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# Specification Verification and Controller Synthesis Using ($\gamma$,$\delta$)-Similarity

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*Abstract*— **We address the problems of specification verification and controller synthesis in the context of ($\gamma$,$\delta$)-similarity, a notion of approximate system comparison that measures to what extent the external behaviors of two potentially non-deterministic systems are similar in an $\mathscr{L}_2$ sense. Expressing specifications in terms of input-output trajectories of a dynamical system, we use ($\gamma$,$\delta$)-similarity to verify whether the external behavior of a system satisfies such specifications in an approximate sense. We characterize this problem as a linear matrix inequality feasibility problem. In case a control system fails to satisfy specifications with a desired accuracy, we synthesize a dynamic controller that enforces specification satisfaction. We characterize the synthesis problem in terms of a bilinear matrix inequality feasibility problem. Aware of the computational costs for solving such problem, we obtain a sufficient condition for the existence of the controller that can be expressed in terms of a linear matrix inequality. Based on this, we propose an algorithm to construct the controller.**

## I. INTRODUCTION

The analysis and synthesis of modern engineering systems, which often appear as interconnections of components, have become increasingly challenging due to a lack of scalability in many existing (control) design methods. These challenges motivate the adoption of modular approaches [1] that enable the verification of global specifications on the basis of component-level (local) specifications. Whereas these methods typically rely on exact relations for comparing the behavior of two systems, in this paper we utilize a novel notion for *approximate* comparison to address the problems of specification verification and controller synthesis.

Various notions of system relation have been exploited for specification verification (and/or controller design). Allowing for discrete abstractions, the notion of (bi)simulation for dynamical systems was developed in the framework of labeled transition systems [2] and has been used extensively for specification verification and controller synthesis [3]. However, as the discrete abstraction of continuous-time dynamical systems suffers from the curse of dimensionality, its use is limited to specification verification and controller synthesis for low-dimensional systems. This motivates the use of (bi)simulation for comparing two continuous-time systems [4], in which the specification itself takes the form of a *dynamical* system. From this perspective, (bi)simulation was deployed for verification [5], assume-guarantee contracts [6], [7], and controller synthesis [8].

This notion of (bi)simulation however fails to address systems whose external behaviors are *close* rather than identical. Requiring external behaviors to be sufficiently close, *approximate* (bi)simulation obtains a bound over the maximum distance between the external trajectories of the systems [9]. Exploiting the framework of labeled transition systems, approximate (bi)simulation has found its application in specification verification [10] and controller synthesis [11].

As far as continuous-time systems are concerned, approximate (bi)simulation measures the "closeness" of external trajectories in terms of the $\mathscr{L}_\infty$ norm [9], which is restrictive in the sense that many existing theories for system analysis and controller synthesis, *e.g.,* in robust control, employ the $\mathscr{L}_2$ norm. Measuring to what extent the external behaviors are similar in an $\mathscr{L}_2$ sense, the notion of ($\gamma$,$\delta$)-similarity formulates behavioral similarity of two potentially *non-deterministic* continuous-time dynamical systems as the sensitivity of the difference between the output trajectories to the external inputs of the two systems [12]. In this framework, systems are non-deterministic in the sense that the evolution of their output trajectories also depend on unknown disturbance signals. Measuring behavioral similarity in terms of the $\mathscr{L}_2$ norm, ($\gamma$,$\delta$)-similarity allows for exploiting a wide range of existing tools for control theory.

Whereas [12] is solely devoted to the development and characterization of ($\gamma$,$\delta$)-similarity, the goal of this paper is to use ($\gamma$,$\delta$)-similarity to address specification verification and controller synthesis. The main contributions of this paper are as follows.

First, considering the verification problem, we express specifications as the input-output trajectories of a "specification" dynamical system and utilize ($\gamma$,$\delta$)-similarity to compare the external behavior of a system with such specification system, *i.e.,* we verify whether the external behavior of the system is approximately contained in that of the specification system.

Second, we consider the synthesis problem for control systems that fail to satisfy the specifications with a desired accuracy imposed by ($\gamma$,$\delta$)-similarity. We therefore formulate the controller synthesis problem in the framework of ($\gamma$,$\delta$)-similarity. Given a plant and a system capturing its specifications, we seek a dynamic output feedback controller such that the specification system is ($\gamma$,$\delta$)-similar to the closed-loop system obtained as the interconnection of the plant and the controller. We show that the existence of such a controller can be characterized as a bilinear matrix inequality (BMI) feasibility problem. We then draw inspiration from

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$H_\infty$ control theory to characterize a condition that allows us to formulate the problem such that it resembles a classical $H_\infty$ synthesis problem. Subsequently, we use ideas from dissipativity theory to express the problem in terms of dissipativity. By doing so, we obtain a sufficient condition for the existence of the controller that can be expressed in terms of an LMI feasibility problem. Accordingly, we propose a step-by-step algorithm to construct the controller.

The rest of this paper is organized as follows. In Section II we discuss the notion of (γ,δ)-similarity in the context of specification verification and obtain an LMI characterization for the verification problem. We then formulate the synthesis problem in Section III accordingly. While in Section III-A we address the existence of the controller, in Section III-B we obtain an algorithm to construct the controller. We further demonstrate our results in an illustrative example in Section IV and, finally, conclude the paper in Section V.

