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Generalized Discrete Adaptive Control Allocation for Over-Actuated Systems

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Abstract— **Actuators degradation negatively impacts the system's stability and performance. Unknown control degradation can be addressed with continuous or discrete adaptive control techniques. Discrete-time methods have several advantages over continuous ones, such as they can be executed without discretization requirements, and they are counted as derivative-free controllers. Motivating from this, in this paper, a generalized discrete adaptive control allocation architecture for over-actuated systems is presented for dealing with unknown control effectiveness degradation while ensuring reference tracking. Lyapunov stability analysis is performed to guarantee asymptotic tracking error convergence of the closed-loop system. Furthermore, a hexacopter model is simulated to demonstrate the effectiveness of this architecture.**

I. Introduction

Fault tolerance is important in various autonomous systems (e.g., aerial vehicles, robot manipulators, underwater vehicles, and ground wheeled robots) [1]–[3]. By increasing the number of actuators, redundancy can be achieved, guaranteeing continued operation even in the event of degraded actuators [4]–[11]. The presence of control degradation for fully actuated systems has been dealt with through continuous-time adaptive control methods [12]–[17].

In the literature, control of vehicle operation relies on an effective mapping from desired (i.e., virtual) inputs to the vehicle's actuators. Finding this mapping is the objective of the control allocation problem and is particularly studied in over-actuated systems. Under nominal and ideal conditions (i.e., in the absence of the control degradation), a solution for control allocation is trivial. However, in the presence of degraded actuators, providing a stable control method is essential for an over-actuated system's safety since the performance of electric motors is often heavily dependent on external factors such as temperature and battery charge levels. Damage to an actuator, such as a damaged propeller, can have a significant impact on the control allocation.

Several continuous-time adaptive control techniques are developed for systems to deal with unknown degradation related to actuators. Specifically, in [6], [8], adaptive control allocation methods are presented by combining a sliding mode outer-loop control for fault tolerability and reference tracking. Later in [7], the actuator constraints are added to the problem of [6] and solved with an adaptive control allocation algorithm. The authors of [9] propose a continuous adaptive control architecture for a linear hexacopter system applied to a position-tracking problem. The authors of [11] add to [9] by considering modeling uncertainties alongside the control effort uncertainty. While these approaches can improve the fault tolerability of autonomous systems, they are all proposed in continuous-time settings. Moreover, these continuous-time control methods cannot be directly applied or readily extended to the discrete-time due to changes in Lyapunov analysis.

Discrete-time algorithms offer the advantage of direct execution in embedded code, while continuous-time ones require discretization that potentially leads to stability margin loss [18]. In addition, current embedded systems and microcontrollers inherently operate in discrete time, processing signals and making decisions at specific intervals. Furthermore, discrete-time control means a derivative-free update law since the control laws are typically formulated based on difference equations rather than differential equations. However, designing discrete-time algorithms, often based on quadratic Lyapunov frameworks, can be complex due to intricate Lyapunov difference expressions. Authors in various studies address this issue by employing logarithmic Lyapunov function candidates to establish asymptotic stability of controlled systems [19]–[28].

There are only a few papers that address control allocation for an over-actuated discrete-time model in the presence of actuation degradation. Specifically in [10], the authors design a discrete adaptive control allocation method for an over-actuated system with a guarantee of asymptotic error convergence. They further extend this architecture to have tracking capabilities through the use of an integrator state. However, they make use of an inertial measurement unit (IMU) to generate the desired forces and moments, and they have certain requirements for the structure of the physical

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system (i.e., a reduced system should be stable).

In this paper, we present a discrete-time adaptive control architecture for an over-actuated system in the presence of actuator degradation. A Lyapunov function that includes logarithmic and quadratic terms is utilized to show asymptotic convergence of state tracking error. A numerical example is given to show the efficacy of the proposed algorithm, where we used hexacopter dynamics.

