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The power series method to compute backstepping kernel gains: theory and practice

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Abstract— **Obtaining PDE backstepping controller or observer gains requires the solution of kernel PDEs - one or more hyperbolic PDEs on a triangular (Goursat) domain with non-standard boundary conditions. The numerical solution of these equations is a challenge that every designer applying backstepping must eventually face, except for the simplest of cases where explicit solutions are available. In addition, recent backstepping designs for coupled systems exhibit discontinuous behavior which must be accurately captured with the numerical approximation. In this paper, we propose a power series method as an alternative to other approaches. This method, which was introduced years ago in combination with convex optimization but whose convergence has only been recently formally established, offers several advantages, most of all simplicity, as it is quite easy to grasp and implement. Other features of the method include precision, adaptability to settings with discontinuous kernels, and the ability to produce symbolical kernels depending on parameters. The paper provides the necessary theoretical background, which borrows fundamental results from complex analysis and leverages already written kernel well-posedness proofs to show the existence of a power series solution. Links to the codes used in the simple examples are given; these are easily adaptable Mathematica notebooks. A complex multi-kernel, multiple-discontinuity example used in the stabilizing feedback law of a multilayer Timoshenko beam is given at the end, demonstrating the applicability of the method to some challenging families of kernel equations appearing in recent backstepping designs.**

I. Introduction

The backstepping method is an ubiquitous design technique for PDE control. First developed to design feedback control laws and observers for one-dimensional reaction-diffusion PDEs [18], it has since been generalized to multiple dimension [33] and applied to many other systems including, among others, flow control [29], [35], thermal loops [31], thermoacustic instabilities [1], nonlinear PDEs [30], hyperbolic 1-D systems [8], [12], [20], multi-agent deployment [24], wave equations [26], beams [6] and delays [19]. Some of the more striking features of backstepping include the possibility of finding explicit control laws in some cases (see e.g. [32]) or even designing adaptive controllers [27].

The application of the method requires the computation of backstepping controller or observer gains. Those require the solution of the so-called backstepping kernel PDEs—one or more linear hyperbolic PDEs (first order or second order) on a triangular (Goursat [13]) domain with non-standard boundary conditions. Except for the simplest of cases (typically, constant coefficient plants) where explicit solutions are available, the numerical solution of these equations is a challenge that every designer applying backstepping must eventually face. In addition, recent backstepping designs for coupled systems (both parabolic [34] and hyperbolic [14]) exhibit discontinuous behavior which must be accurately captured with the numerical approximation. Nevertheless, there have been no publications specifically devoted to this topic. Only a few sections and appendices spread through the backstepping literature give some clues about numerical algorithms, which include finite difference approximations of the kernel equations [2], [11], [17], [18], the use of symbolical successive approximation series [31], or the numerical solution of the integral version of the kernel equations [4], [15]. Given its importance in nonlinear wave propagation phenomena [16], more sophisticated methods for Goursat problems exist in the literature, see, e.g. [10] but have never been applied to backstepping kernel equations. In any case, all these methods may not be easy to adapt to a particular set of kernel equations, specially if discontinuity lines are present.

This paper deals with a power series method to obtain solutions for the backstepping kernel equations. The idea of using a power series to compute backstepping kernels was first seen in [3] (without much analysis of the convergence of the method itself, but rather using ideas from convex optimization to approach the kernels as best as possible while obtaining stability) and later, without proof, in [5] for a problem involving coupled parabolic equations, where piecewise-analytic kernels require the use of several series to account for discontinuities. In [36], we presented the first rigorous proof showing that the method provides a unique converging solution, for a multi-dimensional case that presents singularities at the origin and thus is not amenable to other methods. However, in most cases the already existing proofs of kernel existence can be leveraged, using complex analysis, to easily derive sufficient conditions for analyticity of the resulting kernels.

The power series approach is a solid alternative to more traditional approaches. The main advantages of the method are its simplicity (it does not require the sometimes cumbersome conversion to integral equations or any consideration about discrete meshes or boundary points, thus preventing mistakes), precision (one reaches a simple polynomial in one variable for the gain at the boundary that does not require interpolation), adaptability (it can be adapted to settings with discontinuous kernels by breaking the domain in pieces, see [5]), and capacity to produce kernels depending on parameters (by symbolically solving the kernel equations). The main drawback is the analyticity requirement of the system coefficients (even though many physical systems and most examples seen in backstepping papers indeed posses analytic coefficients) and some additional requirements on space-varying transport and diffusion coefficients arising

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from the underlying complex analysis theory. Convergence may be slow in some cases, requiring very high-order series, even if modern computing systems can reach rather high orders of the series in seconds, the computational complexity of the problem grows quadratically with the order.

