem 1 for any $\rho \in (0,1)$. The simulation results for the system state x and the corresponding spiking control input u, obtained with the initial conditions $x(0,0) = 20$, $\xi_1(0,0) = \xi_2(0,0) = 0$, are shown in Fig. 2 in blue. A zoom of the ultimate bound of the state convergence is also shown in Fig. 2.

Fig. 2 shows that the proposed neuromorphic controller satisfies a practical stability property of the closed-loop system. The ultimate bound obtained in simulations is 0.375, which is smaller than the guaranteed one from Theorem 1, equal to $2\alpha = 1$. In addition, the minimum time between two spikes obtained in the whole simulation is 0.005 and, it is equal 0.341 when considering only the steady-state.

To investigate the effect of disturbances and measurement noises, we also consider the same system parameters with modified state dynamics $\dot{x} = x + v$, where $v \in \mathbb{R}$ is an external disturbance, and modified output equation $y = x + w$, where $w \in \mathbb{R}$ is the measurement noise. Both v and w used in the simulation are piecewise linear signals generated from a random sequence of points, occurring at intervals of 0.01 s, with amplitude ranging between -0.1 and 0.1 (uniform distribution) and with a linear interpolation applied between

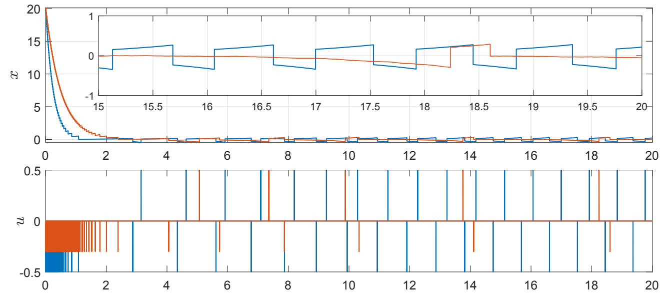

Fig. 2. State x and input u when $\alpha = 0.5$, $\mu = 0.5$, $\Delta = 0.1$ (blue) and $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\mu_1 = 0.2$, $\mu_2 = 0.5$, $\Delta_1 = 0.1$, $\Delta_2 = 0.2$ with noise and disturbances (red).

any two consecutive points. Moreover, we also consider different neuron parameters for the two neuronal variables in (3)-(5). In particular, we select $\alpha_1 = 0.3$, $\alpha_2 = 0.5$, $\mu_1 = 0.2$, $\mu_2 = 0.5$, $\Delta_1 = 0.1$ and $\Delta_2 = 0.2$. The simulation results obtained in this setting are shown Fig. 2 in red, which illustrates that the proposed neuromorphic controller is robust to measurement noise, disturbances and differences on the neuron parameters α_ℓ, μ_ℓ and Δ_ℓ, $\ell \in \{1,2\}$. A formal analysis of the asymmetric case and of the effect of disturbances and measurement noises is left for future work.

VI. Conclusions

We have presented a hybrid system framework to formally analyze a simple neuromorphic controller in closed loop with single-input single-output linear time-invariant systems. The controller is based on two neurons, whose dynamics is inspired by the leaky integrate-and-fire neuron model. Using this hybrid system model, we have proved that the neuromorphic controller can guarantee a practical stability property for the system state in the particular case of single-state linear time-invariant systems and we have ensured the existence of a strictly positive dwell-time between spikes. Moreover, a quantitative relationship between the controller parameters (threshold, amplitude of the spiking control action and leaking parameter) and the guaranteed ultimate bound, region of attraction and dwell-time is provided. Finally, a numerical example confirms the obtained analysis and design.

We believe that the presented framework can be the starting point for many interesting future work directions such as the generalization of the results to higher-order linear time-invariant systems and/or nonlinear systems. This might require multiple pairs of neurons in the controller, and therefore new designs and structures. We would also like to include measurement noise and disturbances in the system model and provide a formal analysis for the resulting closed-loop system. Moreover, in future work we plan to relax the assumption that the parameters of the two neuronal variables are the same, in the sense that we would like to consider possible asymmetric spike amplitudes, leaking parameters and thresholds for the different neurons. Considering the case where the controller receives spiking measurements from neuromorphic sensors, and not continuous-time ones, is an additional challenging research direction. Finally, it would be interesting to run experiments on physical systems implementing the proposed neuromorphic controller.

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