II. Notation

The mathematical notation used in this paper is as follows. $\mathbb{N}$ denotes the set of non-negative integers. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}_+$ denotes the set of real positive numbers. $\mathbb{R}^n$ denotes the set of $n\times 1$ real column vectors. $\mathbb{R}^{n\times m}$ denotes the set of $n\times m$ real matrices, $\mathbb{R}_+^{n\times n}$ denotes the set of positive definite real matrices, $\overline{\mathbb{R}}_+^{n\times n}$ denotes the set of positive semi-definite real matrices, $\mathbb{D}^{n\times n}$ denotes the set of $n\times n$ diagonal matrices, I_n denotes the $n\times n$ identity matrix, and 0_n denotes the $n\times n$ zero matrix. The trace operator is denoted by $\mathrm{tr}(\cdot)$, the inverse operator is denoted by $(\cdot)^{-1}$, the pseudo-inverse operator is denoted by $(\cdot)^{\dagger}$, the set of matrix eigenvalues is denoted by spec(A), the Euclidean norm of a vector, a, is denoted by $||a||$, and the 2-norm of a matrix, A, is denoted by $||A||$. Additionally, the notations $\overline{\lambda}(\cdot)$ and $\underline{\lambda}(\cdot)$ represent the maximum and minimum eigenvalues, respectively, of a matrix. Furthermore, $\triangleq$ is used for definitions.

III. Discrete Adaptive Control Allocation

A. Discrete Control Allocation

In over-actuated systems, a mapping is required to transform control inputs, $u(k)\in\mathbb{R}^l$, to virtual control inputs, $v(k)\in\mathbb{R}^n$. In most physical dynamic systems, the virtual control inputs are the desired force and moment commands. The mapping matrix, denoted by $M\in\mathbb{R}^{n\times l}$, defines this transformation where $l>n$; thus, in general, rank$(M)\le n$. In an ideal case, the virtual control input is calculated from a control law provided that M is full rank. (i.e. rank$(M)=n$). This relationship is represented as

$$v(k)=Mu(k),\ k\in\mathbb{N}. \tag{1}$$

However, certain non-ideal conditions can degrade the actuator's effectiveness of providing the system with the desired virtual control inputs; resulting in an actual virtual control input, $v_{\mathrm{act}}(k)\in\mathbb{R}^n$. To capture this degradation, an unknown constant positive definite diagonal matrix, $\Lambda\in\mathbb{D}^{l\times l}$, is included in (2). The diagonal entries of Λ are restricted to the set $(0,1]$, where 1 represents nominal operation of the actuator and 0 represents full actuator failure. Note that this paper is concerned with the case of actuator degradation and not full actuator failure. Specifically, the actual virtual control input is given as

$$v_{\mathrm{act}}(k)=M\Lambda u(k),\ k\in\mathbb{N}. \tag{2}$$

Ideally, calculating the control input from (2) requires the inverse of the unknown matrix, $M\Lambda\in\mathbb{R}^{n\times l}$, which can be *expressed* as in [9]

$$\begin{aligned} u(k)&=(M\Lambda)^{\dagger}v_{\mathrm{act}}(k)\\ &=(M^{\dagger}+\theta)v_{\mathrm{act}}(k),\ k\in\mathbb{N}, \end{aligned} \tag{3}$$

where $\theta\in\mathbb{R}^{l\times n}$. For the unknown control effectiveness case, in implementation, (3) cannot be used because the true value of Λ, and consequently the value θ is unknown. This results in the control law becoming a desired virtual control input, $v_{\mathrm{d}}(k)\in\mathbb{R}^n$, when the value of θ is approximated. The control input can now be *designed* as

$$u(k)\triangleq(M^{\dagger}+\hat{\theta}(k))v_{\mathrm{d}}(k),\ k\in\mathbb{N}, \tag{4}$$

where $\hat{\theta}(k)\in\mathbb{R}^{l\times n}$ is the estimated value of θ and it's update law is provided in subsection III-B. By substituting (4) into (2) and subtracting the result from $v_{\mathrm{d}}(k)$ one can obtain the control error as

$$\begin{aligned} v_{\mathrm{err}}(k)&\triangleq v_{\mathrm{d}}(k)-v_{\mathrm{act}}(k)\\ &=v_{\mathrm{d}}(k)-M\Lambda(M^{\dagger}+\hat{\theta}(k))v_{\mathrm{d}}(k)\\ &=\Big(I_n-M\Lambda(M^{\dagger}+\hat{\theta}(k))\Big)v_{\mathrm{d}}(k),\ k\in\mathbb{N}. \end{aligned} \tag{5}$$

Noting that in (5), I_n can be replaced with ideal case, $M\Lambda(M^{\dagger}+\theta)$, and the virtual control input error can be expressed as

$$\begin{aligned} v_{\mathrm{err}}(k)&=\Big(M\Lambda(M^{\dagger}+\theta)-M\Lambda(M^{\dagger}+\hat{\theta}(k))\Big)v_{\mathrm{d}}(k)\\ &=-M\Lambda\tilde{\theta}(k)v_{\mathrm{d}}(k), \end{aligned} \tag{6}$$

where $\tilde{\theta}(k)\in\mathbb{R}^{l\times n}$ represents the approximation error that is defined as $\tilde{\theta}(k)\triangleq\hat{\theta}(k)-\theta$.