*Notation*: Given a matrix $M$, we denote a matrix whose columns form a basis for $\ker M$ by $M^\perp$. We define the operator $\langle\cdot\rangle_s$ as $\langle M\rangle_s = M + M^T$, for any square matrix $M$. We also define the operator $\mathrm{col}(\cdot,\cdot)$ such that $\mathrm{col}(x_1,x_2) = [x_1^T, x_2^T]^T$, for any vectors $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$. Given a vector $x \in \mathbb{R}^n$ and a matrix $M \succ 0$, $|x| = (x^T x)^{1/2}$ and $|x|_M = (x^T M x)^{1/2}$ denote the Euclidean norm and the weighted Euclidean norm, respectively. We define the function space $\mathscr{L}_2 = \{u : [0,\infty) \to \mathbb{R}^n \mid \int_0^\infty |u(t)|^2 \mathrm{d}t < \infty\}$, endowed with the norm $\|u\| = (\int_0^\infty |u(t)|^2 \mathrm{d}t)^{1/2}$. Accordingly, given a matrix $M \succ 0$, we define $\|u\|_M = (\int_0^\infty |u(t)|_M^2 \mathrm{d}t)^{1/2}$, for any $u \in \mathscr{L}_2$.

## II. Specification verification and (γ,δ)-Similarity

Consider the continuous-time linear system

$$\boldsymbol{\Sigma} : \begin{cases} \dot{x} = Ax + Ew + Gd, \\ z = Hx, \end{cases} \tag{1}$$

with state $x \in \mathbb{R}^n$, external input $w \in \mathbb{R}^k$, disturbance $d \in \mathbb{R}^q$, and output $z \in \mathbb{R}^p$. We assume that (1) is asymptotically stable, *i.e.,* the state matrix $A$ is Hurwitz. We denote by $z(t; x_0, w, d)$ the output solution, at time $t$, of (1) for initial condition $x(0) = x_0$, external input $w$, and disturbance $d$. The external input $w$ and the output $z$ are the variables through which (1) interacts with its environment. The disturbance $d$, on the other hand, accounts for non-determinism, as the trajectories do not solely depend on the initial condition $x_0$ and external input $w$.

We are interested in verifying whether the input-output behavior of (1) satisfies certain specifications in an approximate sense. Specifically, the specifications take the form of the input-output trajectories of the *specification* system

$$\boldsymbol{S} : \begin{cases} \dot{x}_s = A_s x_s + E_s w_s + G_s d_s, \\ z_s = H_s x_s, \end{cases} \tag{2}$$

where $x_s \in \mathbb{R}^{n_s}$, $w_s \in \mathbb{R}^{k_s}$, $d_s \in \mathbb{R}^{q_s}$, and $z_s \in \mathbb{R}^{p_s}$. We also assume that (2) is asymptotically stable, *i.e.,* the matrix $A_s$ is Hurwitz. By $z_s(t; x_{s,0}, w_s, d_s)$ we denote the output solution of (2) at time $t$ and for initial condition $x_s(0) = x_{s,0}$, external input $w_s$, and driving variable $d_s$. In view of the fact that (2) captures specifications, we regard $d_s$ in (2) as a driving variable responsible for generating the desired input-output trajectories rather than as a physical disturbance.

We compare (1) and (2) according to the following definition, which is taken from [12].

*Definition 1:* Given asymptotically stable systems $\boldsymbol{\Sigma}$ and $\boldsymbol{S}$, for $\gamma, \delta > 0$, the system $\boldsymbol{S}$ is said to be (γ,δ)-similar to the system $\boldsymbol{\Sigma}$, denoted by $\boldsymbol{\Sigma} \preccurlyeq_{\gamma,\delta} \boldsymbol{S}$, if there exist constants $\varepsilon, \eta, \mu > 0$ such that for every external input $w, w_s \in \mathscr{L}_2$ and every disturbance $d \in \mathscr{L}_2$, there exists a driving variable $d_s \in \mathscr{L}_2$ such that

$$\begin{aligned} \|z - z_s\|^2 \leq \gamma \|w - w_s\|^2 + (\delta - \varepsilon) \left\| \begin{bmatrix} w \\ w_s \end{bmatrix} \right\|^2 \\ + (\mu - \varepsilon) \|d\|^2 - \eta \|d_s\|^2, \end{aligned} \tag{3}$$

where $z(t) = z(t; 0, w, d)$ and $z_s(t) = z_s(t; 0, w_s, d_s)$.

Definition 1 states that for every trajectory of $\boldsymbol{\Sigma}$ (determined by $w$ and $d$), there exists a trajectory of $\boldsymbol{S}$ (determined by $w_s$ and $d_s$) that approximates it according to (3). In other words, $\boldsymbol{\Sigma}$ satisfies the specification given by $\boldsymbol{S}$ in an approximate sense. In (3), the (small) parameter $\varepsilon$ is present merely due to technical resasons (for details, see [12, Remark 3]). The parameter $\gamma$, however, measures to what extent a dissimilarity in external inputs gives rise to a deviation in outputs. On the other hand, the parameter $\delta$ reflects the effect that each external input has on the output deviation. For this reason, the notion of (γ,δ)-similarity serves as a criterion that measures to what extent the input-output behavior of $\boldsymbol{\Sigma}$ is contained in that of $\boldsymbol{S}$.

