2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Distributed Online Learning Algorithms for Aggregative Games Over Time-Varying Unbalanced Digraphs.

Xiaolong Zuo and Zhenhua Deng*

***Abstract*— In this paper, online aggregative games over time-varying unbalanced digraphs are studied, where the cost functions of players are time-varying and are gradually revealed to corresponding players only after decisions are made. Moreover, in the problems, players are subject to local convex set constraints and time-varying coupled nonlinear inequality constraints. To the best of our knowledge, no result about online games with unbalanced digraphs has been reported, let alone constrained online games. To solve the problem, a distributed online algorithm based on primal-dual, mirror descents and push-sum methods is developed. With the algorithm, sublinear dynamic regrets and constraint violations are established. Finally, online electricity market games illustrate the algorithm.**

I. INTRODUCTION

Aggregative games (AGs) appear in various fields, such as environmental economies [1], smart grids [2] and communication networks [3]. In AGs, each player aims to find an ideal decision to selfishly optimize its individual cost function, which relies on its own decision as well as the aggregate of the decisions of all players. Hence, generalized Nash Equilibrium (GNE) is a desirable decision for the players, which portrays the state that no player can benefit more by deviating from the equilibrium unilaterally (see [4], [5]).

To seek the (G)NE of AGs, numerous distributed strategies have been developed in recent years(see [6], [7], [8], [9]). However, these works typically assumed that the cost functions are fixed and time-invariant, which is not in line with most of engineering realities. For example, in many practical situations such as radio resource allocation [10], robust portfolio selection [11] and wind power grids [12], the environments are dynamic and evolving, which often lead to cost functions and constraint varying over time. Moreover, the cost functions and constraints only can be known to players after decisions are made. Accordingly, it is necessary to investigate AGs with time-varying and gradually revealing cost functions and constraints, which are called online AGs.

The communication networks are required to be static and/or undirected in most of existing works for online games (see [13], [14]). Nevertheless, in many situations, the communication among players are time-varying and unbalanced, such as transportation networks [15] and social networks [16]. Moreover, it is well known that undirected graphs have limitations compared to unbalanced digraphs which have wider applications and are easier to implement [17]. Besides, static and undirected graphs are special cases of time-varying unbalanced digraphs. As far as we know, there are no results about online AGs over unbalanced digraphs so far, let alone time-varying unbalanced digraphs.

X. Zuo and Z. Deng are with the School of Automation, Central South University, Changsha, 410075, China zhdeng@amss.ac.cn

This paper studies online AGs over time-varying networks and design a distributed online algorithm to seek the GNE sequences. The main contributions are outlined below:

- We study online AGs with local convex set constraints and coupled nonlinear inequality constraint over time-varying unbalanced digraphs. Unlike the traditional static AGs, such as [6], [7], [8], [9], the cost functions and coupled constraints are time-varying and unknown before decisions are made in our problems. Besides, our problems are extensions of existing online games, such as [13], [14], by considering time-varying unbalanced digraphs and time-varying coupled inequality constraints.
- We design a distributed online learning algorithm for AGs over time-varying unbalanced digraphs based on mirror descent and push-sum methods. It is not easy to develop online algorithms with sublinear dynamic regrets (see [18] and references therein). We prove that under our algorithm, the decisions made by players are no-regret and the dynamic regrets and constraint violations are sublinear.

The rest of the paper is organized as follows. Section II recalls some preliminary knowledge and introduces problem formulation. Section III develops a distributed online learning algorithm and presents the main results. Section IV gives a simulation example. Section V summarizes the conclusion.

II. PRELIMINARIES AND FORMULATION

This section introduces some preliminary knowledge and presents the problem.

A. Preliminaries

a) Notations: $\mathbb{R}$ is the set of real numbers. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\otimes$ and $\times$ denote the Kronecker product and the Cartesian product, respectively. $\|x\|$ is the standard Euclidean norm of vector. $\|X\|$ is the spectral norm of matrix X. $col(x_1,\dots,x_n)=[x_1^T,\dots,x_n^T]^T$. x_i is the ith element of vector x. 1_n is the column vectors of n ones. I_n is an $n\times n$ identity matrix. For a vector or matrix M, the transpose of M is denoted by M^T. Communication graph between agents at time t is denoted by $\mathscr{G}_t=\{\mathscr{V},\mathscr{E}_t\}$, where $\mathscr{V}$ is the vertex set of $\mathscr{G}_t$ and $\mathscr{E}_t$ is the edge set. The adjacency matrix is A_t and $[A_t]_{ij}$ denotes the weighting of (i,j) at time t. For a vector $v\in\mathbb{R}^n$, $[v]_+=max(v,0)$. $h_1=\mathscr{O}(h_2)$ means that h_1 grow sublinearly with respect to h_2.

b) Bregman Divergence: The Bregman divergence of x, y with $x, y \in \Omega \subset \mathbb{R}^n$ is defined as (see [19])

$$D_\varphi(x,y) := \varphi(x) - \varphi(y) - \langle \nabla\varphi(y), x-y \rangle,$$

where $\varphi : \Omega \to \mathbb{R}$ is differentiable and δ-strongly convex distance-measuring function, i.e., $\varphi(x) \geq \varphi(y) + \langle \nabla\varphi(y), x-y \rangle + \frac{\delta}{2}\|x-y\|^2$. Therefore, it can be easily derived that $D_\varphi(x,y) \geq \frac{\delta}{2}\|x-y\|^2$ and

$$\langle x-y, \nabla\varphi(y) - \nabla\varphi(z) \rangle = D_\varphi(x,z) - D_\varphi(x,y) - D_\varphi(y,z). \quad (1)$$

A mild assumption on the Bregman divergence are listed as follows, which were widely used in [19], [20].

Assumption 1: The Bregman divergence satisfies the following conditions:

$$D_\varphi(x, \sum_{i=1}^{N} a_i y_i) \leq \sum_{i=1}^{N} a_i D_\varphi(x, y_i)$$
,
$$\|D_\varphi(x,z) - D_\varphi(y,z)\| \leq l_D \|x-y\|$$

where $x, y_i, z \in \mathbb{R}^n$ and $\sum_{i=1}^{N} a_i = 1$ with $a_i \in \mathbb{R}$ and l_D is a positive constant.

