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Quadratic abstractions for k-contraction

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***Abstract*—k-contraction is a generalization of the classical contraction property. It allows the study of complex behaviors in partially stable systems. However, existing conditions for k-contraction are often intractable. This work proposes efficiently solvable sufficient conditions for k-contraction verification in partially linear systems. Our findings are derived by exploiting particular quadratic abstractions arising from classical Lur'e systems analysis. We specialize our result to nonlinearities satisfying shifted monotonicity and differential sector-bound properties. We showcase the potential of our method by designing nonlinear controllers for linear systems, achieving complex closed-loop behaviors.**

I. Introduction

Contraction theory has garnered substantial attention within the nonlinear control research community due to its usefulness in proving strong convergence and robustness properties [1], [2]. Nonetheless, many stable systems (such as multi-stable and orbitally stable ones) do not present classical contractivity properties, as distances between different trajectories do not decrease exponentially in time. To address this limitation, various generalizations of contraction theory have emerged. Among others, k-contraction [3]–[5] appeared as the natural generalization to k-dimensional objects of the standard contraction concept for distances. The geometric interpretation of k-contraction is relatively straightforward: the volume of any k-dimensional set of initial conditions contracts to zero. Therefore, k-contraction includes classical contraction as the special case $k = 1$.

However, for $k > 1$, k-contractive systems can show more complex asymptotic behaviors. This is because k-contraction implies the system's dominant behavior evolves in a subspace of dimension strictly lower than k. For instance, in 2-contractive systems, the area of any surface of initial conditions shrinks to zero, and the dominant behavior is at most 1-dimensional. The connection between the dominant behavior of a system and its asymptotic properties [5], [6] has led to results linking k-contraction with complex steady-states. For example, in 2-contractive time-invariant systems, every bounded solution can be proven to converge to an equilibrium point (not necessarily unique) [7]. Similarly, [5] connected 3-contraction and attractive limit cycles.

Sufficient conditions for k-contraction were originally given in the seminal work by Muldowney [3] and were recently re-proposed in the works [4], [5], [8]. However, existing results either rely on complex mathematical objects known as matrix compounds [4], [8], [9], they involve an infinite number of matrix inequalities [5], [10] or they are excessively conservative [11]. Hence, inspired by classical literature on Lur'e systems, some works focused on specific classes of systems to refine the conditions [12]–[14]. Unfortunately, these findings still depend on matrix compounds, leading to increased computational complexity and hindering practical applications [5], [11]. Additionally, structural properties may be lost with the use of matrix compounds. This last point is of particular interest since exploiting the system structure can provide efficient methods for analysis and feedback design in nonlinear systems [15]–[17].

With this in mind, in this work we depart from the use of matrix compounds and we leverage on the compound-free conditions introduced in [10] and [5, Section IV]. Our focus is on systems represented by a linear component in a feedback interconnection with a nonlinear one. This structure allows incorporating quadratic abstractions arising from classical Integral Quadratic Constraints-based analysis of systems with isolated nonlinearities [18], [19]. By imposing specific properties on the nonlinearities, we show that the infinite set of constraints in [5] can be replaced by a single, efficiently solvable matrix inequality. We further specialize our results to derive sufficient k-contraction conditions for two classes of systems: i) those with nonlinearities satisfying a generalized monotonicity assumption, and ii) those with nonlinearities satisfying a differential sector condition. To validate our findings, we present numerical results showcasing potential applications to the design of nonlinear controllers for achieving nontrivial behaviors in linear time-invariant systems. These behaviors include multi-stability and oscillations around an equilibrium point.

Notation: $\mathbb{R}_{\geqslant 0} := [0, \infty)$. The inertia of a matrix P is defined by the triplet of integers $\mathrm{In}(P) := (\pi_-(P), \pi_0(P), \pi_+(P))$, where $\pi_-(P)$, $\pi_+(P)$ and $\pi_0(P)$ denote the numbers of eigenvalues of P with negative, positive and zero real part, respectively, counting multiplicities. For a symmetric matrix $A = A^\top$, $A \succ 0$ (resp. $A \succeq 0$) denotes A being a positive definite (resp. positive semidefinite) matrix. We define $\mathrm{He}\{A\} := (A + A^\top)$. We denote $\mathrm{Im}(\Phi)$ as the image of the function Φ. We denote $\texttt{col}(x, y) := \begin{bmatrix} x^\top & y^\top \end{bmatrix}^\top$.