B. Proposed Architecture

A discrete-time over-actuated dynamical system with unknown degraded actuator control effectiveness, shown in (2), is considered in this paper. Specifically, the discrete-time system model is defined as

$$x(k+1)=Ax(k)+Bv_{\mathrm{act}}(k),\ k\in\mathbb{N},\qquad x(0)=x_0, \tag{7}$$

where $x(k)\in\mathbb{R}^n$ is the measurable states of the system, $A\in\mathbb{R}^{n\times n}$ is the known state matrix, and $B\in\mathbb{R}^{n\times n}$ is the known control input matrix with (A,B) being controllable. Additionally, a discrete reference model is chosen of the form

$$x_{\mathrm{r}}(k+1)=A_{\mathrm{r}}x_{\mathrm{r}}(k)+B_{\mathrm{r}}r(k),\ k\in\mathbb{N},\qquad x_{\mathrm{r}}(0)=x_{\mathrm{r}0}, \tag{8}$$

where $x_{\mathrm{r}}(k)\in\mathbb{R}^n$ is the reference model state vector, $r(k)\in\mathbb{R}^m$ is the bounded reference command with $\|r(k)\|\le r^*\in$

$\mathbb{R}_+$, $A_\mathrm{r} \in \mathbb{R}^{n\times n}$ is the Schur reference state matrix, and $B_\mathrm{r} \in \mathbb{R}^{n\times m}$ is the reference control input matrix.

The objective of the discrete adaptive controller is to achieve a desired closed-loop stability of discrete system (7) in the presence of unknown control effectiveness degradation, while ensuring discrete reference model (8) tracking. To this end, the desired virtual control law is designed as

$$v_\mathrm{d}(k) \triangleq -K_1 x(k) + K_2 r(k),\ k \in \mathbb{N}, \tag{9}$$

where $K_1 \in \mathbb{R}^{n\times n}$ and $K_2 \in \mathbb{R}^{n\times m}$ are the constant gain matrices chosen such that $A_\mathrm{r} \triangleq A - BK_1$ and $B_\mathrm{r} \triangleq BK_2$. The proposed update law for the estimated control effectiveness degradation parameter $\hat{\theta}(k)$ in control input given by (4) is chosen as

$$\hat{\theta}(k+1) = \hat{\theta}(k) - \frac{\eta_0 M^\mathrm{T} B^{-1}\big(e(k+1) - A_\mathrm{r} e(k)\big) v_\mathrm{d}^\mathrm{T}(k)}{1 + e^\mathrm{T}(k) P e(k)}, \quad k \in \mathbb{N} \tag{10}$$

with learning rate $\eta_0 \in \mathbb{R}_+$ and matrix $P \in \mathbb{R}_+^{n\times n}$ that satisfies the discrete Lyapunov equation $A_\mathrm{r}^\mathrm{T} P A_\mathrm{r} + R - P = 0$, with $R \in \mathbb{R}_+^{n\times n}$. In (10), $e(k) \in \mathbb{R}^n$ represents the tracking error between the discrete dynamic system states and the discrete reference model states defined as

$$e(k) \triangleq x(k) - x_\mathrm{r}(k),\ k \in \mathbb{N}. \tag{11}$$

Then, the tracking error dynamics yields

$$\begin{aligned} e(k+1) &= x(k+1) - x_\mathrm{r}(k+1) \\ &= Ax(k) + Bv_\mathrm{act}(k) - A_\mathrm{r} x_\mathrm{r}(k) - B_\mathrm{r} r(k) \\ &= Ax(k) + B\big(v_\mathrm{d}(k) - v_\mathrm{err}(k)\big) - A_\mathrm{r} x_\mathrm{r}(k) \\ &\quad - B_\mathrm{r} r(k) \\ &= A_\mathrm{r} e(k) + BM\Lambda\tilde{\theta}(k) v_\mathrm{d}(k). \end{aligned} \tag{12}$$

Next, from (10), the approximation error dynamics can be obtained as

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{\eta_0 M^\mathrm{T} B^{-1}\big(e(k+1) - A_\mathrm{r} e(k)\big) v_\mathrm{d}^\mathrm{T}(k)}{1 + e^\mathrm{T}(k) P e(k)}, \quad k \in \mathbb{N}. \tag{13}$$

Using (12), (13) can be mathematically expressed as

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) - \frac{\eta_0 M^\mathrm{T} M\Lambda\tilde{\theta}(k) v_\mathrm{d} v_\mathrm{d}^\mathrm{T}(k)}{1 + e^\mathrm{T}(k) P e(k)}, \quad k \in \mathbb{N}. \tag{14}$$

Note that the tracking error dynamics in (12) and approximation error dynamics in (14) will be used in the the stability analysis of this paper.