B. Problem Formulation

Consider an online AG of N players over time-varying unbalanced digraphs. At each time $t \in \{1, \ldots, T\}$, where $T \in \mathbb{N}_+$ denotes the running time, only after player i has made a decision x_t^i from its decision set $\Omega_i \subset \mathbb{R}^n$, it receives its cost function $f_t^i(x_t^i, x_t^{-i})$, where $x_t^{-i} = col(x_t^1, \ldots, x_t^{i-1}, x_t^{i+1}, \ldots, x_t^N)$ denotes the decisions of other players. Moreover, the decisions of players are coupled by the following time-varying nonlinear inequality:

$$K_t = \{x_t \in \mathbb{R}^{Nn} | \sum_{i=1}^{N} g_t^i(x_t^i) \leq 0_m\} \quad (2)$$

where $g_t^i = col(g_t^{i1}, \ldots, g_t^{im})$ and $g_t^{ij} : \mathbb{R}^n \to \mathbb{R}$. Specially, since the cost functions in AGs depend on the aggregate of the decisions of all players, the function $f_t^i(x_t^i, x_t^{-i})$ could be rewritten as $J_t^i(x_t^i, \sigma(x_t))$, where the aggregative function $\sigma(x_t)$ is defined as $\sigma(x_t) = \frac{1}{N}\sum_{i=1}^{N} \psi_i(x_t^i)$. $\psi_i(x_t^i) : \mathbb{R}^n \to \mathbb{R}^d$ is differentiable and Lipschitz continuous, i.e., there exists a positive constant l_ψ such that $||\psi_i(x_t^i) - \psi_i(y_t^i)|| \leq l_\psi ||x_t^i - y_t^i||$ for any x_t^i, $y_t^i \in \mathbb{R}^n$. Accordingly, for player i, it faces the following aggregative game at time t:

$$\begin{aligned} \min_{x_t^i \in \Omega_i} \ & J_t^i(x_t^i, \sigma(x_t)) \\ s.t. \ & \sum_{i=1}^{N} g_t^i(x_t^i) \leq 0_m. \end{aligned} \quad (3)$$

The GNE of the AG (3) is defined as follows (see [5]):

Definition 1: A decision $x_t^* = (x_t^{i*}, x_t^{-i*})$ is a GNE of the AG (3) at time $t \in \{1, 2, \ldots, T\}$ if $f_t^i(x_t^{i*}, x_t^{-i*}) \leq f_t^i(x_t^i, x_t^{-i*})$, $\forall x_t : (x_t^i, x_t^{-i*}) \in \Omega \cap K_t, i \in \mathscr{V}$, where $\Omega = \Omega_1 \times \ldots \times \Omega_N$.

Similarly to [13], [14], [19], [20], [21], the path length of the GNE sequence is defined as $V_T = \sum_{t=1}^{T} \|x_{t+1}^* - x_t^*\|$.

In the following, some widely used assumptions in online games are listed.

Assumption 2: The nonempty set Ω_i is convex and compact.

Assumption 3: The cost functions and coupled constraint satisfy: (i) The function $f_t^i(x_t^i, x_t^{-i})$ and $g_t^i(x_t^i)$ are convex and differentiable on x_t^i. (ii) The gradient $\nabla J_t^i(x_t^i, \sigma(x_t))$ is Lipschitz continuous, i.e., $\|\nabla J_t^i(x_t^i, \hat{\sigma}_{t+1}^i) - \nabla J_t^i(x_t^i, \sigma(x_t))\| \leq l\|\hat{\sigma}_{t+1}^i - \sigma(x_t)\|$, where l is a positive constant and

$$\nabla J_t^i(x_t^i, \sigma(x_t)) := \left(\frac{\partial}{\partial \mu} J_i(\mu, \nu) + \frac{\partial}{\partial \nu} J_i(\mu, \nu) \frac{\partial \nu}{\partial x_t^i} \right)\Bigg|_{\substack{\mu = x_t^i \\ \nu = \sigma(x_t)}}.$$

(iii)The pseudo-gradient $F_t(x_t)$ is μ-strong monotone, i.e., $(F_t(x_t) - F_t(y_t))^T(x_t - y_t) \geq \mu\|x-y\|^2$, where $F_t(x_t)$ is defined as $F_t(x_t) = col(\nabla f_t^i(x_t^1, x_t^{-1}), \ldots, \nabla f_t^N(x_t^N, x_t^{-N}))$ with $\nabla f_t^i(x_t^i, x_t^{-i}) = \frac{\partial f_t^i(x_t^i, x_t^{-i})}{\partial x_t^i}$.

Assumption 4: The graph $\mathscr{G}_t$, $\forall t \in \{1, \ldots, T\}$, satisfies the following conditions: (i) (Weight Rule) $[A_t]_{ij} \geq \varpi$ with $\varpi > 0$, if $(i,j) \in \mathscr{E}_t$; $[A_t]_{ii} \geq \varpi$, $\forall i \in \mathscr{V}$; (ii) (Column-stochasticity) $\sum_{i=1}^{N} [A_t]_{ij} = 1$; (iii) (Uniformly connectivity) There exists an integer $B > 0$ such that the jointly graph $\{\mathscr{V}, \cup_{k=0,\ldots,B-1} \mathscr{E}_{t+k}\}$ is strongly connected.

The following lemma holds based on Assumption 4.

Lemma 1: (see [22, Corollary 2]) Suppose Assumption 4 holds, there exists a sequence $\{\phi_t\}$ of stochastic vectors $\phi_t \in \mathbb{R}^n$, $\theta > 0$ and $\eta \in (0,1)$, such that $\left|[A(t:k)]_{ij} - \phi_t^i\right| \leq \theta\eta^{t-k}$, where $A(t:k) = A_t A_{t-1} \ldots A_{k+1} A_k$.

The dynamic regret and constraint violation are widely used to evaluate the performance of online decision-making process (see [13], [14], [20], [21]) and are defined as follows:

$$Reg_i(T) = \sum_{t=1}^{T} (f_t^i(x_t^i, x_t^{-i*}) - f_t^i(x_t^{i*}, x_t^{-i*})) \quad (4)$$

$$Rg(T) = \left\| \left[\sum_{t=1}^{T} \sum_{i=1}^{N} g_t^i(x_t^i) \right]^+ \right\|. \quad (5)$$

where $x_t^* = (x_t^{i*}, x_t^{-i*})$ is the GNE of AG (3) at time t.

The sublinear dynamic regret and constraint violation are expected to established in online problems because they implies, in the terms of time average, the errors of the algorithm approach zero and the coupled constraint is largely satisfied, i.e., $\lim_{T\to\infty} Reg_i(T)/T = 0$ and $\lim_{T\to\infty} Rg(T)/T = 0$.

Therefore, the goal of this paper is to design an algorithm for the AG (3) over time-varying graphs $\{\mathscr{G}_t\}$ such that sublinear dynamic regret (4) and constraint violation (5) are obtained.

III. MAIN RESULTS

This section presents a distributed online algorithm (i.e., Algorithm 1) for the AG (3) and analyzes its performance.

A. Distributed Online Learning Algorithm Design

This subsection presents a distributed online algorithm (i.e., Algorithm 1).