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II. Preliminaries

In this section, we consider a nonlinear system of the form

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{1}$$

where f is sufficiently smooth. The flow of f is denoted by ψ^t, and $\psi^t(x_0)$ is the trajectory of (1) at time t. By definition, $\psi^0(x_0) = x_0$. Following classical contraction theory [20], the property of 1-contraction expresses the fact that the length of any C^1 curve between two arbitrary initial conditions decreases along the system trajectories. This curve is usually described by a parametrized smooth function $\Phi : [0,1] \to \mathbb{R}^n$ associating each point on the path to a scalar value between 0 and 1. k-contraction extends this notion to k-dimensional volumes [4], [5]. More formally, consider a set of sufficiently smooth parametrized functions $\mathcal{I}_k$ defined on $[0,1]^k$, namely

$$\mathcal{I}_k := \left\{\Phi : [0,1]^k \to \mathbb{R}^n \mid \ \Phi \text{ is a smooth immersion}\right\}. \tag{2}$$

Let $P \in \mathbb{R}^{n\times n}$ be a positive definite symmetric matrix. For each Φ in $\mathcal{I}_k$, we define the volume $V_P^k(\Phi)$ of Φ as

$$V_P^k(\Phi) := \int_{[0,1]^k} \sqrt{\det\left\{\frac{\partial \Phi}{\partial r}(r)^\top P \frac{\partial \Phi}{\partial r}(r)\right\}}\,\mathrm{d}r\,. \tag{3}$$

With this definition in mind, we now define the k-contraction property for nonlinear systems of the form (1), which will be used throughout the article. Let $k \in \{1,\ldots,n\}$.

Definition 1 (k-contraction)**.** *System* (1) *is k-contractive on a forward invariant set $\mathcal{S} \subseteq \mathbb{R}^n$ if there exist real numbers $a, b > 0$ such that, for every $\Phi \in \mathcal{I}_k$ satisfying $\mathrm{Im}(\Phi) \subseteq \mathcal{S}$, the following holds*

$$V_P^k(\psi^t \circ \Phi) \leqslant b e^{-at}\, V_P^k(\Phi), \qquad \forall t \in \mathbb{R}_{\geqslant 0}. \tag{4}$$

In plain words, a system is k-contractive if, for any parametrized submanifold in $\mathcal{S}$ of initial conditions, its volume is exponentially shrinking along the system dynamics. More details can be found in [4], [5].

Sufficient conditions for k-contraction that do not involve matrix compounds were recently proposed in [5], [10]. We now recall the main result, which involves a pair of matrix inequalities to be verified on a compact set $\mathcal{S}$. An intuition behind the required conditions is provided in [5].

Theorem 1 ([5])**.** *Let $\mathcal{S} \subsetneq \mathbb{R}^n$ be a compact forward invariant set. Suppose there exist two symmetric matrices $P_0, P_1 \in \mathbb{R}^{n\times n}$ of respective inertia $(0,0,n)$ and $(k-1, 0, n-k+1)$, and $\mu_0, \mu_1 \in \mathbb{R}$ such that for all $x \in \mathcal{S}$*

$$\frac{\partial f}{\partial x}(x)^\top P_0 + P_0 \frac{\partial f}{\partial x}(x) \preceq 2\mu_0 P_0, \tag{5a}$$

$$\frac{\partial f}{\partial x}(x)^\top P_1 + P_1 \frac{\partial f}{\partial x}(x) \prec 2\mu_1 P_1, \tag{5b}$$

$$\mu_1 + (k-1)\mu_0 < 0, \tag{5c}$$

Then, system (1) *is k-contractive on $\mathcal{S}$.*

A major drawback of Theorem 1 lies in the fact that it requires solving (5) for all $x \in \mathcal{S}$. In other words, conditions in (5) describe an infinite set of matrix inequalities. To circumvent this obstacle, the main objective of this paper is to specialize conditions (5) to specific classes of nonlinear systems, i.e., Lur'e systems. As shown in the next section, this structural assumption allows the derivation of sufficient conditions that can be efficiently checked with modern matrix inequalities solvers. More specifically, it will be shown that the infinite set of inequalities of (5) can be reduced to a finite set of matrix inequalities, thus providing numerically tractable conditions for k-contraction based on Theorem 1.

III. Main result

In this paper, we focus on systems of the form

$$\dot{x} = f(x) = Ax + B\phi(y), \quad y = Cx \tag{6}$$

with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{p\times n}, \phi : \mathbb{R}^p \to \mathbb{R}^p$. We assume ϕ is a decentralized, memory-less, C^1 nonlinear function. Moreover, we suppose that the nonlinearity $\phi(\cdot)$ satisfies a particular set of inequalities, i.e., *quadratic abstractions*. We first propose an initial analysis exploiting general quadratic abstractions. The obtained results will be subsequently exploited to derive specialized conditions for the particular frameworks of shifted-monotonic and differentially sector-bounded nonlinearities.

We highlight that the only differences between (5a) and (5b) are the inertias of the matrices P_0, P_1 and the values of μ_0, μ_1. Moreover, we highlight that they can be solved separately. For this reason, throughout the rest of the section, we will focus on generalized Lyapunov inequalities of the form

$$\frac{\partial f}{\partial x}(x)^\top P + P\frac{\partial f}{\partial x}(x) \prec 2\mu P, \quad \forall x \in \mathcal{S} \tag{7}$$

with arbitrary invertible symmetric matrix P, arbitrary constant μ, vector fields f of the form (6) and $\mathcal{S} \subseteq \mathbb{R}^n$. The proposed techniques can be used to solve any of the inequalities in (5). While our analysis is general and will not assume compactness and forward invariance of the set $\mathcal{S}$, we remark that these assumptions are required for applying the results of Theorem 1.