The focus of this paper is on providing an overview of both the theory and the practice of this method. To this end, in Section II, the paper first focuses on the kernel equations required to stabilize a simple example and provides the required theoretical background including proofs of existence, uniqueness and convergence, giving the sufficient conditions for a power series solution to exist; the idea is to leverage proofs that already exist in the literature. Further examples are given in Section III, including a case with discontinuous kernels; for these, only sufficient conditions are provided. The symbolical codes used in the examples, which are easily adaptable Mathematica notebooks, are also given. A complex multi-kernel, multiple-discontinuity example used in the stabilizing feedback law of a multilayer Timoshenko beam is then shown in Section IV, demonstrating the applicability of the method to some challenging families of kernel equations appearing in recent backstepping designs. We finish in Section V with some concluding remarks.

II. The theory: sufficient conditions for existence, uniqueness and convergence

This section is devoted to theoretical aspects of the method proposed to solve the backstepping kernel equations. The exposition is based on a simple particular case often used when learning backstepping.

Consider the reaction-diffusion equation

$$u_t = \epsilon u_{xx} + \lambda(x) u, \tag{1}$$

for $t > 0$, with $\epsilon > 0$ and $\lambda(x)$ a smooth function in the domain $x \in [0, L]$ (more specific conditions are given subsequently), and with boundary conditions

$$u(t, L) = U(t), \quad u(t, 0) = 0, \tag{2}$$

where U is the actuation variable. For sufficiently large $\lambda(x) > 0$, (1)–(2) is open-loop unstable.

Applying backstepping [18], one chooses a tuning parameter $c \geq 0$; then, the stabilizing control law is

$$U = \int_0^L K(L, \xi) u(\xi) d\xi, \tag{3}$$

where the function $K(x, \xi)$ is obtained by solving

$$K_{xx}(x,\xi) - K_{\xi\xi}(x,\xi) = \frac{\lambda(\xi) + c}{\epsilon} K(x,\xi), \tag{4}$$

with boundary conditions

$$K(x,x) = -\frac{1}{2\epsilon}\int_0^x (\lambda(\xi) + c)\, d\xi, \tag{5}$$

$$K(x,0) = 0, \tag{6}$$

in the *triangular* domain $\mathcal{T} = \{(x,\xi) : 0 \leq \xi \leq x \leq L\}$.

A. A power series solution of the kernel equations

To solve (4)–(6) consider a solution given as the *double* power series

$$K(x,\xi) = \sum_{i=0}^{\infty}\sum_{j=0}^{i} K_{ij} x^{i-j}\xi^j. \tag{7}$$

Note that the way the series (7) is written, for fixed i one gets a polynomial in two variables with all the possible monomials of degree i. This is a convenient representation since for practical purposes one needs to truncate (7); truncating at $i = N$ will produce a polynomial of degree N.

The next step requires the expansion of $\lambda(x)$ into its power series. For convenience we consider instead $\frac{\lambda(x)+c}{\epsilon}$, namely

$$\frac{\lambda(x) + c}{\epsilon} = \sum_{i=0}^{\infty} \lambda_i x^i. \tag{8}$$

Now, both (7) and (8) are replaced into (4)–(6). Starting with (4) and assuming analyticity of $K(x,\xi)$ one can exchange derivatives and sums, obtaining

$$\begin{aligned} &\sum_{i=2}^{\infty}\sum_{j=0}^{i-2}(i-j)(i-j-1)K_{ij}x^{i-j-2}\xi^j \\ &-\sum_{i=2}^{\infty}\sum_{j=2}^{i} j(j-1)K_{ij}x^{i-j}\xi^{j-2} \\ = &\left(\sum_{i=0}^{\infty}\lambda_i x^i\right)\sum_{i=0}^{\infty}\sum_{j=0}^{i}K_{ij}x^{i-j}\xi^j. \end{aligned} \tag{9}$$