Assumption 2 indicates that the radius of the convex set Ω is bounded, i.e., there exists positive constant r such that $\|x-y\| \leq r$, $\forall x, y \in \Omega$, which together with Assumption 3 indicates $J_t^i(x_t^i, \sigma(x_t))$ and $\nabla J_t^i(x_t^i, \sigma(x_t))$ are bounded because $J_t^i(x_t^i, \sigma(x_t))$ is convex and $x_t^i \in \Omega_i$ always holds under Algorithm 1, i.e., there exist positive constants B_g, C_g, B_j and C_j such that $\|g_t^i(x_t^i)\| \leq B_g$, $\|\nabla g_t^i(x_t^i)\| \leq C_g$, $\|J_t^i(x_t^i, \sigma(x_t))\| \leq B_j$ and $\|\nabla J_t^i(x_t^i, \sigma(x_t))\| \leq C_j$. Furthermore, since $g_t^i(x_t^i) + \nabla g_t^i(x_t^i)(z_{t+1}^i - x_t^i) \leq g_t^i(z_{t+1}^i)$, by (6e) and $\|g_t^i(x_t^i)\| \leq B_g$, we have $\|y_{t+1}^i\| \le B_g$. Besides, based on Assumption 1, it can be deduced that $D_\varphi(x,y) \le l_D r$, $\forall x,y \in \Omega$.

Algorithm 1 Distributed Online Push-sum Mirror Descent Algorithm

Initialization: z_0^i, $x_0^i \in \mathbb{R}^n$, $\omega_0 = 1_N$, $u_0^i = 0_m$, $s_0^i = \psi_i(x_0^i)$.

for $t = 0,\dots,T-1$ **do**

$$\tilde{s}_{t+1}^i = \sum_{k=1}^N [A_t]_{ik} s_t^k \tag{6a}$$

$$w_{t+1}^i = \sum_{k=1}^N [A_t]_{ik} w_t^k \tag{6b}$$

$$\hat{u}_{t+1}^i = \sum_{k=1}^N [A_t]_{ik} u_t^k \tag{6c}$$

$$z_{t+1}^i = \arg\min_{x\in\Omega_i} \{\alpha_t \langle \nabla J_t^i(x_t^i, \hat{\sigma}_{t+1}^i) + \gamma_t \nabla g_t^i(x_t^i)^T \tilde{u}_{t+1}^i, x\rangle + D_{\varphi_i}(x, x_t^i)\} \tag{6d}$$

$$y_{t+1}^i = g_t^i(x_t^i) + \nabla g_t^i(x_t^i)(z_{t+1}^i - x_t^i) \tag{6e}$$

$$u_{t+1}^i = [(1-\gamma_t^2)\sum_{k=1}^N [A_t]_{ik} u_t^k + \gamma_t y_{t+1}^i]^+ \tag{6f}$$

$$x_{t+1}^i = (1-\alpha_t) x_t^i + \alpha_t z_{t+1}^i \tag{6g}$$

$$s_{t+1}^i = \sum_{k=1}^N [A_t]_{ik} s_t^k + \psi_i(x_{t+1}^i) - \psi_i(x_t^i) \tag{6h}$$

end for

where $\hat{\sigma}_{t+1}^i = \frac{\tilde{s}_{t+1}^i}{w_{t+1}^i}$, $\tilde{u}_{t+1}^i = \frac{\hat{u}_{t+1}^i}{w_{t+1}^i}$, and $\alpha_t, \gamma_t \in (0,1)$ are non-increasing stepsizes.

B. Performance Analysis

In this subsection, we analyze the bounds of dynamic regret (4) and constraint violation (5) of Algorithm 1. Before stating our main results, some necessary lemmas are given as follows.

Lemma 2: (see [21, Lemma 4]) Under Assumptions 2 and 4, $\forall i \in \mathcal{V}, t \in \{1,\dots,T\}$, it can be obtained that $\delta \le \omega_t^i \le N$, $\delta \le 1$ and $\|\hat{u}_{t+1}^i\| \le \frac{\omega_{t+1}^i B_g}{\gamma_t \delta^2}$.

Lemma 3: Under Assumptions 2 and 4, $\forall i \in \mathcal{V}$, $t \in \{1,\dots,T\}$, we have

$$\|\hat{\sigma}_{t+1}^i - \sigma(x_t)\| \le \frac{2\theta}{\delta}\eta^t \|s_0\| + \frac{2\theta\sqrt{N} r l_\psi}{\delta}\sum_{k=1}^t \eta^{t-k}\alpha_{k-1}$$

$$\|\tilde{u}_{t+1}^i - \bar{u}_t\| \le \frac{4\theta\sqrt{N} N B_g}{\delta^3}\sum_{k=1}^t \eta^{t-k}\gamma_{k-1}.$$

where $\delta := \inf_{t=0,1,\dots}(\min_{1\le i\le N}[A_t A_{t-1}\dots A_0 1_N]_i) \ge \frac{1}{N^{NB}}$, $s_0 = col(s_0^1,\dots,s_0^N)$ and $\bar{u}_t = \sum_{i=1}^N u_t^i$.

Proof: Denote $\xi_{t+1}^i = \psi_i(x_{t+1}^i) - \psi_i(x_t^i)$. Under Assumption 2, by (6g) and the Lipschitz continuity of $\psi_i(x_t^i)$, we have $\|\xi_{t+1}^i\| \le l_\psi \|x_{t+1}^i - x_t^i\| \le l_\psi \alpha_t r$. Similarly, denote $\varepsilon_{t+1}^i = [(1-\gamma_t^2)\hat{u}_{t+1}^i + \gamma_t y_{t+1}^i]^+ - \hat{u}_{t+1}^i$, based on $\|y_{t+1}^i\| \le B_g$, Lemma 2 in [4] and Lemma 2, we have $\varepsilon_{t+1}^i \le \frac{2\gamma_t \omega_{t+1}^i B_g}{\delta^2}$. For easy of notation, we consider $n=1$ and $m=1$. Then the compact form of s_{t+1}^i and u_{t+1}^i are $s_{t+1} = A_t s_t + \xi_{t+1}$ and $u_{t+1} = A_t u_t + \varepsilon_{t+1}$, respectively, where $s_t = col(s_t^1,\dots,s_t^N)$, $u_t = col(u_t^1,\dots,u_t^N)$, $\xi_t = col(\xi_t^1,\dots,\xi_t^N)$ and $\varepsilon_t = col(\varepsilon_t^1,\dots,\varepsilon_t^N)$. Hence, we can deduced that $\bar{s}_t = \sigma(x_t)$, where $\bar{s}_t = \frac{1}{N}\sum_{t=1}^N s_t^i$(see [23, Lemma 1]). Then Lemma 3 can be obtained based on Lemmas 1 and the detailed proof is given in [22]. ∎