We will characterize (γ,δ)-similarity by deriving an alternative formulation of (3). For this purpose, we let $x_a = \mathrm{col}(x, x_s)$, $w_a = \mathrm{col}(w, w_s, d)$, and $z_a = \mathrm{col}(z - z_s, d_s)$ and take together the dynamics of (1) and (2) to obtain the *augmented* system

$$\begin{aligned} \dot{x}_a &= A_a x_a + B_a d_s + E_a w_a, \\ z_a &= H_a x_a + D_a d_s, \end{aligned} \tag{4}$$

where

$$A_a = \begin{bmatrix} A & 0 \\ 0 & A_s \end{bmatrix}, \quad B_a = \begin{bmatrix} 0 \\ G_s \end{bmatrix}, \quad E_a = \begin{bmatrix} E & 0 & G \\ 0 & E_s & 0 \end{bmatrix},$$
$$H_a = \begin{bmatrix} H & -H_s \\ 0 & 0 \end{bmatrix}, \quad D_a = \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

The following result characterizes (γ,δ)-similarity in terms of strict dissipativity of (4), see [12, Lemma 4] for details.

*Lemma 1:* For $\gamma, \delta > 0$, $\boldsymbol{S}$ is (γ,δ)-similar to $\boldsymbol{\Sigma}$ if and only if there exist constants $\mu, \eta > 0$ and a matrix $F_a$ such that

$$\begin{aligned} \dot{x}_a &= (A_a + B_a F_a) x_a + E_a w_a, \\ z_a &= (H_a + D_a F_a) x_a, \end{aligned} \tag{5}$$

is asymptotically stable and strictly dissipative with respect to the supply rate

$$s(w_a, z_a) = \begin{bmatrix} w_a \\ z_a \end{bmatrix}^T \begin{bmatrix} Q(\mu) & 0 \\ 0 & -R(\eta) \end{bmatrix} \begin{bmatrix} w_a \\ z_a \end{bmatrix}, \tag{6}$$

where

$$Q(\mu)=\begin{bmatrix}(\gamma+\delta)I & -\gamma I & 0\\ -\gamma I & (\gamma+\delta)I & 0\\ 0 & 0 & \mu I\end{bmatrix},\ R(\eta)=\begin{bmatrix}I & 0\\ 0 & \eta I\end{bmatrix}. \quad (7)$$

For the definition of strict dissipativity, we refer to [13, Definition 2.5].

*Remark 1:* An important consequence of Lemma 1 is that one can restrict attention to signals $d_s$ that are obtained as a static state feedback. Namely, we have $d_s = F_a x_a$. This will prove to be crucial in the rest of the paper.

By equivalently expressing the result of Lemma 1 in terms of a matrix inequality, we give an algebraic characterization of ($\gamma$,$\delta$)-similarity that is solely in terms of the system matrices, see [12, Theorem 2] for details.

*Theorem 2:* For $\gamma,\delta>0$, $\boldsymbol{S}$ is ($\gamma$,$\delta$)-similar to $\boldsymbol{\Sigma}$ if and only if there exist matrices $X\succ 0$, $F_a$, and scalars $\mu,\eta>0$ such that

$$\begin{bmatrix}\langle X(A_a+B_aF_a)\rangle_s & XE_a & (H_a+D_aF_a)^T\\ E_a^TX & -Q(\mu) & 0\\ H_a+D_aF_a & 0 & -R(\eta)\end{bmatrix}\prec 0. \quad (8)$$

*Remark 2:* One may utilize a congruence transformation to equivalently represent (8) as an LMI in new decision variables and then employ elimination of variables (see, *e.g.,* [13]) to eliminate $\mu$ and $\eta$, see [12, Theorem 3].

## III. Controller Synthesis

In this section, we intend to enforce ($\gamma$,$\delta$)-similarity through control. More specifically, given a *plant* and a specification system (that captures the required input-output specifications on the plant), we design a controller such that the specification system is ($\gamma$,$\delta$)-similar to the closed-loop system comprising the plant and the controller.

Consider a *plant* of the form

$$\boldsymbol{\Sigma}_p:\begin{cases}\dot{x}_p=A_px_p+B_pu_p+E_pw_p+G_pd_p,\\ z_p=H_px_p,\end{cases} \quad (9)$$

where $x_p\in\mathbb{R}^{n_p}$, $u_p\in\mathbb{R}^{m_p}$, $w_p\in\mathbb{R}^{k_p}$, $d_p\in\mathbb{R}^{q_p}$, and $z_p\in\mathbb{R}^{p_p}$ respectively denote the plant state, control input, external input, disturbance, and output. Supposing the output $z_p$ is measurable, we consider a dynamic output feedback controller of the form

$$\boldsymbol{\Sigma}_c:\begin{cases}\dot{x}_c=A_cx_c+B_cu_c,\\ y_c=C_cx_c+D_cu_c,\end{cases} \quad (10)$$

where $x_c\in\mathbb{R}^{n_c}$. Taking $u_c=z_p$ and $u_p=y_c$, we let $x_{cl}=\mathrm{col}(x_p,x_c)$ and obtain the closed-loop system