Rearranging the sums in (9) one gets

$$\begin{aligned} &\sum_{i=0}^{\infty}\sum_{j=0}^{i}(i-j+2)(i-j+1)K_{(i+2)j}x^{i-j}\xi^j \\ &-\sum_{i=0}^{\infty}\sum_{j=0}^{i}(j+2)(j+1)K_{(i+2)(j+2)}x^{i-j}\xi^j \\ = &\sum_{i=0}^{\infty}\sum_{j=0}^{i}B_{ij}x^{i-j}\xi^j, \end{aligned} \tag{10}$$

where

$$B_{ij} = \sum_{k=j}^{i} K_{kj}\lambda_{i-k}. \tag{11}$$

Equating all terms in (10) having the same powers of x and ξ, and slightly modifying the indexes to start at $i = 2$, one gets for $0 \leq j \leq i-2$ the following recursion

$$(i-j)(i-j-1)K_{ij} - (j+2)(j+1)K_{i(j+2)} = B_{(i-2)j}. \tag{12}$$

Proceeding similarly with the boundary conditions (5)–(6) one gets

$$\sum_{i=0}^{\infty}\left(\sum_{j=0}^{i}K_{ij}\right)x^i = -\frac{1}{2}\sum_{i=1}^{\infty}\frac{\lambda_{i-1}}{i}x^i, \tag{13}$$

$$\sum_{i=0}^{\infty}K_{i0}x^i = 0, \tag{14}$$

thus obtaining

$$\left(\sum_{j=0}^{i} K_{ij}\right) = -\frac{1}{2}\frac{\lambda_{i-1}}{i}, \quad \forall i \geq 1, \tag{15}$$

$$K_{i0} = 0, \qquad \forall i. \tag{16}$$

Considering (12) and (15)–(16) one may wonder if there always exists a solution. The first terms are easy enough to compute: $K_{00} = 0$, $K_{10} = 0$, $K_{01} = -\frac{\lambda_0}{2}$, $K_{20} = 0$, $K_{02} = 0$, $K_{11} = -\frac{\lambda_1}{4}$, and so on.

One could follow the lengthy procedure of [36] to show that the recurrence (12) and (15)–(16) has a unique solution, and that the corresponding power series (7) is convergent with a certain radius of convergence. However, *there is a simpler route based on the already-existing proofs of kernel existence and a few facts of complex analysis*. The most interesting aspect of the proof is that it can be easily generalized to most cases by using the proofs of kernel existence and uniqueness already contained in the papers, quickly providing sufficient conditions required for the method. The main goal of this paper, along with the computational recipes, is to show how these strong theoretical foundations allow to use the method without fears of non-existence, non-uniqueness, or non-convergence, its possible limitations (see several examples of Section III) notwithstanding.

B. From kernel existence to power series existence

In this section, we extend the domain where the kernel is defined so that the independent variables are now *complex-valued*, which is the natural setting to talk about power series convergence and analyticity. Denote as $\mathcal{D}_L$ the complex-valued open disc centered at the origin and of radius L, i.e., $\mathcal{D}_L = \{x \in \mathbb{C} : |x| < L\}$. The new domain where (7) is defined is considered to be the polydisc[1] $\mathcal{D}_{L+\delta} \times \mathcal{D}_{L+\delta}$ of $\mathbb{C}^2$, for some $\delta > 0$; we require such δ since it is essential to evaluate the kernel at $x = L$ for the control law (3).

Theorem 1: If there exists $\delta > 0$ such that λ is analytic on $\mathcal{D}_{L+\delta}$, then the solution K of the kernel equations (4)–(6) can be extended to an analytic function in the polydisc $\mathcal{D}_{L+\delta/2} \times \mathcal{D}_{L+\delta/2}$.

Proof: The classical proof of kernel existence found in most backstepping papers relies on transformation to an integral equation, see e.g. [18]. Indeed, defining $K(x,\xi) = G\left(\frac{x+\xi}{2}, \frac{x-\xi}{2}\right)$ (which is just a rotation of the variables that does not affect the result), one finds that G verifies an integral equation, whose solution can be posed as the successive approximation series

$$G(x,\xi) = \sum_{i=0}^{\infty} G_i(x,\xi), \tag{17}$$

with G_i recursively defined as

$$G_0 = -\frac{1}{4\epsilon}\int_0^x \left(\lambda\left(\frac{s}{2}\right) + c\right) ds, \tag{18}$$

$$G_{i+1} = \frac{1}{4\epsilon}\int_\xi^x \int_0^\xi \left(\lambda\left(\frac{\tau - s}{2}\right) + c\right) G_i(\tau,s) ds d\tau. \tag{19}$$