Lemma 4: Under Assumptions 2-4, for $\forall u \in \mathbb{R}_+^m$, we have

$$\begin{aligned}
&-\sum_{t=1}^T \gamma_t \sum_{i=1}^N \langle \tilde{u}_{t+1}^i, y_{t+1}^i\rangle \\
\le& \frac{1}{2}N(1+\sum_{t=1}^T \gamma_t^2)\|u\|^2 - u^T\sum_{t=1}^T\gamma_t\sum_{i=1}^N g_t^i(x_t^i) + NC_g r\|u\|\sum_{t=1}^T\gamma_t \\
&+ \frac{6N^3B_g^2}{\delta^5}\sum_{t=1}^T\gamma_t^2 + \frac{2NB_g}{\delta^2}\sum_{t=1}^T\gamma_t\sum_{i=1}^N\|\tilde{u}_{t+1}^i - \bar{u}_t\|.
\end{aligned} \tag{7}$$

Proof: See Appendix V-A. ∎

Lemma 5: Under Assumptions 1-4, there holds

$$\begin{aligned}
&\mu\sum_{t=1}^T\|x_t - x_t^*\|^2 \\
\le& lr\sum_{t=1}^T\sum_{i=1}^N\|\hat{\sigma}_{t+1}^i - \sigma(x_t)\| + B_g\sum_{t=1}^T\gamma_t\sum_{i=1}^N\|\tilde{u}_{t+1}^i - \bar{u}_t\| \\
&-\frac{\delta}{4}\|z_{t+1} - x_t\|^2 + NC_g B_u r\sum_{t=1}^T\gamma_t + \frac{NC_j^2}{2\delta}\sum_{t=1}^T\alpha_t \\
&-\sum_{t=1}^T\gamma_t\sum_{i=1}^N\langle\tilde{u}_{t+1}^i, y_{t+1}^i\rangle + \frac{Nl_D r}{\alpha_T^2} + \frac{\sqrt{N}l_D}{\alpha_T^2}V_T.
\end{aligned}$$

Proof: See Appendix V-B. ∎

Based on above analysis, we have the following theorem.

Theorem 1: Suppose Assumptions 1-4 hold. Under Algorithm 1, the dynamic regret and constraint violation satisfy

$$Reg_i(T) \le \mathcal{O}(\sqrt{T(\frac{V_T+1}{\alpha_T^2} + \sum_{t=1}^T\alpha_t + \sum_{t=1}^T\gamma_t)}) \tag{8}$$

$$Rg(T) \le \mathcal{O}(\frac{1}{\gamma_T}\sqrt{(\frac{V_T+1}{\alpha_T^2} + \sum_{t=1}^T\alpha_t + \sum_{t=1}^T\gamma_t)(1+\sum_{t=1}^T\gamma_t^2)}). \tag{9}$$

Proof: By the convexity of the cost function, it is apparent that

$$\begin{aligned}
Reg_i(T) \le& \sum_{t=1}^T \nabla f_t^i(x_t^i, x_t^{-i*})^T(x_t^i - x_t^{i*}) \\
\le& \sum_{t=1}^T C_j\|x_t^i - x_t^{i*}\| \le C_j\sqrt{T\sum_{t=1}^T\|x_t - x_t^*\|^2}.
\end{aligned} \tag{10}$$

Based on Lemma 3, we have

$$\begin{aligned}
&\sum_{t=1}^T\sum_{i=1}^N\|\hat{\sigma}_{t+1}^i - \sigma(x_t)\| \\
\le& \frac{2\theta}{\delta}\|s_0\|\frac{\eta}{1-\eta} + \frac{2\theta\sqrt{N}rl_\psi}{\delta}\sum_{t=1}^T\sum_{k=1}^t\eta^{t-k}\alpha_{k-1}
\end{aligned} \tag{11}$$

where $\sum_{t=1}^T\sum_{k=1}^t\eta^{t-k}\alpha_{k-1} = \sum_{k=1}^T\alpha_{k-1}\sum_{t=0}^{T-k}\eta^t \le \mathcal{O}(\sum_{t=1}^T\alpha_t)$. Therefore, it can be derived that

$\sum_{t=1}^{T}\sum_{i=1}^{N}\|\hat{\sigma}_{t+1}^{i}-\sigma(x_t)\| \leq \mathscr{O}(\sum_{t=1}^{T}\alpha_t)$. Similarly, we have $\sum_{t=1}^{T}\sum_{i=1}^{N}\|\tilde{u}_{t+1}^{i}-\bar{u}_t\| \leq \mathscr{O}(\sum_{t=1}^{T}\gamma_t)$.

Based on above analysis, it can be deduced from Lemmas 4 and 5 that

$$\mu\sum_{t=1}^{T}\|x_t-x_t^*\|^2-\sum_{t=1}^{T}E(t)\|u\|^2+u^T\sum_{t=0}^{T}\gamma_t\sum_{i=1}^{N}g_t^i(x_t^i)$$
$$\leq\mathscr{O}(\frac{V_T+1}{\alpha_T^2}+\sum_{t=1}^{T}\alpha_t+\sum_{t=1}^{T}\gamma_t) \quad (12)$$

where $E(t)=\frac{N}{2\delta}(\delta+\delta\sum_{t=1}^{T}\gamma_t^2+2NC_g^2\sum_{t=1}^{T}\gamma_t^2)$.

Due to the arbitrariness of $u\in\mathbb{R}_+^m$, let $u=0_m$, we have

$$\mu\sum_{t=1}^{T}\|x_t-x_t^*\|^2\leq\mathscr{O}(\frac{1+V_T}{\alpha_T^2}+\sum_{t=1}^{T}\alpha_t+\sum_{t=1}^{T}\gamma_t). \quad (13)$$

Then (8) is obtained by substituting (13) into (10).

Let $u=\frac{[\sum_{t=1}^{T}\gamma_t\sum_{i=1}^{N}g_t^i(x_t^i)]_+}{E(t)}$, we have

$$u^T\sum_{t=1}^{T}\gamma_t\sum_{i=1}^{N}g_t^i(x_t^i)-E(t)\|u\|^2=\frac{[\sum_{t=1}^{T}\gamma_t\sum_{i=1}^{N}g_t^i(x_t^i)]_+^2}{E(t)}\geq\frac{\gamma_T^2Rg^2}{E(t)}$$

which, together with (12), results in (9).

This completes the proof of Theorem 1. ■

Remark 1: Theorem 1 indicates that the bound of the dynamic regret and constraint violation depend on the algorithm parameter α_t, γ_t and V_T. Thus, provided that V_T satisfies certain conditions, sublinear dynamic regret and constraint violation can be obtained by choosing appropriate α_t and γ_t.