$$\boldsymbol{\Sigma}_{cl}:\begin{cases}\dot{x}_{cl}=A_{cl}x_{cl}+E_{cl}w_p+G_{cl}d_p,\\ z_{cl}=H_{cl}x_{cl},\end{cases} \quad (11a)$$

where

$$A_{cl}=\begin{bmatrix}A_p+B_pD_cH_p & B_pC_c\\ B_cH_p & A_c\end{bmatrix},\quad E_{cl}=\begin{bmatrix}E_p\\ 0\end{bmatrix},$$
$$H_{cl}=\begin{bmatrix}H_p & 0\end{bmatrix},\quad G_{cl}=\begin{bmatrix}G_p\\ 0\end{bmatrix}. \quad (11b)$$

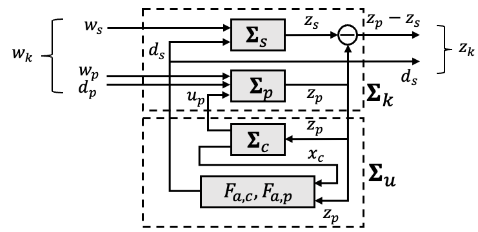



Fig. 1: We structure $F_a$ such that $d_s$ can be interpreted as an output of (10).

Noting that (11a) is of the form (1), we may now formally state the design problem as follows.

*Problem Statement*: Given the plant (9), the specification system (2), and an integer $n_c>0$, for $\gamma,\delta>0$, find a controller (10) of order $n_c$ such that $\boldsymbol{\Sigma}_{cl}$ is asymptotically stable and $\boldsymbol{\Sigma}_{cl}\preccurlyeq_{\gamma,\delta}\boldsymbol{S}$.

We will study the problem in two steps. Fixing the order of the controller (*i.e.,* for a given $n_c$), we first address the *existence* of the controller $\boldsymbol{\Sigma}_c$, *i.e.,* the existence of matrices $A_c$, $B_c$, $C_c$, and $D_c$, such that $\boldsymbol{\Sigma}_{cl}$ is asymptotically stable and $\boldsymbol{\Sigma}_{cl}\preccurlyeq_{\gamma,\delta}\boldsymbol{S}$. Subsequently, we derive an algorithm to *construct* $\boldsymbol{\Sigma}_c$, *i.e.,* to compute the matrices $A_c$, $B_c$, $C_c$, and $D_c$.

### A. Existence of the Controller

As a first step towards controller synthesis, we aim to characterize the existence of $\boldsymbol{\Sigma}_c$ exclusively in terms of the parameters of $\boldsymbol{\Sigma}_p$ and $\boldsymbol{S}$. Taking $A_c$, $B_c$, $C_c$, and $D_c$ as unknown parameters, we obtain the augmented system (4) with

$$A_a=\begin{bmatrix}A_{cl} & 0\\ 0 & A_s\end{bmatrix},\quad B_a=\begin{bmatrix}0\\ G_s\end{bmatrix},\ E_a=\begin{bmatrix}E_{cl} & 0 & G_{cl}\\ 0 & B_s & 0\end{bmatrix},$$
$$H_a=\begin{bmatrix}H_{cl} & -H_s\\ 0 & 0\end{bmatrix},\ D_a=\begin{bmatrix}0\\ I\end{bmatrix}. \quad (12)$$

We can then utilize Theorem 2 to characterize $\boldsymbol{\Sigma}_{cl}\preccurlyeq_{\gamma,\delta}\boldsymbol{S}$ as the feasibility problem (8), where $A_c$, $B_c$, $C_c$, and $D_c$ now appear as unknowns in addition to $X$ and $F_a$. Using elimination of variables, one may eliminate the controller variables $A_c$, $B_c$, $C_c$, and $D_c$, which would lead to a *bilinear* matrix inequality in terms of $X$ and $F_a$, that cannot be further simplified into an LMI. However, we show that by imposing restrictions on the structure of $F_a$, we may obtain a *sufficient* condition for the existence of $\boldsymbol{\Sigma}_c$ that can be expressed in terms of an LMI. This thus leads to an easily verifiable condition for existence of a controller.

We seek the controller $\boldsymbol{\Sigma}_c$ such that $\boldsymbol{\Sigma}_{cl}$ is asymptotically stable and $\boldsymbol{\Sigma}_{cl}\preccurlyeq_{\gamma,\delta}\boldsymbol{S}$ . In light of Remark 1, we thus seek a controller $\boldsymbol{\Sigma}_c$ and a static state feedback gain $F_a$ such that the augmented system (4) with parameters (12), comprising (11a) and (2), is asymptotically stable and strictly dissipative with respect to (6). To enable this, we draw inspiration from $H_\infty$ synthesis problems to propose a structure for $F_a$ that allows $d_s$ to be interpreted as an output of the dynamics (10), see Figure 1. In this fashion, we may write (5) as the interconnection of a *known* system (whose dynamics are in terms of the parameters of $\mathbf{\Sigma}_p$ and $\boldsymbol{S}$) and an *unknown* system (whose dynamics are in terms of the unknowns $A_c$, $B_c$, $C_c$, $D_c$, and $F_a$).