Remark 1. *Although we focus on k-contraction, an inequality of the form* (7) *also appears in other properties such as p-dominance [6] and orbital stability analysis [21].*

A. Preliminary analysis

As mentioned before, we consider a nonlinear system of the form

$$\dot{x} = \mathbf{f}(x) = \mathbf{A}x + \mathbf{B}\boldsymbol{\phi}(y), \quad y = \mathbf{C}x, \tag{8}$$

with $\mathbf{A} \in \mathbb{R}^{n\times n}, \mathbf{B} \in \mathbb{R}^{n\times p}, \mathbf{C} \in \mathbb{R}^{p\times n}, \boldsymbol{\phi} : \mathbb{R}^p \to \mathbb{R}^p$. We assume $\boldsymbol{\phi}$ is a decentralized, memory-less and C^1 nonlinearity. The notation in bold $\mathbf{A}, \mathbf{B}, \mathbf{C}, \boldsymbol{\phi}(\cdot)$ is used to distinguish (8) from (6), and it will be specialized in Sections III-B and III-C. Moreover, let $\mathcal{S} \subseteq \mathbb{R}^n$ and define the following set

$$\mathcal{Y} := \{y \in \mathbb{R}^p : y = \mathbf{C}x,\, x \in \mathcal{S}\} \subseteq \mathbb{R}^p. \tag{9}$$

Then, the condition in (7) for system (8) reads

$$\frac{\partial \mathbf{f}}{\partial x}(x)^\top P + P\frac{\partial \mathbf{f}}{\partial x}(x) \prec 2\mu P, \quad \forall x \in \mathcal{S}. \tag{10}$$

$$\begin{pmatrix} -2\beta M + \mathrm{He}\{N_1\mathbf{A}\} - \mathbf{C}^\top \sum_{i=1}^m \alpha_i X_i \mathbf{C} & M - N_1 + \mathbf{A}^\top N_2^\top & N_1\mathbf{B} + \mathbf{A}^\top N_3^\top - \mathbf{C}^\top \sum_{i=1}^m \alpha_i Y_i^\top \\ M + N_2\mathbf{A} - N_1^\top & -\mathrm{He}\{N_2\} & N_2\mathbf{B} - N_3^\top \\ N_3\mathbf{A} + \mathbf{B}^\top N_1^\top - \sum_{i=1}^m \alpha_i Y_i \mathbf{C} & -N_3 + \mathbf{B}^\top N_2^\top & \mathrm{He}\{N_3\mathbf{B}\} - \sum_{i=1}^m \alpha_i Z_i \end{pmatrix} \prec 0 \tag{12}$$

We also assume that the nonlinearity $\boldsymbol{\phi}$, satisfies a set of $m \geqslant 1$ quadratic abstractions of the form

$$\begin{pmatrix} I_p \\ \frac{\partial \boldsymbol{\phi}}{\partial y}(y) \end{pmatrix}^\top \begin{pmatrix} X_i & Y_i^\top \\ Y_i & Z_i \end{pmatrix} \begin{pmatrix} I_p \\ \frac{\partial \boldsymbol{\phi}}{\partial y}(y) \end{pmatrix} \preceq 0, \tag{11}$$

for all $y \in \mathcal{Y}$ and $i \in \{1, \ldots, m\}$, with X_i, Y_i, Z_i being square matrices of dimension $p \times p$ and X_i, Z_i being symmetric, for all $i \in \{1, \ldots, m\}$.

We now propose a sufficient condition for the verification of (10) under the dynamics (8) and conditions (11).

Theorem 2. *Consider system* (8) *and the set* $\mathcal{Y}$ *in* (9)*. If there exist* m *square matrices* $X_i = X_i^\top, Y_i, Z_i = Z_i^\top$*,* m *positive scalars* $\alpha_i > 0$ *with* $i \in \{1, \ldots, m\}$*, matrices* $N_1, N_2 \in \mathbb{R}^{n \times n}, N_3 \in \mathbb{R}^{p \times n}$*, a scalar* $\beta \in \mathbb{R}$ *and a nonsingular matrix* $M = M^\top \in \mathbb{R}^{n \times n}$ *such that* (12) *and* (11) *hold for all* $i \in \{1, \ldots, m\}$ *and for all* $y \in \mathcal{Y}$*, then* (10) *holds for all* $x \in \mathcal{S}$ *with* $P = M$ *and* $\mu = \beta$*.*