Corollary 1: Suppose Assumptions 1-4 hold. Under Algorithm 1 with $\alpha_t=\gamma_t=\frac{1}{t^a}$ and $a\in(0,\frac{1}{2})$, the dynamic regret (8) and constraint violation (9) are bounded by

$$Reg_i(T)\leq\mathscr{O}\left(T^{\frac{1}{2}+a}\sqrt{V_T}+T^{1-\frac{a}{2}}\right)$$
$$Rg(T)\leq\mathscr{O}\left(T^{\frac{1}{2}+a}\sqrt{V_T}+T^{1-\frac{a}{2}}\right)$$

Proof: By choosing $\alpha_t=\frac{1}{t^a}$, we have

$$\sum_{t=1}^{T}\alpha_t=\sum_{t=1}^{T}\frac{1}{t^a}\leq\int_0^T t^{-a}dt\leq\mathscr{O}(T^{1-a}). \quad (14)$$

Similarly, we have $\sum_{t=1}^{T}\gamma_t\leq\mathscr{O}(T^{1-a})$ and $\sum_{t=1}^{T}\gamma_t^2\leq\mathscr{O}(T^{1-2a})$. Then based on Theorem 1, the Corollary 1 can be proved. ■

Remark 2: In Corollary 1, it is clearly that $T^{\frac{1}{2}+a}$ and $T^{1-\frac{a}{2}}$ are sublinear with T because $a\in(0,\frac{1}{2})$. Hence, the sublinear dynamic regret and constraint violation can be established, i.e., $\lim_{T\to\infty}Reg_i(T)/T=0$ and $\lim_{T\to\infty}Rg(T)/T=0$ if V_T satisfies $\lim_{T\to\infty}V_T/T^{1-2a}=0$.

IV. SIMULATION

Consider the online electricity market games of six wind farms over time-varying unbalanced communication networks (e.g., Fig. 1). Every wind farm faces the following problem (see [24]).

$$\begin{aligned}\min_{P_{i,t}\in\mathbb{R}}\quad & J_{i,t}(P_{i,t},P_{-i,t})\\ s.t.\quad & P_{i,t}\in[P_{G_i}^{min},P_{G_i}^{max}]\\ & \sum_{i=1}^{6}P_{i,t}\geq\sum_{i=1}^{6}P_{D_{i,t}}.\end{aligned} \quad (15)$$

where $J_{i,t}$ is the cost function of wind farm i at time t, in $M\$$; $P_{i,t}$ is the generated power of wind farm i at time t, in MW; The load demand of wind farm i at time t is denoted by $P_{D_{i,t}}$ with $P_{D_t}=[12,11+0.5sin(t/4),8,7.5,9,10]^T$. The generated power $P_{i,t}$ cannot exceed the corresponding minimum output power $P_{G_i}^{min}$ and maximum output power $P_{G_i}^{max}$ ($P_G^{min}=[2,3,3,6,5,4]$ and $P_G^{max}=[65,58,73,35,66,45]$). Moreover, the sum of the generated power is expected to equal or greater than the total load demand. Particularly, the cost function $J_{i,t}$ is $J_{i,t}(P_{i,t},P_{-i,t})=c_{i,t}(P_{i,t})-p_{i,t}(P_{i,t},P_{-i,t})P_{i,t}$, where $c_{i,t}(P_{i,t})$ and $p_{i,t}(P_{i,t},P_{-i,t})$ are the production cost and the electricity price, respectively. $p_{i,t}(P_{i,t},P_{-i,t})=\iota_{i,t}-\frac{1}{6}\sum_{k=1}^{6}P_{k,t}$ with $\iota_{i,t}=80+0.5sin(t/12)$ and $c_{i,t}(P_{i,t})$ is $c_{i,t}(P_{i,t})=\tilde{\alpha}_{i,t}+\tilde{\beta}_{i,t}P_{i,t}+\tilde{\gamma}_{i,t}(P_{i,t})^2$, where $\tilde{\alpha}_t=[2,5+0.5sin(t/12),4,3,7,2]$, $\tilde{\beta}_t=[35+3.0sin(t/3),20+1.5sin(t/4),52+2.0sin(t/7),38+3.5sin(t/8),15+2.5sin(t/10),47+4.0sin(t/11)]$ and $\tilde{\gamma}_t=[2,4,3,1.5,3+0.5cos(t/3),2]$.

In Algorithm 1, we choose $\varphi(x)=\frac{1}{2}\|x\|^2$ and the corresponding Bregman divergence is $D(x,y)=\frac{1}{2}\|x-y\|^2$. Setting $\alpha_t=1/\sqrt[4]{t}$ and $\beta_t=1/\sqrt[4]{t}$. The simulation results of average dynamic regret Reg_i/t and average constraint violation Rg/t are presented in Fig. 2 and Fig. 3, respectively.

It can be observed from Figs. 2 and 3 that both Reg_i/t and Rg/t approach zero as time tends to infinity, which implies the dynamic regret and constraint violation are sublinear with T, thus verifying the effectiveness of the algorithm.

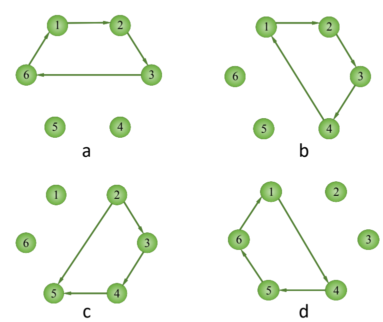

Fig. 1. The time-varying unbalanced communication networks

V. CONCLUSIONS

This paper has investigated constrained online AGs over time-varying unbalanced digraphs. To seek the GNE sequence of the online AG, a distributed online algorithm based on push-sum and mirror descent approach has been proposed. To the best of our knowledge, this is the first work to solve online AGs over time-varying unbalanced digraphs. With the algorithm, sublinear dynamic regret and constraint violation have been established. Finally, simulation results of online market games verify the effectiveness of our method.

APPENDIX

A. Proof of Lemma 4

Multiplying both sides of $u_{t+1}=A_tu_t+\varepsilon_{t+1}$ by $\frac{1}{N}1_N^T$, because A_t is column stochastic, we have $\bar{u}_{t+1}=\bar{u}_t+$

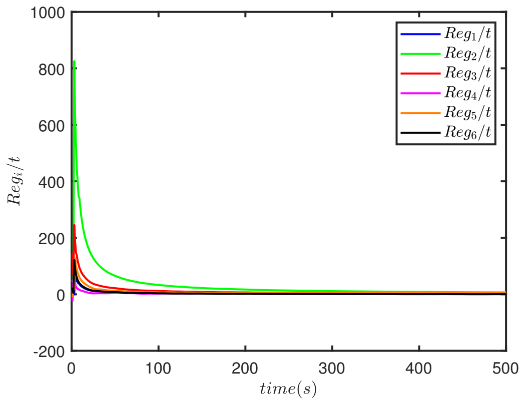

Fig. 2. The trajectories of Reg_i/t

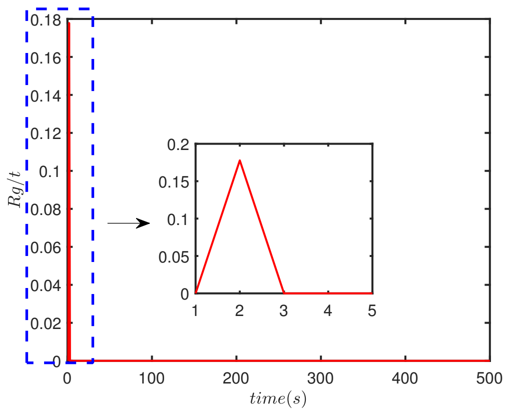

Fig. 3. The trajectories of Rg/t

$\frac{1}{N}\sum_{i=1}^{N}\varepsilon_{t+1}^{i}$. Therefore, for $\forall u \in \mathbb{R}_{+}^{m}$,

$$\begin{aligned}&\|\bar{u}_{t+1}-u\|^2\\ \leq&\|\bar{u}_t-u\|^2+\frac{1}{N}\sum_{i=1}^{N}\|\varepsilon_{t+1}^{i}\|^2+\frac{2}{N}\sum_{i=1}^{N}\varepsilon_{t+1}^{iT}(\bar{u}_t-u).\end{aligned}\tag{16}$$