Now, consider extending (18)–(19) to $(x,\xi) \in \mathcal{D}_{L+\delta/2} \times \mathcal{D}_{L+\delta/2}$, which requires considering the integrals as line integrals in the complex plane, see e.g. [28, p.44]. Such integrals of analytic functions are analytic and do not depend on the integration path. Noting that the argument of λ in (19) is always inside $\mathcal{D}_{L+\delta/2}$, then one can recursively show that G_i is analytic for all i, being integrals and products of analytic functions. Using the results of complex analysis, see e.g. [21, Proposition 1.2.3], if one can show uniform convergence for (17) in compact subsets of $\mathcal{D}_{L+\epsilon/2} \times \mathcal{D}_{L+\epsilon/2}$ one can conclude analyticity of G and therefore of K in that domain, which is sufficient to compute the kernel gain appearing in (3). Now, since λ is analytic in $\mathcal{D}_{L+\delta}$ then it is continuous on $\mathcal{D}_{L+\delta/2}$, and therefore bounded by some value, which we denote as $\bar{\lambda}$. Define $M = \frac{\bar{\lambda}+c}{\epsilon}$. Denote $L' = L + \delta/2$. As in [18], the following bound can be proved recursively:

$$|G_i(x,\xi)| \leq \frac{(L'M)^{n+1}}{4}\frac{|x|^n + |\xi|^n}{n!} \leq \frac{L'M}{4n!}\left(2L'^2 M\right)^n, \tag{20}$$

and by Weierstrass' M-test uniform convergence of (17) is obtained thus finishing the proof. It remains to prove the bound (20), by induction.

For what follows, note that for f analytic in $\mathcal{D}_L$ and $z \in \mathcal{D}_L$ the line integral[2] $\int_0^z f(s)ds$ can be taken along a straight line from the origin such that the modulus of s varies from 0 to $|z|$, and if one knows some bound of the type $|f(s)| \leq h(|s|)$ for a positive function h one can bound the modulus of the line integral as $\left|\int_0^z f(s)ds\right| \leq \int_0^{|z|} h(r)dr$. Thus, for (18), we get

$$|G_0| \leq \frac{1}{4\epsilon}\int_0^{|x|} (\bar{\lambda} + c)ds \leq \frac{L'M}{4}. \tag{21}$$

Now, noting $\int_\xi^x = \int_0^x - \int_0^\xi$, we get

$$|G_{i+1}| \leq \left|\int_0^\xi \int_0^\xi G_i(\tau,s) ds d\tau\right| + \left|\int_0^x \int_0^\xi G_i(\tau,s) ds d\tau\right|. \tag{22}$$

Now, one has, again performing the integrals along straight lines

$$\int_0^{|\xi|} (|\tau|^n + |s|^n)\, ds = |\xi||\tau|^n + \frac{|\xi|^{n+1}}{n+1}, \tag{23}$$

$$\int_0^{|x|} \left(|\xi||\tau|^n + \frac{|\xi|^{n+1}}{n+1}\right) d\tau \leq L'\frac{|x|^{n+1} + |\xi|^{n+1}}{n+1}, \tag{24}$$

$$\int_0^{|\xi|} \left(|\xi||\tau|^n + \frac{|\xi|^{n+1}}{n+1}\right) d\tau \leq 2L'\frac{|\xi|^{n+1}}{n+1}, \tag{25}$$

thus, using the induction hypothesis,

$$|G_{i+1}| \leq \frac{3}{4}\frac{(L'M)^{n+2}}{4}\frac{|x|^{n+1} + |\xi|^{n+1}}{(n+1)!}, \tag{26}$$

[1] Note a polydisc $\mathcal{D}_L \times \mathcal{D}_L$ is different from the 2-ball of radius L of $\mathbb{C}^2$, but polydiscs are the proper setting for analytic functions of multiple variables [21].

[2] See [22, Chapter 2] for the definition of line (or contour) integrals of complex numbers, which are more classically writen as $\int_\gamma f(s)ds$, where γ is the path along which the integral is computed. Here $\gamma = [0, z]$. Note that line integrals of analytic functions are independent of the path joining the initial and final points, thus we can consider simple straight lines without any loss of generality.

which immediately gives (20). ■

The following corollary becomes an immediate consequence that gives us the necessary properties for the power series solution.