Partitioning $F_a$ according to $n_p$, $n_c$, and $n_s$, we structure $F_a$ as

$$F_a = \begin{bmatrix} F_{a,p}H_p & F_{a,c} & 0 \end{bmatrix}, \tag{13}$$

such that $F_a x_a = F_{a,p}H_p x_p + F_{a,c}x_c = F_{a,p}z_p + F_{a,c}x_c$. Taking $x_k = \mathrm{col}(x_p, x_s)$, $u_k = \mathrm{col}(u_p, d_s)$, $w_k = \mathrm{col}(w_p, w_s, d_p)$, $y_k = z_p$, and $z_k = \mathrm{col}(z_p - z_s, d_s)$, we collect the dynamics of $\mathbf{\Sigma}_p$ and $\boldsymbol{S}$ to obtain the known system as

$$\mathbf{\Sigma}_k : \begin{cases} \dot{x}_k = A_k x_k + B_k u_k + E_k w_k, \\ y_k = C_k x_k, \\ z_k = H_k x_k + D_k u_k, \end{cases} \tag{14}$$

where

$$\begin{aligned} A_k &= \begin{bmatrix} A_p & 0 \\ 0 & A_s \end{bmatrix}, B_k = \begin{bmatrix} B_p & 0 \\ 0 & G_s \end{bmatrix}, \quad E_k = \begin{bmatrix} E_p & 0 & G_p \\ 0 & E_s & 0 \end{bmatrix}, \\ C_k &= \begin{bmatrix} H_p & 0 \end{bmatrix}, \quad H_k = \begin{bmatrix} H_p & -H_s \\ 0 & 0 \end{bmatrix}, D_k = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}. \end{aligned} \tag{15}$$

On the other hand, we take $x_u = x_c$ and obtain the unknown system as

$$\begin{aligned} \dot{x}_u &= A_u x_u + B_u u_u, \\ y_u &= C_u x_u + D_u u_u, \end{aligned} \tag{16a}$$

*where*

$$A_u = A_c,\ B_u = B_c,\ C_u = \begin{bmatrix} C_c \\ F_{a,c} \end{bmatrix},\ D_u = \begin{bmatrix} D_c \\ F_{a,p} \end{bmatrix}. \tag{16b}$$

Taking $u_k = y_u$ and $u_u = y_k$, we may now write (5) as

$$\begin{aligned} \begin{bmatrix} \dot{x}_k \\ \dot{x}_u \end{bmatrix} &= \begin{bmatrix} A_k + B_k D_u C_k & B_k C_u \\ B_u C_k & A_u \end{bmatrix} \begin{bmatrix} x_k \\ x_u \end{bmatrix} + \begin{bmatrix} E_k \\ 0 \end{bmatrix} w_k, \\ z_k &= \begin{bmatrix} H_k + D_k D_u C_k & D_k C_u \end{bmatrix} \begin{bmatrix} x_k \\ x_u \end{bmatrix}, \end{aligned} \tag{17}$$

*i.e.,* the feedback interconnection of $\mathbf{\Sigma}_k$ and $\mathbf{\Sigma}_u$. This formulation resembles a classical $H_\infty$ synthesis problem in the sense that we may treat $\mathbf{\Sigma}_k$ and $\mathbf{\Sigma}_u$ respectively as a given "plant" and a "controller" to be found.

In the following proposition, upon which the main result of this paper will rely, we address the existence of $\mathbf{\Sigma}_c$ in terms of the dissipativity of (17).

*Proposition 3:* Given an integer $n_c > 0$, for $\gamma, \delta > 0$, there exist matrices $A_c$, $B_c$, $C_c$, and $D_c$ such that $\mathbf{\Sigma}_{cl}$ is asymptotically stable and $\mathbf{\Sigma}_{cl} \preccurlyeq_{\gamma,\delta} \boldsymbol{S}$ if there exist matrices $A_u$, $B_u$, $C_u$, $D_u$, and constants $\mu, \eta > 0$ such that (17) is asymptotically stable and strictly dissipative with respect to the supply rate

$$s(w_k, z_k) = \begin{bmatrix} w_k \\ z_k \end{bmatrix}^T \begin{bmatrix} Q(\mu) & 0 \\ 0 & -R(\eta) \end{bmatrix} \begin{bmatrix} w_k \\ z_k \end{bmatrix}, \tag{18}$$

where $Q(\mu)$ and $R(\eta)$ are given as in (7).

*Proof:* Suppose there exist $A_u$, $B_u$, $C_u$, $D_u$, and $\mu, \eta > 0$ such that (17) is asymptotically stable and strictly dissipative with respect to (18). This, however, implies that there exist $A_c$, $B_c$, $C_c$, $D_c$, $F_a$, and $\mu, \eta > 0$ such that (5), considered for system matrices (12) and feedback gain (13), is asymptotically stable and strictly dissipative with respect to (6). It then follows from Lemma 1 that $\mathbf{\Sigma}_{cl}$ is asymptotically stable and $\mathbf{\Sigma}_{cl} \preccurlyeq_{\gamma,\delta} \boldsymbol{S}$. ■

Proposition 3 formalizes the idea that the problem of simultaneously finding the controller (10) and the feedback gain $F_a$ in Lemma 1 structured as (13) can be replaced by finding the output feedback "controller" (16a). In the following result, we exploit this to give a sufficient condition for the existence of a controller (10) for sufficiently large $n_c$.