C. Stability Analysis

Theorem 1. *Consider the uncertain over-actuated discrete dynamical system with actuator degradation given by* (7) *and the discrete reference model given by* (8)*. The control input designed in* (9) *along with the adaptive update law* (10) *guarantees boundedness of the pair* $(e, \tilde{\theta}) \in \mathbb{R}^n \times \mathbb{R}^{l\times n}$*. Moreover,* $e(k) \longrightarrow 0$ *as* $k \longrightarrow \infty$*.*

Proof. The positive definite Lyapunov candidate function that consists of logarithmic and quadratic functions for proving stability of the closed-loop system error dynamics is chosen as

$$\begin{aligned} V(e,\tilde{\theta}) &= \ln\left[1 + e^\mathrm{T}(k) P e(k)\right] \\ &\quad + \xi\eta_0^{-1} \mathrm{tr}\left[\tilde{\theta}^\mathrm{T}(k) \Lambda^{\frac{1}{2}\mathrm{T}} \Lambda^{\frac{1}{2}} \tilde{\theta}(k)\right], \end{aligned} \tag{15}$$

where $\xi \in \mathbb{R}_+$. The corresponding Lyapunov difference equation is then given as

$$\begin{aligned} \Delta V(\cdot) &= \ln\big[1 + e^\mathrm{T}(k+1) P e(k+1)\big] - \ln\big[1 + e^\mathrm{T}(k) P e(k)\big] \\ &\quad + \xi\eta_0^{-1} \mathrm{tr}\Big[\tilde{\theta}^\mathrm{T}(k+1) \Lambda \tilde{\theta}(k+1)\Big] \\ &\quad - \xi\eta_0^{-1} \mathrm{tr}\Big[\tilde{\theta}^\mathrm{T}(k) \Lambda \tilde{\theta}(k)\Big]. \end{aligned} \tag{16}$$

The two logarithmic terms in (16) can be combined using the logarithmic property $\ln[a] - \ln[a] = \ln\left[\frac{a}{b}\right]$, using (12) and (13), using the trace property $\mathrm{tr}[ba^\mathrm{T}] = a^\mathrm{T} b$ where $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$, and using the logarithmic property $\ln[1+a] \le a$, an upper bound results in

$$\begin{aligned} \Delta V(\cdot) &\le \frac{e^\mathrm{T}(k) A_\mathrm{r}^\mathrm{T} P A_\mathrm{r} e(k) - e^\mathrm{T}(k) P e(k)}{1 + e^\mathrm{T}(k) P e(k)} \\ &\quad + \frac{2e^\mathrm{T}(k) A_\mathrm{r}^\mathrm{T} P B M \Lambda \tilde{\theta}(k) v_\mathrm{d}(k)}{1 + e^\mathrm{T}(k) P e(k)} \\ &\quad + \frac{v_\mathrm{d}^\mathrm{T}(k) \tilde{\theta}^\mathrm{T}(k) \Lambda M^\mathrm{T} B^\mathrm{T} P B M \Lambda \tilde{\theta}(k) v_\mathrm{d}(k)}{1 + e^\mathrm{T}(k) P e(k)} \\ &\quad - \frac{2\xi v_\mathrm{d}^\mathrm{T}(k) \tilde{\theta}^\mathrm{T}(k) \Lambda M^\mathrm{T} M \Lambda \tilde{\theta}(k) v_\mathrm{d}(k)}{1 + e^\mathrm{T}(k) P e(k)} \\ &\quad + \frac{\xi\eta_0 v_\mathrm{d}^\mathrm{T}(k) \tilde{\theta}^\mathrm{T}(k) \Lambda M^\mathrm{T} M \Lambda M^\mathrm{T} M \Lambda \tilde{\theta}(k)}{1 + e^\mathrm{T}(k) P e(k)} \\ &\quad \cdot \frac{v_\mathrm{d}(k) v_\mathrm{d}^\mathrm{T}(k) v_\mathrm{d}(k)}{1 + e^\mathrm{T}(k) P e(k)}. \end{aligned} \tag{17}$$