Proof. Consider the matrix inequality (12), which can be equivalently written as $\Omega - \sum_{i=1}^m \alpha_i \Lambda_i \preceq 0$ where

$$\Omega := \begin{pmatrix} \Omega_{1,1} & \Omega_{1,2} & \Omega_{1,3} \\ \Omega_{1,2}^\top & \Omega_{2,2} & \Omega_{2,3} \\ \Omega_{1,3}^\top & \Omega_{2,3}^\top & \Omega_{3,3} \end{pmatrix}, \tag{13}$$

with

$$\begin{aligned} &\Omega_{1,1} := -2\beta M + \mathrm{He}\{N_1\mathbf{A}\} && \Omega_{1,2} := M - N_1 + \mathbf{A}^\top N_2^\top \\ &\Omega_{2,2} := -(N_2 + N_2^\top) && \Omega_{1,3} := N_1\mathbf{B} + \mathbf{A}^\top N_3^\top \\ &\Omega_{3,3} := N_3\mathbf{B} + \mathbf{B}^\top N_3^\top && \Omega_{2,3} := N_2\mathbf{B} - N_3^\top \end{aligned}$$

and

$$\Lambda_i := \begin{pmatrix} \mathbf{C}^\top X_i \mathbf{C} & 0 & \mathbf{C}^\top Y_i^\top \\ 0 & 0 & 0 \\ Y_i \mathbf{C} & 0 & Z_i \end{pmatrix}. \tag{14}$$

Consider now (14) and note that $\frac{\partial \boldsymbol{\phi}}{\partial x}(y) = \frac{\partial \boldsymbol{\phi}}{\partial y}(\mathbf{C}x)\mathbf{C}$ by the chain rule. Let $\nu := \mathtt{col}\left(I_n, \frac{\partial \mathbf{f}}{\partial x}(x), \frac{\partial \boldsymbol{\phi}}{\partial y}(\mathbf{C}x)\mathbf{C}\right)$. Left and right multiplication of (11) by $\mathbf{C}^\top$ and $\mathbf{C}$, respectively, yields for all $x \in \mathcal{S}$

$$\nu^\top \Lambda_i \nu \preceq 0, \qquad \forall i = 1, \ldots, m.$$

Consequently, by the positivity of the multipliers α_i, we obtain $\nu^\top \Omega \nu \preceq \nu^\top(\Omega - \sum_{i=1}^m \alpha_i \Lambda_i)\nu \preceq 0$ for all $x \in \mathcal{S}$. Consider now (13). It can be equivalently formulated as

$$\Omega = \begin{pmatrix} -2\beta M & M & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \mathrm{He}\left\{ \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \begin{pmatrix} \mathbf{A} & -I_n & \mathbf{B} \end{pmatrix} \right\}. \tag{15}$$

From the dynamics (8), one gets the following constraint

$$\begin{pmatrix} \mathbf{A} & -I_n & \mathbf{B} \end{pmatrix} \nu = 0. \tag{16}$$

Then, from (15), it follows that, for all $x \in \mathcal{S}$, it must hold

$$\nu^\top \begin{pmatrix} -2\beta M & M & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \nu \prec 0. \tag{17}$$

Under the selection $P = M$ and $\mu = \beta$, the proof is concluded by developing the product in (17). □

Remark 2. *By fixing* β *and* α_i*,* $i = 1, \ldots, m$*, condition* (12) *is a linear matrix inequality (LMI) efficiently solvable by standard LMI-solvers. Additionally, for a fixed* β*, all the solutions of* (12)*, if they exist, will present a matrix* M *with the same inertia. This is a consequence of two facts. First,* $P = M$ *and* $\mu = \beta$ *is a solution of* (10) *by means of Theorem 2. Second, a solution of* (10) *satisfies* $\mathrm{In}(-P) = \mathrm{In}(\frac{\partial f}{\partial x}(x) - \mu I)$ *for all* $x \in \mathcal{S}$ *[5, Lemma 6]. Consequently, by fixing* β*, we are indirectly fixing the inertia of* M*.*

In the following subsections, we will exploit the results of Theorem 2 to derive results for two main frameworks: i) ϕ in (6) satisfies a *shifted monotonicity* condition or ii) ϕ in (6) satisfies a *differential sector* condition.

B. Shifted-monotonic nonlinearity

In this subsection, we assume that the nonlinearity ϕ in system (6) satisfies the following shifted monotonic condition.