*Theorem 4:* Given an integer $n_c \geq n_p + n_s$, for $\gamma, \delta > 0$, there exist matrices $A_c$, $B_c$, $C_c$, and $D_c$ such that $\mathbf{\Sigma}_{cl}$ is asymptotically stable and $\mathbf{\Sigma}_{cl} \preccurlyeq_{\gamma,\delta} \boldsymbol{S}$ if there exist matrices $P, R \succ 0$ and constants $\mu, \eta > 0$ such that

$$\left[\begin{array}{c|c} N_1 & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{c|cc} \langle PA_k \rangle_s & PE_k & H_k^T \\ \hline E_k^T P & -Q(\mu) & 0 \\ H_k & 0 & -R(\eta) \end{array}\right] \left[\begin{array}{c|c} N_1 & 0 \\ \hline 0 & I \end{array}\right] \prec 0, \tag{19a}$$

$$\left[\begin{array}{c|c} N_2 & 0 \\ \hline 0 & I \end{array}\right]^T \left[\begin{array}{cc|c} \langle A_k R \rangle_s & R H_k^T & E_k \\ H_k R & -R(\eta) & 0 \\ \hline E_k^T & 0 & -Q(\mu) \end{array}\right] \left[\begin{array}{c|c} N_2 & 0 \\ \hline 0 & I \end{array}\right] \prec 0, \tag{19b}$$

$$\begin{bmatrix} P & I \\ I & R \end{bmatrix} \succcurlyeq 0, \tag{19c}$$

where $N_1$ and $N_2$ are matrices whose columns form a basis for $\ker C_k$ and $\ker \begin{bmatrix} B_k^T & D_k^T \end{bmatrix}$, respectively.

*Proof:* We show that the feasibility of (19) implies the existence of $\mathbf{\Sigma}_u$, *i.e.,* there exist $A_u$, $B_u$, $C_u$, $D_u$, and $\mu, \eta > 0$, such that (17) is asymptotically stable and strictly dissipative with (18). We then utilize Proposition 3 to conclude that the matrices $A_c$, $B_c$, $C_c$, and $D_c$ obtained as in (16) are such that $\mathbf{\Sigma}_{cl}$ is asymptotically stable and $\mathbf{\Sigma}_{cl} \preccurlyeq_{\gamma,\delta} \boldsymbol{S}$.

We follow a similar procedure to that of $H_\infty$ synthesis problems (see, *e.g.,* [14, Theorem 4.3]) to conclude the existence of $\mathbf{\Sigma}_u$. Suppose there exist $P, R \succ 0$ and $\mu, \eta > 0$ such that (19) holds. Since $n_c \geq n_p + n_s$, it follows from (19c) (see, *e.g.,* [14, Section 6]) that there exist matrices $P_2, R_2 \in \mathbb{R}^{(n_p+n_s)\times n_c}$ and $P_3, R_3 \in \mathbb{R}^{n_c \times n_c}$ such that

$$\begin{bmatrix} P & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0, \quad \begin{bmatrix} R & R_2 \\ R_2^T & R_3 \end{bmatrix} = \begin{bmatrix} P & P_2 \\ P_2^T & P_3 \end{bmatrix}^{-1}. \tag{20}$$

Using these matrices $P_2$ and $P_3$, we can define

$$\Lambda = \begin{bmatrix} \langle PA_k \rangle_s & A_k^T P_2 & PE_k & H_k^T \\ P_2^T A_k & 0 & P_2^T E_k & 0 \\ E_k^T P & E_k^T P_2 & -Q(\mu) & 0 \\ H_k & 0 & 0 & -R(\eta) \end{bmatrix}, \tag{21a}$$

as well as

$$U = \begin{bmatrix} B_k^T P_1 & B_k^T P_2 & 0 & D_k^T \\ P_2^T & P_3 & 0 & 0 \end{bmatrix}, V = \begin{bmatrix} 0 & I & 0 & 0 \\ C_k & 0 & 0 & 0 \end{bmatrix}. \tag{21b}$$

Following similar steps as in [14, Theorem 4.2], we obtain

$$U^\perp = \begin{bmatrix} R & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_k^T & 0 & 0 & D_k^T \\ 0 & I & 0 & 0 \end{bmatrix}^\perp.$$

After performing a few rearrangements, we can observe that (19a) and (19b) are respectively equivalent to $U^{\perp T} \Lambda U^\perp \prec 0$

and $V^{\perp T}\Lambda V^{\perp} \prec 0$. It then immediately follows from the projection lemma (see, *e.g.,* [14, Lemma 3.1]) that there exists a matrix $\Omega$ such that

$$\Lambda + U^T \Omega V + V^T \Omega^T U \prec 0. \quad (22)$$

Partitioning $\Omega$ as

$$\Omega = \begin{bmatrix} A_u & B_u \\ C_u & D_u \end{bmatrix}, \quad (23)$$

we immediately realize that (22) is equivalent to (8) whose parameters are now given by (12) and (13). This, however, implies that the matrix

$$\begin{bmatrix} A_k + B_k D_u C_k & B_k C_u \\ B_u C_k & A_u \end{bmatrix}$$

is Hurwitz, implying that (17) is asymptotically stable. More importantly, (22) implies that the matrices $A_u$, $B_u$, $C_u$, $D_u$, and constants $\mu, \eta > 0$ are such that (17) is strictly dissipative with respect to (18) [13, Proposition 3.9]. It now follows from Proposition 3 that there exist matrices $A_c$, $B_c$, $C_c$, and $D_c$ such that $\mathbf{\Sigma}_{cl}$ is asymptotically stable and $\mathbf{\Sigma}_{cl} \preccurlyeq_{\gamma,\delta} \boldsymbol{S}$. ■