The first two terms in the numerator of the natural log in (17) can be expressed using the discrete Lyapunov equation defined before that is $A_\mathrm{r}^\mathrm{T} P A_\mathrm{r} - P = -R$. For sufficiently small values of γ, $R \ge Q + \gamma A_\mathrm{r}^\mathrm{T} P B B^\mathrm{T} P A_\mathrm{r}$ holds, where $Q \in \mathbb{R}_+^{n\times n}$ and $\gamma \in \mathbb{R}_+$, [27]. Adding and subtracting the term "$\gamma^{-1} v_\mathrm{d}^\mathrm{T}(k) \tilde{\theta}^\mathrm{T}(k) \Lambda M^\mathrm{T} M \Lambda \tilde{\theta}(k) v_\mathrm{d}(k)$" into (17) an upper bound for (17) can be written as

$$\begin{aligned} \Delta V(\cdot) &\le \frac{-e^\mathrm{T}(k) Q e(k) - z^\mathrm{T}(k) L z(k)}{1 + e^\mathrm{T}(k) P e(k)} \\ &\quad + \Big(\gamma^{-1} + \overline{\lambda}(B^\mathrm{T} P B) - \xi\big(2 - \eta_0 \overline{\lambda}(M M^\mathrm{T}) m^*\big)\Big) \end{aligned}$$

$$\cdot \frac{||M\Lambda\tilde{\theta}(k)v_{\rm d}(k)||^2}{1+e^{\rm T}(k)Pe(k)}.$$

with $z(k) \triangleq \begin{bmatrix} B^{\rm T}PA_{\rm r}e(k) \\ M\Lambda\tilde{\theta}(k)v_{\rm d}(k) \end{bmatrix}$ and $L \triangleq \begin{bmatrix} \gamma I_{\rm n} & -I_{\rm n} \\ -I_{\rm n} & \gamma^{-1}I_{\rm n} \end{bmatrix}$, where $z(k) \in \mathbb{R}^{2n}$ and $L \in \overline{\mathbb{R}}_+^{2n\times 2n}$ and m^* is an upper bound for $\frac{v_{\rm d}^{\rm T}(k)v_{\rm d}(k)}{1+e^{\rm T}Pe(k)}$ as given in Appendix A.

By defining $\xi \triangleq \frac{\gamma^{-1}+\overline{\lambda}(B^{\rm T}PB)}{2-\eta_0\overline{\lambda}(MM^{\rm T})m^*}$, the maximum bound for (16) is shown to be negative semi-definite, that is

$$\Delta V(\cdot) \leq \frac{-e^{\rm T}(k)Qe(k) - z^{\rm T}(k)Lz(k)}{1+e^{\rm T}(k)Pe(k)} \leq 0. \quad (18)$$

Note that in order for (15) to remain positive definite, ξ must be positive. This puts an upper bound on the learning rate, η_0, described as

$$\eta_0 < \frac{2}{\overline{\lambda}(MM^{\rm T})m^*}. \quad (19)$$

Here, (18) proves that the pair $(e(k), \tilde{\theta}(k))$ is Lyapunov stable. It can be shown from Theorem 13.10 of [29] that $e(k) \longrightarrow 0$ as $k \longrightarrow \infty$. Because of the asymptotic convergence of $e(k)$, if follows that system states, $x(k)$, approach the reference model states, $x_{\rm r}(k)$, as $k \longrightarrow \infty$. ■

IV. Illustrative Numerical Examples

A. Simulation Model

The discrete-time system used for simulating the adaptive architecture is the linearized state-space rotational equations for the hexacopter. Specifically, we considered

$$x(k+1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} 0.01 & 0 & 0 \\ 0 & 0.01 & 0 \\ 0 & 0 & 0.0071 \end{bmatrix} M\Lambda u(k), \quad (20)$$

where the discretization is obtained in 10 [Hz], the state vector, $x(k) = [p(k), q(k), r(k)]^{\rm T}$, consists of the hexacopter's angular rates about the three body-fixed axes and the control input vector, $u(k) = [\omega_1^2(k), \omega_2^2(k), \omega_3^2(k), \omega_4^2(k), \omega_5^2(k), \omega_6^2(k)]^{\rm T}$, consists of the squared motor speeds for the hexacopter's six motors. M is the motor mixing matrix, which represents the linear mapping from (1). Specifically,

$$M \triangleq \begin{bmatrix} 0 & \frac{\sqrt{3}}{2}lk_l & \frac{\sqrt{3}}{2}lk_l & 0 & -\frac{\sqrt{3}}{2}lk_l & -\frac{\sqrt{3}}{2}lk_l \\ lk_l & \frac{1}{2}lk_l & -\frac{1}{2}lk_l & -lk_l & -\frac{1}{2}lk_l & \frac{1}{2}lk_l \\ -k_d & k_d & -k_d & k_d & -k_d & k_d \end{bmatrix}. \quad (21)$$