$$-\left(\Gamma_1^\top + \Gamma_1\right) \prec \frac{\partial \phi}{\partial y}(y)^\top + \frac{\partial \phi}{\partial y}(y) \preceq \Gamma_2, \quad \forall y \in \mathcal{Y} \tag{18}$$

with matrices $\Gamma_1, \Gamma_2 \in \mathbb{R}^{p \times p}, \mathrm{He}\{\Gamma_1\} \succeq 0, \Gamma_2 = \Gamma_2^\top \succ 0$. This condition generalizes the classical differential description of monotonicity (e.g., [15], [22]) where $\Gamma_1 = 0$ to enforce positivity. Indeed, (18) can be interpreted as requiring the *shifted* nonlinearity $\bar{\phi}(y) = \phi(y) + \Gamma_1 y$ to be monotonic. Notice that, by continuity arguments, there always exists $\varepsilon > 0$ small enough such that

$$\varepsilon \frac{\partial \bar{\phi}}{\partial y}(y)^\top \frac{\partial \bar{\phi}}{\partial y}(y) \preceq \frac{\partial \bar{\phi}}{\partial y}(y)^\top + \frac{\partial \bar{\phi}}{\partial y}(y) \preceq \Gamma, \quad \forall y \in \mathcal{Y}, \tag{19}$$

with $\Gamma := \Gamma_2 + \mathrm{He}\{\Gamma_1\}$. Then, the condition (19) can be represented as a quadratic abstraction (11) by means of two sets of matrices X_1, Y_1, Z_1 and X_2, Y_2, Z_2, namely, $m = 2$. This allows the particularization of Theorem 2 to the case of shifted monotonic nonlinearities. The result is formalized in the next proposition.

Proposition 1. *Consider system* (6) *and the set* $\mathcal{Y}$ *in* (9)*. If there exist square matrices* $\Gamma_1, \Gamma_2 = \Gamma_2^\top \succ 0$*, positive scalars* $\alpha_1, \alpha_2 > 0$*, matrices* $N_1, N_2 \in \mathbb{R}^{n \times n}, N_3 \in \mathbb{R}^{p \times n}$*,* $\gamma \in (0, \varepsilon], \beta$ *and a nonsingular matrix* $M = M^\top$ *such that* (18) *and* (12) *hold with* $m = 2$*,* $\mathbf{A} = A - B\Gamma_1 C$*,* $\mathbf{B} = B$*,*

$\mathbf{C} = C$, $X_1 = -(\Gamma_1^\top + \Gamma_1 + \Gamma_2)$, $X_2 = 0$, $Y_1 = I_p$, $Y_2 = -I_p$, $Z_1 = 0$ and $Z_2 = \gamma I_p$ for all $y \in \mathcal{Y}$, then (7) *holds with $P = M$ and $\mu = \beta$ for all $x \in \mathcal{S}$.*

Proof. The proof is based on the results of Theorem 2. Hence, we aim at reworking condition (19) such that it fits the formulation (11). To this aim, recall that $\bar{\phi}(y) := \phi(y) + \Gamma_1 y$. By (6), we have

$$\frac{\partial f}{\partial x}(x) = \frac{\partial \bar{f}}{\partial x}(x) = A - B\Gamma_1 C + B\frac{\partial \bar{\phi}}{\partial x}(x)C.$$

Consider now the right inequality of constraint (19). It can be equivalently written as

$$\begin{pmatrix} I_p \\ \frac{\partial \bar{\phi}}{\partial x}(y) \end{pmatrix}^\top \begin{pmatrix} -\Gamma & I_p \\ I_p & 0 \end{pmatrix} \begin{pmatrix} I_p \\ \frac{\partial \bar{\phi}}{\partial x}(y) \end{pmatrix} \preceq 0, \quad \forall y \in \mathcal{Y}.$$

Thus, it imposes a first constraint of the form (11) with $X_1 = -\Gamma$, $Y_1 = I_p$ and $Z_1 = 0$. Similarly, the left inequality in (19), we obtain a second constraint of the form (11) with $X_2 = 0$, $Y_2 = -I_p$ and $Z_2 = \varepsilon I_p$. Then, if (12) holds with $\mathbf{A} = A - B\Gamma_1 C$, $\mathbf{B} = B$, $\mathbf{C} = C$, $X_1 = -(\Gamma_1^\top + \Gamma_1 + \Gamma_2) = -\Gamma$, $X_2 = 0$, $Y_1 = I_p$, $Y_2 = -I_p$, $Z_1 = 0$ and $Z_2 = \gamma I_p$ with $\gamma \in (0, \varepsilon]$, the result follows by Theorem 2. □

C. Differentially sector-bounded nonlinearity

In this subsection, we assume that the nonlinearity ϕ in (6) satisfies the differential sector condition

$$\text{He}\left\{ \left(\frac{\partial \phi}{\partial y}(y) + S_1\right)^\top D \left(\frac{\partial \phi}{\partial y}(y) + S_2\right) \right\} \preceq 0, \qquad (20)$$

for all $y \in \mathcal{Y}$, with $D \succ 0$ and $S_1, S_2 \in \mathbb{R}^{p\times p}$. This condition generalizes the one proposed in [15], by allowing arbitrary sector boundaries S_1, S_2.

Remark 3. *Note that condition* (5) *doesn't require the nonlinearity ϕ to be square, differently from* (19)*. However, we maintain such an assumption for consistency reasons.*

Similarly to Proposition 1, we now specialize Theorem 2.