Corollary 1: If there exists $\delta > 0$ such that λ is analytic on $\mathcal{D}_{L+\delta}$, the recurrence (12) and (15)–(16) is well-defined, has a unique solution, and the power series (7) converges and defines an analytic function in the polydisc $\mathcal{D}_{L+\delta/2}\times\mathcal{D}_{L+\delta/2}$ that is the unique solution of the kernel equations (4)–(6).

Proof: Since, from Theorem 1, the kernel equations' solution is analytic, all formal operations of Section II-A are valid; by the identity theorem [22, p. 365] and its extension to multiple complex variables, e.g., [21, Theorem 1.2.6], there cannot be two different analytic functions that agree on parts of domain containing an accumulation point (such as the boundary condition (6)). The only possible conclusion is that (7) has unique coefficients that can be numerically computed from (12) and (15)–(16). ■

III. The Practice: Finding Kernel Power Series with Symbolic Software. Examples. Limitations.

Section II-A introduced the equations one needs to solve to find the power series representation of the kernel, which turn out to be a set of linear equations recursively defined. In general, for a truncation of (7) of order N, namely

$$K(x,\xi)=\sum_{i=0}^{N}\sum_{j=0}^{i}K_{ij}x^{i-j}\xi^{j}, \tag{27}$$

one needs $\frac{(N+1)(N+2)}{2}$ coefficients. From (12) one obtains $\frac{N(N-1)}{2}$ equations, (15) gives N equations and (16) gives $N+1$ equations. Since

$$\frac{N(N-1)}{2}+N+N+1=\frac{(N+1)(N+2)}{2}$$

one has exactly the required number of equations. However, it might be argued that obtaining and programming (12) and (15)–(16) is a burdensome task, which in addition requires computing the expansion of $\lambda(x)$ into its power series.

Fortunately, modern symbolic software can help with these tasks, automatizing almost all the required steps. We have chosen Mathematica [37], a general purpose computer algebra system which was initially released in 1988, which integrates symbolic and numerical calculations, visualization, programming, and documentation. In particular, it can carry out power series expansions of known functions, deduce the set of equations (12) and (15)–(16) and solve it, with just a few commands. All the problems solved in this paper have been translated into Mathematica 13.2 notebooks[3]. An additional advantage of these methods is that some coefficients can be kept as parameters, obtaining a symbolic solution in terms of these parameters. Next, we give some examples of our methodology together with additional sufficient conditions for the coefficients (in the form of Theorem statements) when required, also outlining some limitations of the method (non-convergence when these conditions are not verified, or slow convergence in some cases).

[3]All code downloadable at `http://aero.us.es/rvazquez/powerseries.zip`, containing all notebooks used in Section III.

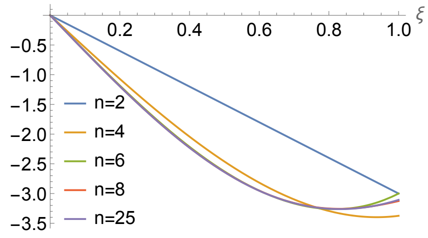

Fig. 1. Convergent example with $\lambda(x)=3+x^2\sin(3x)$ (Example 1a).

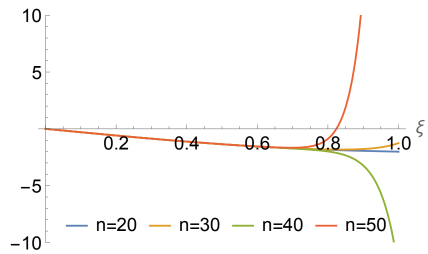

Fig. 2. Divergent example with $\lambda(x)=\sqrt{0.5+x^2}$ (Example 1b).

A. Example 1: parabolic equation with space-varying reaction

Consider $\lambda(x)=3+x^2\sin(3x)$, $L=\epsilon=1$, $c=3$. This verifies Theorem 1 and Fig. 1 shows the resulting kernel gain for several orders of approximation. It can be seen that order 8 is enough in this case, as it is indistinguishable from order 25. However, even if we have stated the requirements of Theorem 1 as sufficient, in general they are also necessary, as an example shown in Fig. 2 shows. Choosing $\lambda(x)=\sqrt{0.5+x^2}$ which is not analytic on the unit disc, the kernel gain does not converge as the order increases as it can clearly be seen in the figure by inspecting the gain for different orders of the series.