According to Theorem 4, the feasibility of LMI (19) guarantees the existence of the controller $\mathbf{\Sigma}_c$. We now exploit (19) to propose an algorithm to construct $\mathbf{\Sigma}_c$, accordingly.

### B. Controller Design

Having characterized the existence of $\mathbf{\Sigma}_c$, *i.e.,* the matrices $A_c$, $B_c$, $C_c$, and $D_c$, we now propose a step-by-step algorithm to construct it. We therefore suppose that (19) is feasible, *i.e.,* matrices $P, R \succ 0$ and constants $\mu, \eta > 0$ exist such that (19) holds. We construct $P_2$, $P_3$, and find $A_u$, $B_u$, $C_u$, and $D_u$ such that (22) holds, implying that (17) is asymptotically stable and strictly dissipative with respect to (18). We then obtain $A_c$, $B_c$, $C_c$, and $D_c$ according to (16b) and utilize Proposition 3 to conclude that $\mathbf{\Sigma}_{cl}$ is asypmtotically stable and $\mathbf{\Sigma}_{cl} \preccurlyeq_{\gamma,\delta} \boldsymbol{S}$.

As a first step, following the proof of Theorem 4, we utilize (19c) to construct $P_2, R_2 \in \mathbb{R}^{(n_p+n_s)\times n_c}$ and $P_3, R_3 \in \mathbb{R}^{n_c \times n_c}$ such that (20) holds. Since $R \succ 0$, it follows from the Schur complement of the matrix in (19c), see, *e.g.,* [15, Proposition 8.2.4], that $P - R^{-1} \succcurlyeq 0$. We construct $P_2$ and $P_3 \succ 0$ such that

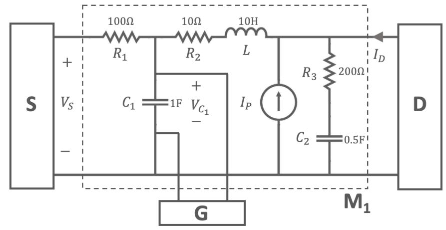
$$P - R^{-1} = P_2 P_3^{-1} P_2^T. \quad (24)$$

To do so, we perform a Cholesky decomposition (see, *e.g.,* [15, Fact 8.9.37]) to find a lower-triangular matrix $L \in \mathbb{R}^{(n_p+n_s)\times(n_p+n_s)}$, with positive diagonal entries, such that $LL^T = P_2 P_3^{-1} P_2^T$. Taking $P_2 \in \mathbb{R}^{(n_p+n_s)\times n_c}$ as $P_2 = \begin{bmatrix} L & 0 \end{bmatrix}$ and $P_3 = I$, it follows from (24) and $R \succ 0$ that

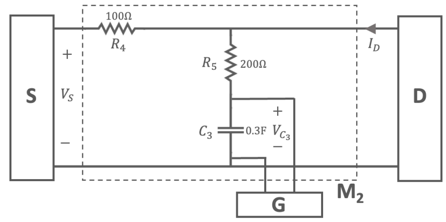
$$\begin{bmatrix} P & P_2 \\ P_2^T & P_3 \end{bmatrix} \succ 0.$$

As the second step, we accordingly define the matrices $\Lambda$, $U$, and $V$ as in (21) and solve the feasability problem (22), which is now linear in terms of the variable $\Omega$.

As a final step, we obtain $A_u$, $B_u$, $C_u$, and $D_u$ according to (23), which implies that (17) is asymptotically stable and strictly dissipative with (18). We finally obtain $A_c$, $B_c$, $C_c$, and $D_c$ according to (16b). The procedure is summarized in Algorithm 1.




(a) Component $M_1$




(b) Component $M_2$

Fig. 2: Through the current source $I_P$, we control the component $M_1$ such that $V_{C_1}$ approximates $V_{C_3}$ generated by the ideal component $M_2$.

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**Algorithm 1** Construction of the controller (10) of order $n_c$ for a given plant (9) and a given specification system (2)

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**Require:** $A_p$, $B_p$, $E_p$, $G_p$, $H_p$, $A_s$, $E_s$, $G_s$, $H_s$, and $n_c$;
1: Obtain $A_k$, $B_k$, $E_k$, $C_k$, $H_k$, and $D_k$ according to (15);
2: Solve (19) and obtain $P$, $R$, $\mu$, and $\eta$;
3: Using Cholesky decomposition, obtain $P - R^{-1} = LL^T$;
4: Take $P_2 = \begin{bmatrix} L & 0 \end{bmatrix}$ and $P_3 = I$;
5: Define $U$, $\Lambda$, and $V$ as in (21);
6: Find $\Omega$ such that (22) holds;
7: Partition $\Omega$ as in (23);
8: Obtain $A_c$, $B_c$, $C_c$, and $D_c$ according to (16b);
**Ensure:** $A_c$, $B_c$, $C_c$, and $D_c$.