In (21) k_l and k_d are the lift and drag coefficients of the vehicle's rotors and l is the hexacopter's arm length. The specific values for the parameters in (21) are summarized in Table I. Finally, Λ is the control effectiveness matrix described by (2). A discrete-time reference model, constructed from desired continuous-time stable eigenvalues, spec$(A_{\rm r}) = [-0.5 \pm 2i, -0.1]$ is used to obtain

$$x_{\rm r}(k+1) = \begin{bmatrix} 0.9323 & -0.1890 & 0 \\ 0.1890 & 0.9323 & 0 \\ 0 & 0 & 0.9900 \end{bmatrix} x_{\rm r}(k) + \begin{bmatrix} 0.0677 & 0.1890 & 0 \\ -0.1890 & 0.0677 & 0 \\ 0 & 0 & 0.0100 \end{bmatrix} r(k). \quad (22)$$

That results in the discrete feedback gains used in (9) are

$$K_1 = \begin{bmatrix} 6.7732 & 18.8980 & 0 \\ -18.8980 & 6.7732 & 0 \\ 0 & 0 & 1.3930 \end{bmatrix} \quad (23)$$

$$K_2 = \begin{bmatrix} 6.7732 & 18.8980 & 0 \\ -18.8980 & 6.7732 & 0 \\ 0 & 0 & 1.3930 \end{bmatrix}. \quad (24)$$

The reference command chosen is a constant angular rate, $r = [5\,\text{deg/s}, 5\,\text{deg/s}, 5\,\text{deg/s}]^{\rm T}$. All initial conditions are set to 0 and the diagonal entries of the unknown uncertainty matrix, Λ, is set to $[1, 0.8, 0.7, 0.9, 1, 0.9]$ indicating degradation of 20%, 30%, 10%, and 10% on motors 2, 3, 4, and 6, respectively.

TABLE I
HEXACOPTER PHYSICAL PARAMETERS

Parameter	Value	Units
I_{xx}	10	kg·m^2
I_{yy}	10	kg·m^2
I_{zz}	14	kg·m^2
k_l	8×10^{-5}	-
k_d	6×10^{-6}	-
l	1	m

B. Simulation Results

The system model described in Section IV-A is simulated to show the effectiveness of the adaptive architecture in the face of control effectiveness degradation. The learning rate, η_0, is set to 8.3585×10^5, which is its maximum bound set by (19). Figures 1, 2, and 3 show the results of the successful implementation of the discrete adaptive control architecture. The angular rates track the reference model trajectories which converge to the reference command, with minor overshoot. Convergence of the yaw rate takes longer due to the smaller magnitude of the real eigenvalue of $A_{\rm r}$.

Convergence of the estimated parameter, $\hat{\theta}$, happens faster than the reference tracking convergence. Note that due to the structure of M, the first three rows of $\hat{\theta}$ are the negative of the last three rows of $\hat{\theta}$ (see Figures 4 and 5).

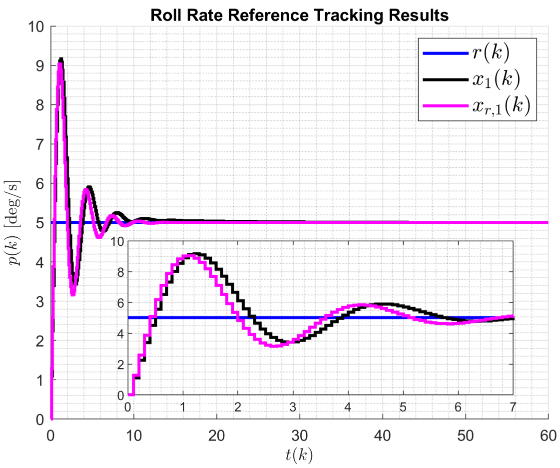

Fig. 1. Roll rate tracking with control effectiveness degradation.