Proposition 2. *Consider system* (6) *and $\mathcal{Y}$ in* (9)*. If there exist matrices $D \succ 0$, $S_1, S_2 \in \mathbb{R}^{p\times p}$, $N_1, N_2 \in \mathbb{R}^{n\times n}$, $N_3 \in \mathbb{R}^{p\times n}$, scalars $\alpha_1 > 0, \beta$ and a nonsingular matrix $M = M^\top$ such that* (20) *and* (12) *hold with $m = 1$, $\mathbf{A} = A - BS_1C$, $\mathbf{B} = B$, $\mathbf{C} = C$, $X_1 = 0$, $Y_1 = D(S_2 - S_1)$ and $Z_1 = 2D$, then* (7) *holds with $P = M$ and $\mu = \beta$ for all $x \in \mathcal{S}$.*

Proof. Similarly to the proof of Proposition 1, we rework condition (20) by rewriting system (6). By defining $\bar{\phi}(y) := \phi(y) + S_1 y$ and $S = S_2 - S_1$ the sector condition (20) reads

$$\text{He}\left\{ \frac{\partial \bar{\phi}}{\partial y}(y)^\top D \left(\frac{\partial \bar{\phi}}{\partial y}(y) + S\right) \right\} \preceq 0, \quad \forall y \in \mathcal{Y}. \qquad (21)$$

It is easily verifiable that (21) implies

$$\begin{pmatrix} I_p \\ \frac{\partial \bar{\phi}}{\partial x}(y) \end{pmatrix}^\top \begin{pmatrix} 0 & S^\top D \\ DS & 2D \end{pmatrix} \begin{pmatrix} I_p \\ \frac{\partial \bar{\phi}}{\partial x}(y) \end{pmatrix} \preceq 0, \quad \forall y \in \mathcal{Y}.$$

Thus, it imposes a constraint of the form (11) with $X_1 = 0$, $Y_1 = S^\top D$ and $Z_1 = 2D$. As in the proof of Proposition 1, if (12) holds with $m = 1$, $\mathbf{A} = A - BS_1C$, $\mathbf{B} = B$, $\mathbf{C} = C$, $X_1 = 0$, $Y_1 = D(S_2 - S_1) = S$ and $Z_1 = 2D$, the result follows by Theorem 2. □

IV. Numerical examples

We now propose two numerical examples exploiting the results of Propositions 1 and 2. In what follows, we solve the inequalities (5) by means of (12) to study 2 and 3-contraction, respectively. In particular, consider the next linear system

$$\begin{aligned} \dot{x} &= \widetilde{A}x + Bu, & y &= Cx, \\ \widetilde{A} &= \begin{bmatrix} 2 & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}, & B &= \begin{bmatrix} 0.5 \\ 0 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}. \end{aligned} \qquad (22)$$

A simple eigenvalue analysis shows that the system (22) is unstable. Also, it is easy to verify that the linear controller $u = -Ky$ exponentially stabilizes the origin of the closed-loop for all $K > 4$. However, we aim at exploiting a nonlinearity to generate more complex behaviors, e.g., multi-stability. More specifically, we consider a feedback term $u = -Ky + v$ with $v = \phi(y)$ to enforce multiple equilibrium points. Under the aforementioned feedback law, system (22) reads

$$\dot{x} = Ax + B\phi(y), \quad y = Cx, \quad A = \begin{bmatrix} 2-0.5K & 1 & -1 \\ 1 & -2 & 0 \\ 1 & 0 & -2 \end{bmatrix}. \qquad (23)$$

with $\phi(y)$ being a feedback term to be designed.

A. 2-contractive example: Multi-stability via nonlinear laws

In this section, we focus on nonlinear functions $\phi(\cdot)$ that satisfy the shifted monotonicity condition in (18). In particular, we select $\phi(\cdot) = -\cos(\cdot)$, which satisfies the shifted-monotonic condition (18) with the proposed constants $\Gamma_1 = 1.1$, $\Gamma_2 = 2.1$. It can be verified that the cosine function induces a number of equilibrium points which depends on the choice of K in (23). Specifically, the equilibrium points, denoted here as x_1^*, x_2^*, x_3^*, satisfy $(2 - 0.5K)x_1^* - 0.5\cos(x_1^*) = 0$ and $x_2^* = x_3^* = 0.5x_1^*$. Therefore, the first coordinate is given by the zeroes of a scaled cosine function oscillating around a line inclined by the coefficient $(2 - 0.5K)$. For instance, $K = 4$ creates infinite equilibrium points, $K = 4.5$ produces a single one and $K = 4.2$ generates three equilibria with $x_1^* \in \{-1.306, -1.977, 3.837\}$. A local analysis around each equilibrium point confirms that the equilibrium related to $x_1^* = 3.837$ is unstable, while the others are stable. Indeed, it represents a point where bifurcation of behavior is happening. As a consequence, by selecting $K = 4.2$, the closed-loop system cannot be 1-contractive, since 1-contraction implies the existence of a unique, globally exponentially stable equilibrium point [1, Theorem 3.8]. Nonetheless, from this local analysis we cannot conclude that any bounded solution of the system will converge to one of the equilibrium points. To conclude this, we follow a 2-contraction analysis. Inequality (12) as in Proposition 1 can be solved for (23) with