---

## IV. Illustrative Example

In order to illustrate our results, consider the electrical circuit depicted in Figure 2a. Supplied with voltage $V_S$ by the component $S$, the component $M_1$ feeds the device $G$ with the voltage $V_{C_1}$, which is controlled through the current source $I_P$. In the meantime, however, $M_1$ experiences the unwanted current $I_D$ injected by the component $D$. The voltage $V_S$ and the current $I_D$ are assumed to be unknown.

Suppose that the device $G$ operates optimally when supplied with the voltage $V_{C_3}$ generated by the "ideal" component $M_2$, see Figure 2b. It is then natural to control $V_{C_1}$ (by applying the suitable current $I_P$) such that it approximates the desired voltage $V_{C_3}$. Regarding $M_1$ as the plant to be controlled and $M_2$ as the specification system, we will cast this problem into the framework of $(\gamma,\delta)$-similarity.

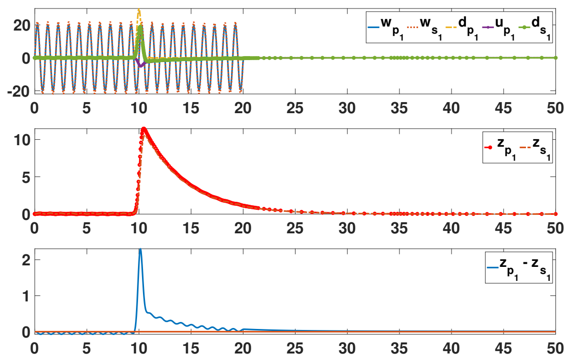


Fig. 3: The top graph depicts the driving variable $d_s$, obtained from (13), and the control input $u_p$, provided by (10). As depicted in the middle graph, subject to such $d_s$, $M_2$ reveals an output solution very similar to that of $M_1$ which is subject to the $u_p$ given in the top graph. The output deviation is also depicted in the bottom graph which further indicates how similar the output solutions are.

Let $I_L$ denote the current passing through the inductor $L$ and $V_{C_i}$ denote the voltage across the capacitor $C_i$, for $i = 1,2,3$. Taking $x_p := \mathrm{col}(V_{C_1}, V_{C_2}, I_L)$, it follows from circuit theory that $M_1$ is governed by dynamics of the form (9) where $u_p$, $w_p$, and $d_p$ respectively denote $I_P$, $V_S$, and $I_D$. In a similar fashion, with $x_s := V_{C_3}$, $M_2$ is seen to be governed by the dynamics (2) where $w_s$ and $d_s$ denote $V_S$ and $I_D$, respectively.

Due to environmental noises, the voltages that supply $M_1$ and $M_2$ (*i.e.,* $w_p$ and $w_s$) differ slightly, and therefore, the dissimilarity in external inputs is negligible. The external inputs, however, may be large individually. We are therefore more interested in making $\delta$ small than in finding a small $\gamma$. For this reason, we choose $\gamma = 1$, $\delta = 0.7$ and utilize Theorem 4 to conclude that there exists a controller (10), of order $n_c = n_p + n_s = 4$, such that the closed-loop system $\mathbf{\Sigma}_{cl}$ is asymptotically stable and $\mathbf{\Sigma}_{cl} \preccurlyeq_{\gamma,\delta} M_2$. Accordingly, we follow Algorithm 1 to construct (10). The controller synthesis problem has been solved in 0.765316 seconds in MATLAB R2022b, using YALMIP [16] and SDPT3 [17].

We conduct numerical simulation to compare the controlled behavior of $M_1$ with the behavior of $M_2$. We take $w_p, w_s$ as sinusoidal signals over $[0,20]$ and $d_s$ as a Gaussian signal with a peak at $t = 10$, which indicates that $M_1$ and $M_2$ are supplied over the interval $[0,20]$, while $M_1$ experiences an unwanted surge at $t = 10$, see Figure 3 (top graph). The output solutions and the output deviation are also illustrated in Figure 3 (middle and bottom graph, respectively). It is clear from Figure 3 that when $d_s$ is chosen according to (13), the output solution of $M_2$ is very similar to that of $M_1$ that is subject to a $u_p$ provided by the controller (10). This, however, implies that subject to the controller (10), the component $M_1$ approximately satisfies the specifications captured by the component $M_2$.

## V. Conclusion

This paper addressed specification verification and controller synthesis in the framework of ($\gamma$,$\delta$)-similarity. In this context, we characterized the verification problem in terms of an LMI feasibility problem. Accordingly, we formulated the controller synthesis problem and obtained a BMI characterization of the existence of the controller. We then proposed a sufficient condition for the existence of the controller that could be expressed in terms of an LMI. Having obtained the LMI condition, we subsequently developed a step-by-step algorithm to construct the controller.

Future work will focus on exploring techniques to obtain an LMI characterization for the existence of the controller. On the other hand, we also aim to exploit compositional reasoning to apply our results to decentralized control synthesis.

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