For the term $\varepsilon_{t+1}^{iT}(\bar{u}-u)$, we have

$$\begin{aligned}&\varepsilon_{t+1}^{iT}(\bar{u}_t-u)\\ =&\varepsilon_{t+1}^{iT}(\bar{u}_t-\tilde{u}_{t+1}^{i})+\varepsilon_{t+1}^{iT}(\tilde{u}_{t+1}^{i}-u)\\ \leq&\|\varepsilon_{t+1}^{i}\|\|\tilde{u}_{t+1}^{i}-\bar{u}_t\|+\gamma_t(y_{t+1}^{i}-\gamma_t\hat{u}_{t+1}^{i})^T(\tilde{u}_{t+1}^{i}-u)\\ &+\frac{1}{\omega_{t+1}^{i}}(\varepsilon_{t+1}^{i}-\gamma_t(y_{t+1}^{i}-\gamma_t\hat{u}_{t+1}^{i}))^T(\hat{u}_{t+1}^{i}-u_{t+1}^{i})\\ &+\frac{1}{\omega_{t+1}^{i}}(\varepsilon_{t+1}^{i}-\gamma_t(y_{t+1}^{i}-\gamma_t\hat{u}_{t+1}^{i}))^T(u_{t+1}^{i}-\omega_{t+1}^{i}u)\\ \leq&\|\varepsilon_{t+1}^{i}\|\|\tilde{u}_{t+1}^{i}-\bar{u}_t\|+\gamma_t\langle y_{t+1}^{i},\tilde{u}_{t+1}^{i}-u\rangle\\ &-\gamma_t^2\omega_{t+1}^{i}\tilde{u}_{t+1}^{iT}(\tilde{u}_{t+1}^{i}-u)+\frac{1}{\omega_{t+1}^{i}}\gamma_t(y_{t+1}^{i}-\gamma_t\hat{u}_{t+1}^{i})^T\varepsilon_{t+1}^{i}\\ \leq&\frac{2\gamma_t\omega_{t+1}^{i}B_g}{\delta^2}\|\tilde{u}_{t+1}^{i}-\bar{u}_t\|+\gamma_t\langle y_{t+1}^{i},\tilde{u}_{t+1}^{i}-u\rangle\\ &+\frac{1}{2}\omega_{t+1}^{i}\gamma_t^2\|u\|^2+\frac{4\gamma_t^2\omega_{t+1}^{i2}B_g^2}{\delta^5}\end{aligned}\tag{17}$$

where the second inequality holds for $\varepsilon_{t+1}^{iT}(\hat{u}_{t+1}^{i}-u_{t+1}^{i})=-\|\varepsilon_{t+1}^{i}\|^2\leq 0$ and $(\varepsilon_{t+1}^{i}-\gamma_t(y_{t+1}^{i}-\gamma_t\hat{u}_{t+1}^{i}))^T(u_{t+1}^{i}-\omega_{t+1}^{i}u)\leq 0$ (see [4, Lemma 2]). $-\gamma_t^2\tilde{u}_{t+1}^{iT}(\tilde{u}_{t+1}^{i}-u)\leq\frac{1}{2}\gamma_t^2\|u\|^2$ and $\varepsilon_{t+1}^{i}\leq\frac{2\gamma_t\omega_{t+1}^{i}B_g}{\delta^2}$ are used to get the last inequality.

Substitute (17) into (16) and rearrange (16), we obtain

$$\begin{aligned}&-\gamma_t\sum_{i=1}^{N}\langle\tilde{u}_{t+1}^{i},y_{t+1}^{i}\rangle\\ \leq&\frac{N}{2}\|\bar{u}_t-u\|^2-\frac{N}{2}\|\bar{u}_{t+1}-u\|^2-\gamma_t\sum_{i=1}^{N}\langle u,g_t^{i}(x_t^{i})\rangle\\ &+\frac{2\gamma_t NB_g}{\delta^2}\sum_{i=1}^{N}\|\tilde{u}_{t+1}^{i}-\bar{u}_t\|+\frac{6\gamma_t^2N^3B_g^2}{\delta^5}+\frac{1}{2}N\gamma_t^2\|u\|^2\\ &-\gamma_t\sum_{i=1}^{N}\langle u,\nabla g_t^{i}(x_t^{i})(z_{t+1}^{i}-x_t^{i})\rangle\end{aligned}$$

where $\frac{N}{2}\sum_{t=0}^{T}(\|\bar{u}_t-u\|^2-\|\bar{u}_{t+1}-u\|^2)\leq\frac{1}{2}N\|u\|^2$. Then Lemma 4 can be obtained based on Cauchy-Schwarz inequality and $\|\nabla g_t^{i}(x_t^{i})\|\leq C_g$.

B. Proof of Lemma 5

Based on the optimality condition of z_{t+1}^{i} in (6d) and by (1), the following inequality is yields

$$\begin{aligned}&\alpha_t\langle z_{t+1}^{i}-x_t^{i*},\nabla J_t^{i}(x_t^{i},\hat{\sigma}_{t+1}^{i})+\gamma_t\nabla g_t^{i}(x_t^{i})^T\tilde{u}_{t+1}^{i}\rangle\\ \leq&\langle\nabla\varphi_i(z_{t+1}^{i})-\nabla\varphi_i(x_t^{i}),x_t^{i*}-z_{t+1}^{i}\rangle\\ =&D_\varphi(x_t^{i*},x_t^{i})-D_\varphi(x_t^{i*},z_{t+1}^{i})-D_\varphi(z_{t+1}^{i},x_t^{i}).\end{aligned}\tag{18}$$