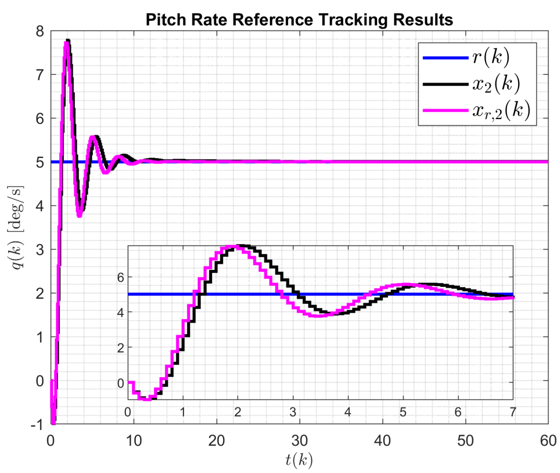

Fig. 2. Pitch rate tracking with control effectiveness degradation.

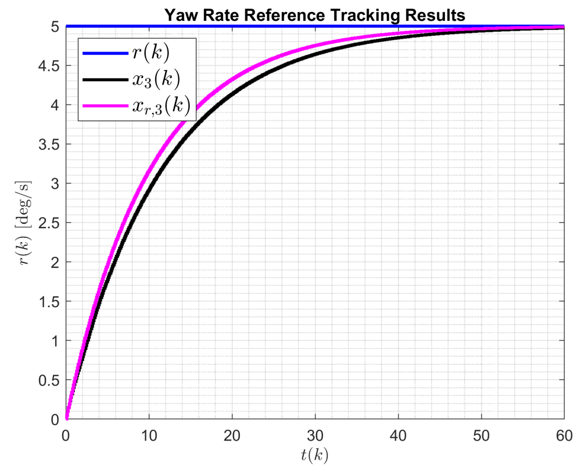

Fig. 3. Yaw rate tracking with control effectiveness degradation.

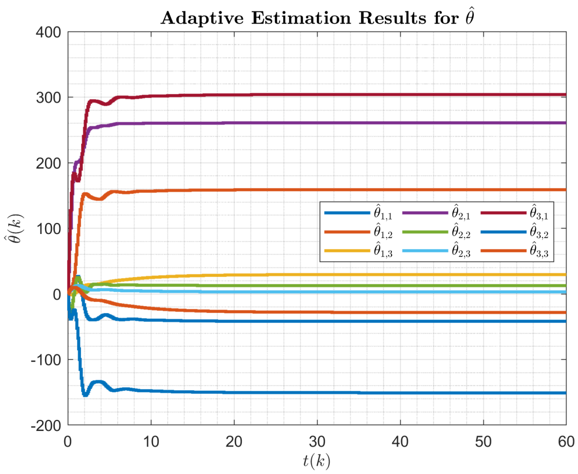

Fig. 4. Convergence of estimated parameter (first 3 rows).

V. Conclusions

This work presented a generalized adaptive control allocation architecture in a discrete setting for the presence of over-actuated systems with unknown control effectiveness degradation. The implemented architecture ensured a desired closed-loop stability and tracking performance. An adaptive update law was designed to deal with the unknown degradation. The Lyapunov analysis proved the asymptotic convergence of the tracking error. Simulation results for the attitude dynamics of a hexacopter system illustrated the performance of the adaptive architecture. In the future, this work can be extended by considering modeling uncertainties and redesigning the control architecture to deal with the modeling uncertainty as well and control effectiveness degradation.

VI. Appendix

The Lyapunov proof relies on the bound of $\frac{v_{\mathrm{d}}^{\mathrm{T}}(k)v_{\mathrm{d}}(k)}{1+e^{\mathrm{T}}(k)Pe(k)}$.

$$v_{\mathrm{d}}^{\mathrm{T}}(k)v_{\mathrm{d}}(k) = \left(-K_1x(k)+K_2r(k)\right)^{\mathrm{T}} \cdot \left(-K_1x(k)+K_2r(k)\right)$$

$$v_{\mathrm{d}}^{\mathrm{T}}(k)v_{\mathrm{d}}(k) = x^{\mathrm{T}}(k)K_1^{\mathrm{T}}K_1x(k) - 2x^{\mathrm{T}}(k)K_1^{\mathrm{T}}K_2r(k) + r^{\mathrm{T}}(k)K_2^{\mathrm{T}}K_2r(k). \tag{25}$$