B. Example 2: parabolic equation with parameterized space-varying reaction

Consider now $\lambda_P(x)=1+Px$,$L=\epsilon=1$, $c=3$, where P is an arbitrary real parameter. One can solve the kernel *symbolically* carrying out the parameter P. The resulting kernel gain $K(L,\xi)$ is displayed in Fig. 3, and written next to ninth order in ξ and third order in P:

$$\begin{aligned}K(L,\xi)\approx&\,(-5.888-0.5867P-0.01044P^2-6.428\cdot10^{-5}P^3)\xi\\&+(3.608-0.02556P^2-(4.904\cdot10^{-4})P^3)\xi^3\\&+(0.4907P+0.04889P^2+(8.698\cdot10^{-4})P^3)\xi^4\\&+(-0.8079-0.004122P^2-(5.673\cdot10^{-4})P^3)\xi^5\\&+(-0.1804P+0.001277P^3)\xi^6\\&(0.09424-0.0115P^2-0.001271P^3)\xi^7\\&+(0.02885P+(1.468\cdot10^{-4})P^3)\xi^8\\&+(-0.006742+0.003092P^2)\xi^9.\end{aligned}$$

C. Example 3: parabolic equation with space-varying diffusion

This example was analyzed in [25]. Consider the reaction-diffusion equation

$$u_t=\epsilon(x)u_{xx}+\lambda(x)u, \tag{28}$$

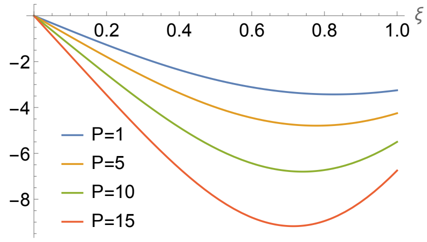

Fig. 3. Kernel gains if $\lambda_P(x) = 1 + Px$ for values of P (Example 2).

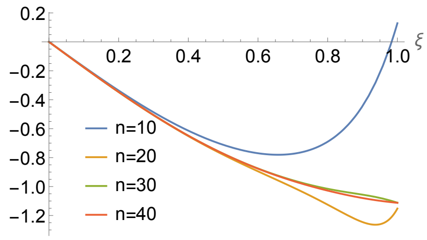

Fig. 4. Convergent space-varying diffusion case, $\lambda(x) = 3 + x\sin(6x)$, $\epsilon(x) = 2 + x^2$ (Example 3a).

for $x \in [0, L]$, with $\epsilon(x)$ and $\lambda(x)$ analytic functions in $\mathcal{D}_L$. Boundary conditions and control law are as in Section II, this is, (2)–(3). Applying backstepping [25], one needs to solve the following kernel equations for some $c > 0$:

$$\epsilon(x)K_{xx}(x,\xi) - \epsilon(\xi)K_{\xi\xi}(x,\xi) = (\lambda(\xi) + c)K(x,\xi), \quad (29)$$

with boundary conditions

$$-2\epsilon(x)\frac{d}{dx}K(x,x) = -\epsilon'(x) - \lambda(x) - c, \quad (30)$$

$$K(x,0) = 0. \quad (31)$$

A result similar to Theorem 1 can be stated.

Theorem 2: If there exists $\delta > 0$ such that λ and ϵ are analytic on $\mathcal{D}_{L+\delta}$, $|\epsilon(z)| > 0$ for all $z \in \mathcal{D}_{L+\delta}$, then there exists a power series solution which converges and defines an analytic function in the polydisc $\mathcal{D}_{L+\delta/2} \times \mathcal{D}_{L+\delta/2}$, that is the unique solution of the kernel equations (29)–(31).

Proof: Using the same ideas of Section II, from the proof in [18, Chapter 4.8] one can observe that with a smart scaling transformation, the space-varying diffusion system can be transformed into a plant with only space-varying reaction; thus the proof of Theorem 1 directly applies. However, the scaling transformation requires inverses and roots of $\epsilon(x)$, which results in the additional requirement of $|\epsilon(z)| > 0$ for all $z \in \mathcal{D}_{L+\delta}$. ■

With $\lambda(x) = 3 + x\sin(6x)$, $L = 1$, $c = 3$, convergent ($\epsilon(x) = 2 + x^2$) and divergent ($\epsilon(x) = 2 + 3x^2$) examples are shown in Fig. 4 and Fig. 5, respectively.