Based on $D_\varphi(x,y)\geq\frac{\delta}{2}\|x-y\|^2$, we further obtain that

$$\begin{aligned}D_\varphi(x_t^{i*},z_{t+1}^{i})\leq&D_\varphi(x_t^{i*},x_t^{i})-\frac{\delta}{2}\|z_{t+1}^{i}-x_t^{i}\|^2\\ &+\alpha_t\langle x_t^{i*}-z_{t+1}^{i},\nabla J_t^{i}(x_t^{i},\hat{\sigma}_{t+1}^{i})\rangle\\ &+\alpha_t\gamma_t\langle x_t^{i*}-z_{t+1}^{i},\nabla g_t^{i}(x_t^{i})^T\tilde{u}_{t+1}^{i}\rangle.\end{aligned}\tag{19}$$

For the second term in the right hand sides of (19), we have

$$\begin{aligned}&\alpha_t\langle x_t^{i*}-z_{t+1}^{i},\nabla J_t^{i}(x_t^{i},\hat{\sigma}_{t+1}^{i})\rangle\\ \leq&\alpha_t\langle x_t^{i*}-x_t^{i},\nabla J_t^{i}(x_t^{i},\sigma(x_t))-\nabla J_t^{i}(x_t^{i*},\sigma(x_t^{*}))\rangle\\ &+\alpha_t\langle x_t^{i*}-x_t^{i},\nabla J_t^{i}(x_t^{i},\hat{\sigma}_{t+1}^{i})-\nabla J_t^{i}(x_t^{i},\sigma(x_t))\rangle\\ &+\alpha_t\langle x_t^{i*}-x_t^{i},\nabla J_t^{i}(x_t^{i*},\sigma(x_t^{*}))+\gamma_t\nabla g_t^{i}(x_t^{i*})^Tu_t^{*}\rangle\\ &-\alpha_t\gamma_t\langle x_t^{i*}-x_t^{i},\nabla g_t^{i}(x_t^{i*})^Tu_t^{*}\rangle\\ &+\alpha_t\langle x_t^{i}-z_{t+1}^{i},\nabla J_t^{i}(x_t^{i},\hat{\sigma}_{t+1}^{i})\rangle\\ \leq&\alpha_t\langle x_t^{i*}-x_t^{i},\nabla J_t^{i}(x_t^{i},\sigma(x_t))-\nabla J_t^{i}(x_t^{i*},\sigma(x_t^{*}))\rangle\\ &+\alpha_t\|x_t^{i*}-x_t^{i}\|\|\nabla J_t^{i}(x_t^{i},\hat{\sigma}_{t+1}^{i})-\nabla J_t^{i}(x_t^{i},\sigma(x_t))\|\\ &+\alpha_t\gamma_t\|x_t^{i*}-x_t^{i}\|\|\nabla g_t^{i}(x_t^{i*})\|\|u_t^{*}\|\\ &+\alpha_t\|x_t^{i}-z_{t+1}^{i}\|\|\nabla J_t^{i}(x_t^{i},\hat{\sigma}_{t+1}^{i})\|\\ \leq&\alpha_t\langle x_t^{i*}-x_t^{i},\nabla J_t^{i}(x_t^{i},\sigma(x_t))-\nabla J_t^{i}(x_t^{i*},\sigma(x_t^{*}))\rangle\\ &+\alpha_t lr\|\hat{\sigma}_{t+1}^{i}-\sigma(x_t)\|+\alpha_t\gamma_t rC_gB_u\\ &+\frac{\delta}{4}\|z_{t+1}^{i}-x_t^{i}\|^2+\frac{1}{\delta}C_j^2\alpha_t^2\end{aligned}\tag{20}$$

where u_t^{*} is the optimal dual variable and it has a upper bound by Lemma 1 in [25], i.e., $\|u_t^{*}\|\leq B_u$. The second inequality is obtained by Cauchy-Schwarz inequality and the optimality of x_t^*, and the third inequality is by Assumptions 2-3 and Jensen's inequality.

For the third term in the right hand sides of (19), we have

$$\begin{aligned}&\alpha_t\gamma_t\langle x_t^{i*}-z_{t+1}^i,\nabla g_t^i(x_t^i)^T\tilde{u}_{t+1}^i\rangle\\ \le&\alpha_t\gamma_t\langle x_t^{i*}-x_t^i,\nabla g_t^i(x_t^i)^T\tilde{u}_{t+1}^i\rangle+\alpha_t\gamma_t\langle x_t^i-z_{t+1}^i,\nabla g_t^i(x_t^i)^T\tilde{u}_{t+1}^i\rangle\\ \le&\alpha_t\gamma_t g_t^i(x_t^{i*})^T\tilde{u}_{t+1}^i-\alpha_t\gamma_t\langle\tilde{u}_{t+1}^i,y_{t+1}^i\rangle\\ \le&\alpha_t\gamma_t B_g\|\tilde{u}_{t+1}^i-\bar{u}_t\|+\alpha_t\gamma_t g_t^i(x_t^{i*})^T\bar{u}_t-\alpha_t\gamma_t\langle\tilde{u}_{t+1}^i,y_{t+1}^i\rangle\end{aligned}\tag{21}$$

where the second inequality is obtain by the convexity of $g_t^i(x_t^i)$ and the definition of y_{t+1}^i.

By Assumptions 1 and (6g), it can be derived that

$$\begin{aligned}&D_\varphi(x_{t+1}^{i*},x_{t+1}^i)\\ \le&D_\varphi(x_t^{i*},x_{t+1}^i)+l_D\|x_{t+1}^{i*}-x_t^{i*}\|\\ \le&(1-\alpha_t)D_\varphi(x_t^{i*},x_t^i)+\alpha_t D_\varphi(x_t^{i*},z_{t+1}^i)+l_D\|x_{t+1}^{i*}-x_t^{i*}\|\end{aligned}\tag{22}$$

Combining with (19)-(22) and summing over $i\in\{1,\ldots,N\}$ yields

$$\begin{aligned}&\mu\|x_t-x_t^*\|^2\\ \le&\langle x_t-x_t^*,F(x_t)-F(x_t^*)\rangle\\ \le&\frac{1}{\alpha_t^2}\sum_{i=1}^N(D_\varphi(x_t^{i*},x_t^i)-D_\varphi(x_{t+1}^{i*},x_{t+1}^i))+\frac{1}{2\delta}NC_j^2\alpha_t\\ &+lr\sum_{i=1}^N\|\hat{\sigma}_{t+1}^i-\sigma(x_t)\|+\gamma_t B_g\sum_{i=1}^N\|\tilde{u}_{t+1}^i-\bar{u}_t\|+\gamma_t NC_gB_ur\\ &-\gamma_t\sum_{i=1}^N\langle\tilde{u}_{t+1}^i,y_{t+1}^i\rangle+\frac{\sqrt{N}l_D}{\alpha_t^2}\|x_{t+1}^*-x_t^*\|\end{aligned}\tag{23}$$

where $\sum_{i=1}^N g_t^i(x_t^{i*})^T\bar{u}_t\le 0$ and the strong monotonicity of $F_t(x_t)$ are used to get the inequalities. Hereto, Lemma 5 can be easily obtained by summing over $t\in\{1,\ldots,T\}$ and by the following inequalities