$$M_0 = \begin{bmatrix} 1.8208 & -0.0039 & 0.0044 \\ -0.0039 & 1.0843 & -0.4171 \\ 0.0044 & -0.4171 & 1.0219 \end{bmatrix}, \quad \beta_0 = 1.3,$$

where $M_0 \succ 0$. Additionally, (12) is also solved with

$$M_1 = \begin{bmatrix} -1.9479 & -0.0043 & 0.0056 \\ -0.0043 & 6.2381 & -5.3810 \\ 0.0056 & -5.3810 & 5.2375 \end{bmatrix}, \quad \beta_1 = -1.5,$$

where M_1 has inertia $(1, 0, 2)$. Combining these results with Proposition 1, we deduce that (5) is satisfied since $\beta_0 + \beta_1 < 0$. Therefore, by Theorem 1, the nonlinear function $\phi(\cdot) = -\cos(\cdot)$ makes the system 2-contractive and any bounded solution will converge to an equilibrium point [7].

Precisely, in Fig. 1, we present the simulation of 100 different randomly initialized trajectories for the system. The simulation confirms that the system presents 2 attractive fixed points and an unstable one. Moreover, we verify that a flat square of initial conditions shrinks to a straight line connecting the three equilibrium points.

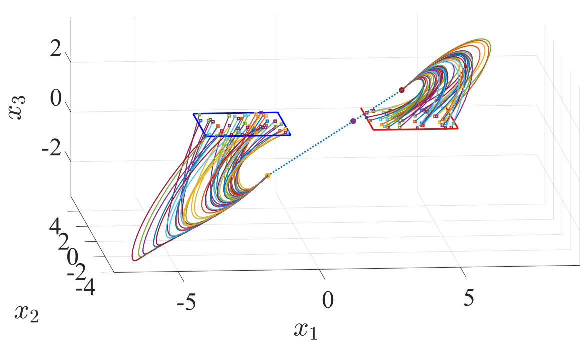

Fig. 1: Evolution of 100 trajectories of (23) with $v = -\cos(y)$. The blue and red squares depict the set of initial conditions. The 3 aligned points depict the fixed points of the system, connected by a straight line (dashed).

B. 3-*contractive example: Oscillations via integral action*

Consider the linear system (22) with the dynamic feedback law $u = -(Ky - \sin(y) + z)$, where $z \in \mathbb{R}$ integrates a constant reference $r \in \mathbb{R}$ according to $\dot{z} = y - r$. The extended system considering the integral action is

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = f(x, z, r) := \begin{bmatrix} Ax - B(z - \sin(y)) \\ y - r \end{bmatrix}, \qquad (24)$$

with A as in (23) and $K = 4.2$. It can be easily verified that (24) presents a unique equilibrium point at

$$\begin{bmatrix} (x^*)^\top & z^* \end{bmatrix} = \begin{bmatrix} r & 0.5r & 0.5r & -0.2r + \sin(r) \end{bmatrix}. \qquad (25)$$

Usually, integral action is designed to regulate the system's output to constant references [23]. This requires (25) to be asymptotically stable, e.g. [15], [23]. However, the equilibrium point in (25) is not asymptotically stable for all possible values of r. This fact can be seen, from the linearization of the vector field in (24) evaluated at (25). Therefore, the system cannot be 1-contractive for all $r \in \mathbb{R}$ [1, Theorem 3.8]. Nonetheless, the system could still be 3-contractive, possibly implying more complex behaviors.

Therefore, we now exploit the differential sector conditions (20) to study 3-contraction. It can be easily verified that $\phi(\cdot) = \sin(\cdot)$ satisfies (20) with $S_2 = -S_1 = 1.1$ for any $D > 0$. By selecting $D = 1$, we can numerically verify that the inequality (12) as in Proposition 2 is solved with

$$M_0 = \begin{bmatrix} 1.5585 & 0.2122 & -0.4469 & -0.1143 \\ 0.2122 & 1.0882 & -0.1989 & -0.1036 \\ -0.4469 & -0.1989 & 0.9948 & 0.0461 \\ -0.1143 & -0.1036 & 0.0461 & 0.7379 \end{bmatrix}, \; \beta_0 = 0.9,$$

where $M_0 \succ 0$. Additionally, (20) is also solved with

$$M_1 = \begin{bmatrix} -0.5269 & 0.4218 & -0.8719 & 0.0211 \\ 0.4218 & 27.4981 & -28.0998 & 0.1017 \\ -0.8719 & -28.0998 & 29.6619 & -0.2471 \\ 0.0211 & 0.1017 & -0.2471 & -0.2486 \end{bmatrix}, \; \beta_1 = -1.9,$$

where M_2 has inertia $(2, 0, 2)$. Combining these results with Proposition 2, we deduce that (5) is satisfied since $2\beta_0 + \beta_1 < 0$. Hence, by Theorem 1, system (24) is 3-contractive for all $r \in \mathbb{R}$. Consequently, since the solutions of (24) remain bounded, the system's trajectories will converge to a fixed point or a limit cycle [5, Lemma 4]. Precisely, for references r such that the linearization of (24) is unstable, the system will converge to a unique limit cycle around the equilibrium point (25). Otherwise, the system will converge to the equilibrium point (25). Both situations are shown in Fig. 2.