Considering that each quadratic term in (25) can be upper bounded, an upper bound on (25) can be expressed as

$$v_{\mathrm{d}}^{\mathrm{T}}(k)v_{\mathrm{d}}(k) \leq ||K_1^{\mathrm{T}}K_1||\,||x(k)||^2 + 2||K_1^{\mathrm{T}}K_2||\,||x(k)||\,||r(k)|| + ||K_2^{\mathrm{T}}K_2||\,||r(k)||^2. \tag{26}$$

Using (11) with the triangle inequality, along with defining bounds on the reference states and reference command as $||x_{\mathrm{r}}(k)|| \leq x_{\mathrm{r}}^*$ and $||r(k)|| \leq r^*$ (26) can be rewritten as

$$\begin{aligned} v_{\mathrm{d}}^{\mathrm{T}}(k)v_{\mathrm{d}}(k) \leq\; & ||K_1^{\mathrm{T}}K_1||\,||e(k)||^2 \\ & +2\left(||K_1^{\mathrm{T}}K_1||x_{\mathrm{r}}^* + ||K_1^{\mathrm{T}}K_2||r^*\right)||e(k)|| \\ & +||K_1^{\mathrm{T}}K_1||x_{\mathrm{r}}^{*2} + ||K_2^{\mathrm{T}}K_2||r^{*2} \\ & +2x_{\mathrm{r}}^*r^*||K_1^{\mathrm{T}}K_2||. \end{aligned} \tag{27}$$

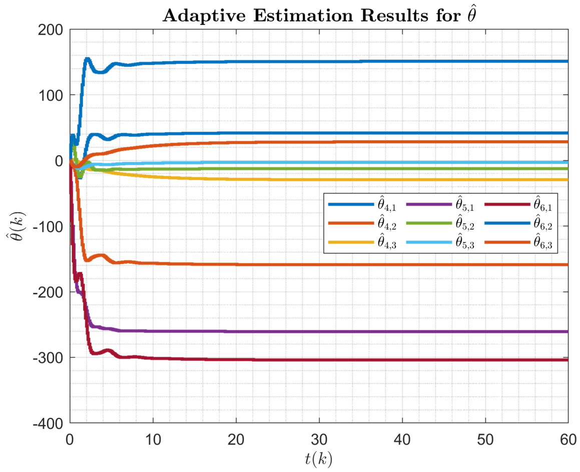

Fig. 5. Convergence of estimated parameter (last 3 rows).

Including the denominator, a further bound can be found by using minimum eigenvalue of P

$$\frac{v_{\mathrm{d}}^{\mathrm{T}}(k)v_{\mathrm{d}}(k)}{1+e^{\mathrm{T}}(k)Pe(k)} \leq \frac{||K_1^{\mathrm{T}}K_1||\,||e(k)||^2}{1+\underline{\lambda}(P)||e(k)||^2} + \frac{2\left(||K_1^{\mathrm{T}}K_1||x_{\mathrm{r}}^* + ||K_1^{\mathrm{T}}K_2||r^*\right)||e(k)||}{1+\underline{\lambda}(P)||e(k)||^2} + \frac{||K_1^{\mathrm{T}}K_1||x_{\mathrm{r}}^{*2} + ||K_2^{\mathrm{T}}K_2||r^{*2}}{1+\underline{\lambda}(P)||e(k)||^2} + \frac{2x_{\mathrm{r}}^* r^*||K_1^{\mathrm{T}}K_2||}{1+\underline{\lambda}(P)||e(k)||^2}. \tag{28}$$

Using the inequalities $\frac{ax^2}{1+bx^2} \leq \frac{a}{b}$, $\frac{ax}{1+bx^2} \leq \frac{a}{2\sqrt{b}}$, and $\frac{a}{1+bx^2} \leq a$, a final bound on (28) can be expressed as

$$\frac{v_{\mathrm{d}}^{\mathrm{T}}v_{\mathrm{d}}}{1+e^{\mathrm{T}}Pe} \leq \frac{||K_1^{\mathrm{T}}K_1||}{\underline{\lambda}(P)} + \frac{||K_1^{\mathrm{T}}K_1||x_{\mathrm{r}}^* + ||K_1^{T}K_2||r^*}{\sqrt{\underline{\lambda}(P)}} + ||K_1^{\mathrm{T}}K_1||x_{\mathrm{r}}^{*2} + ||K_2^{\mathrm{T}}K_2||r^{*2} + 2x_{\mathrm{r}}^* r^*||K_1^{\mathrm{T}}K_2|| \triangleq m^*. \tag{29}$$

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