$$\begin{aligned}&\sum_{t=1}^T\frac{1}{\alpha_t^2}\sum_{i=1}^N(D_\varphi(x_t^{i*},x_t^i)-D_\varphi(x_{t+1}^{i*},x_{t+1}^i))\\ =&\sum_{t=1}^T\sum_{i=1}^N(\frac{1}{\alpha_{t-1}^2}D_\varphi(x_t^{i*},x_t^i)-\frac{1}{\alpha_t^2}D_\varphi(x_{t+1}^{i*},x_{t+1}^i))\\ &+\sum_{t=1}^T\sum_{i=1}^N(\frac{1}{\alpha_t^2}-\frac{1}{\alpha_{t-1}^2})D_\varphi(x_{t+1}^{i*},x_{t+1}^i)\\ \le&\sum_{i=1}^N\frac{1}{\alpha_0^2}D_\varphi(x_1^{i*},x_1^i)+(\frac{1}{\alpha_T^2}-\frac{1}{\alpha_0^2})Nl_Dr\le\frac{Nl_Dr}{\alpha_T^2}\end{aligned}\tag{24}$$

where $D_\varphi(x,y)\le l_Dr$ is used to obtain the inequalities.

References

[1] R. Cornes, "Aggregative environmental games," *Environmental and Resource Economics*, vol. 63, no. 2, pp. 339–365, 2016.

[2] H. Chen, Y. Li, R. H. Y. Louie, and B. Vucetic, "Autonomous demand side management based on energy consumption scheduling and instantaneous load billing: An aggregative game approach," *IEEE Transactions on Smart Grid*, vol. 5, no. 4, pp. 1744–1754, Jul. 2014.

[3] J. Barrera and A. Garcia, "Dynamic incentives for congestion control," *IEEE Transactions on Automatic Control*, vol. 60, no. 2, pp. 299–310, Aug. 2015.

[4] Z. Deng, Y. Liu, and T. Chen, "Generalized Nash equilibrium seeking algorithm design for distributed constrained noncooperative games with second-order players," *Automatica*, vol. 141, p. 110317, 2022.

[5] F. Facchinei and C. Kanzow, "Generalized Nash equilibrium problems," *Annals of Operations Research*, vol. 175, no. 1, pp. 177–211, 2010.

[6] Z. Deng, "Distributed Nash equilibrium seeking for aggregative games with second-order nonlinear players," *Automatica*, vol. 135, p. 109980, 2022.

[7] S. Huang, J. Lei, and Y. Hong, "A linearly convergent distributed Nash equilibrium seeking algorithm for aggregative games," *IEEE Transactions on Automatic Control*, vol. 68, no. 3, pp. 1753–1759, Mar. 2023.

[8] D. Gadjov and L. Pavel, "Distributed GNE seeking over networks in aggregative games with coupled constraints via forward-backward operator splitting," in *2019 IEEE 58th Conference on Decision and Control (CDC)*, 2019, pp. 5020–5025.

[9] Z. Deng, "Distributed generalized Nash equilibrium seeking algorithm for nonsmooth aggregative games," *Automatica*, vol. 132, p. 109794, 2021.

[10] M. Maskery, V. Krishnamurthy, and Q. Zhao, "Decentralized dynamic spectrum access for cognitive radios: cooperative design of a non-cooperative game," *IEEE Transactions on Communications*, vol. 57, no. 2, pp. 459–469, Feb. 2009.

[11] D. Goldfarb and G. Iyengar, "Robust portfolio selection problems," *Mathematics of Operations Research*, vol. 28, no. 1, pp. 1–38, 2003.

[12] S. You, S. W. Hadley, M. Shankar, and Y. Liu, "Co-optimizing generation and transmission expansion with wind power in large-scale power grids—implementation in the us eastern interconnection," *Electric Power Systems Research*, vol. 133, pp. 209–218, 2016.

[13] Z. Deng and X. Zuo, "Distributed online learning algorithm for noncooperative games over unbalanced digraphs," *IEEE Transactions on Neural Networks and Learning Systems*, pp. 1–11, 2023.

[14] K. Lu, G. Li, and L. Wang, "Online distributed algorithms for seeking generalized Nash equilibria in dynamic environments," *IEEE Transactions on Automatic Control*, vol. 66, no. 5, pp. 2289–2296, Jun. 2021.

[15] L. Yang and X. Zhou, "Optimizing on-time arrival probability and percentile travel time for elementary path finding in time-dependent transportation networks: Linear mixed integer programming reformulations," *Transportation Research Part B: Methodological*, vol. 96, pp. 68–91, 2017.

[16] X. Qiu, L. Zhao, J. Wang, X. Wang, and Q. Wang, "Effects of time-dependent diffusion behaviors on the rumor spreading in social networks," *Physics Letters A*, vol. 380, no. 24, pp. 2054–2063, 2016.

[17] S. Liang, X. Zeng, G. Chen, and Y. Hong, "Distributed sub-optimal resource allocation via a projected form of singular perturbation," *Automatica*, vol. 121, p. 109180, 2020.

[18] Y. Li, G. Qu, and N. Li, "Online optimization with predictions and switching costs: Fast algorithms and the fundamental limit," *IEEE Transactions on Automatic Control*, vol. 66, no. 10, pp. 4761–4768, Nov. 2021.

[19] S. Shahrampour and A. Jadbabaie, "Distributed online optimization in dynamic environments using mirror descent," *IEEE Transactions on Automatic Control*, vol. 63, no. 3, pp. 714–725, Aug. 2018.

[20] D. Yuan, Y. Hong, D. W. C. Ho, and S. Xu, "Distributed mirror descent for online composite optimization," *IEEE Transactions on Automatic Control*, vol. 66, no. 2, pp. 714–729, Apr. 2021.

[21] X. Li, X. Yi, and L. Xie, "Distributed online optimization for multi-agent networks with coupled inequality constraints," *IEEE Transactions on Automatic Control*, vol. 66, no. 8, pp. 3575–3591, Sep. 2021.

[22] A. Nedić and A. Olshevsky, "Distributed optimization over time-varying directed graphs," *IEEE Transactions on Automatic Control*, vol. 60, no. 3, pp. 601–615, Mar. 2015.

[23] X. Li, X. Yi, and L. Xie, "Distributed online convex optimization with an aggregative variable," *IEEE Transactions on Control of Network Systems*, vol. 9, no. 1, pp. 438–449, Aug. 2022.

[24] B. F. Hobbs and J.-S. Pang, "Nash-cournot equilibria in electric power markets with piecewise linear demand functions and joint constraints," *Operations Research*, vol. 55, no. 1, pp. 113–127, Feb. 2007.

[25] A. Nedić and A. Ozdaglar, "Approximate primal solutions and rate analysis for dual subgradient methods," *SIAM Journal on Optimization*, vol. 19, no. 4, pp. 1757–1780, 2009.