Finally, we highlight that the induced oscillatory behavior inherits robustness properties from the integral action. Indeed, consider now the perturbed system

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} Ax - B(z - \sin(x_1)) + d \\ x_1 - r \end{bmatrix}, \qquad (26)$$

where $d \in \mathbb{R}^3$ is a vector of constant disturbances. Numerical simulations confirm that the projection of the limit cycle onto the x_1 axis is robust to the constant disturbances. In particular, the evolution of a trajectory of (26) for reference $r = 3.5$ and disturbance $d = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^\top$ is depicted in Bottom-Right of Fig. 2. Since, the linearization of (24) is stable with this reference, the typical scenario of integral action is obtained. That is, the system converges to an equilibrium point with $x_1^* = r$, independently from the disturbance d. Alternatively, the evolution of a trajectory of (26) for the unstable reference $r = 1$ and disturbance $d = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^\top$ is depicted in Bottom-Left of Fig. 2. In this scenario, it can be seen that the component x_1 still converges to the same oscillatory behavior.

We remark that this robustness is due to two facts. First, we are generating self-sustained oscillations through feedback. Second, the feedback contains an integral action. Indeed, this robustness would not be present if oscillations were to be generated by external driving signals.

V. Conclusions

We discussed efficient methods for k-contraction verification of partially linear systems. By exploiting properties of isolated nonlinearities, we showed that recent sufficient conditions based on an infinite set of matrix inequalities can be reduced to a single, efficiently solvable one. We specialized our findings to the scenarios of shifted-monotonic and differentially sector-bounded nonlinearities. We validated the method by designing nonlinear feedback laws achieving nontrivial asymptotic behaviors in linear systems. Future works will focus on dealing with explicit local formulations, embedding $\mathcal{S}$ in the LMI conditions.

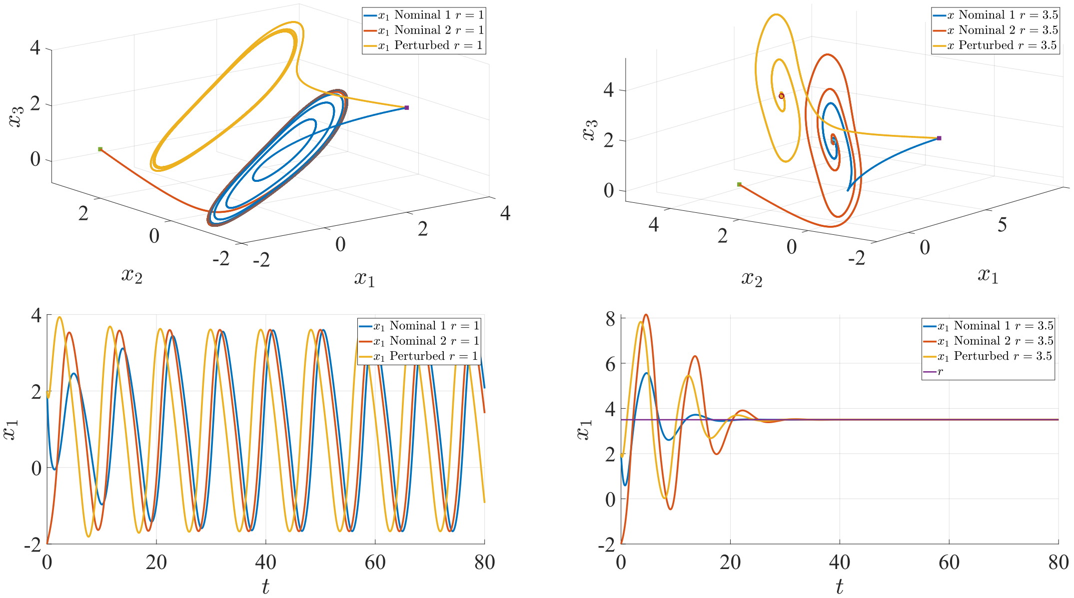

Fig. 2: Top-left) Evolution of 2 trajectories of system (24) with $r = 1$ and one trajectory of system (26) with $r = 1$ and $d = \begin{bmatrix} 3 & 3 & 3 \end{bmatrix}^\top$. Bottom-left) Projection of these trajectories into x_1. Top-right) and Bottom-right) The same with $r = 3.5$.

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