D. Example 4: 2×2 1-D linear hyperbolic system with space-varying coefficients

Consider the following hyperbolic 1-D system [8]

$$u_t = -\epsilon(x)u_x + c_1(x)u + c_2(x)v, \quad (32)$$

$$v_t = \mu(x)v_x + c_3(x)u + c_4(x)v, \quad (33)$$

for $x \in [0, L]$, and assume that $\epsilon(x), \mu(x), c_i(x)$ are analytic in $\mathcal{D}_L$. The boundary conditions are:

$$u(t,0) = qv(t,0), \quad v(t,L) = U(t), \quad (34)$$

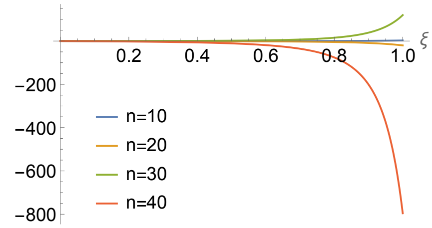

Fig. 5. Divergent space-varying diffusion case, $\lambda(x) = 3 + x\sin(6x)$, $\epsilon(x) = 2 + 3x^2$ (Example 3b).

with U the actuation variable. Then, the stabilizing control law is

$$U = \int_0^L K^{vu}(L,\xi)u(\xi)d\xi + \int_0^L K^{vv}(L,\xi)v(\xi)d\xi. \quad (35)$$

In this case, backstepping requires finding the solution for two kernels, which we denote as $K^{vv}(x,\xi)$ and $K^{vu}(x,\xi)$ in the domain $\mathcal{T}$. The kernel equations are

$$\mu(x)K_x^{vv} + \mu(\xi)K_\xi^{vv} = -\mu'(\xi)K^{vv} + c_2(\xi)K^{vu} + [c_4(x) - c_4(\xi)]K^{vu}, \quad (36)$$

$$\mu(x)K_x^{vu} - \epsilon(\xi)K_\xi^{vu} = \epsilon'(\xi)K^{vu} + c_3(\xi)K^{vv} + [c_4(x) - c_1(\xi)]K^{vv}, \quad (37)$$

with boundary conditions

$$K^{vv}(x,0) = \frac{q\epsilon(0)}{\mu(0)}K^{vu}(x,0), \quad (38)$$

$$(\epsilon(x) + \mu(x))K^{vu}(x,x) = -c_3(x), \quad (39)$$

where (39) is expressed in a way to avoid computing additional power series of fractions. An additional requirement of the power series method is that $|\epsilon(x)|, |\mu(x)| > 0$ in $\mathcal{D}_L$, when considered as functions with complex arguments, for the same reason of Example 3. We state the following theorem which is obtained by complexifying the kernel well-posedness proof of [8] and having the same considerations as in Example 3.

Theorem 3: If there exists $\delta > 0$ such that c_1, c_2, c_3, c_4, ϵ and μ are analytic on $\mathcal{D}_{L+\delta}$, and $|\epsilon(z)| > 0$ and $|\mu(z)| > 0$ for all $z \in \mathcal{D}_{L+\delta}$, then there exists a pair of power series solution for K^{vv} and K^{vu} in the form of (7) which converge and define analytic functions in the polydisc $\mathcal{D}_{L+\delta/2} \times \mathcal{D}_{L+\delta/2}$, that are the unique solution of the kernel equations (36)–(39).

Figure 6 shows the convergent series of the corresponding gain kernels for $\mu(x) = 1.5 + x^2$, $\epsilon(x) = 1.2 + x^3$, $L = 1$, $c_1(x) = 3\cos(x)$,$c_2(x) = \sin(2x)$,$c_3(x) = 1 + 2\exp(x)$,$c_4(x) = \frac{1}{3+y^2}$,$q = 1$, which verify the required assumptions. The presence of space-varying transport slows down the computation.

E. Example 5: Motion planning kernels for $(0+2) \times (0+2)$ 1-D linear hyperbolic system with space-varying coupling

This example is directly extracted from [14] and is the simplest possible discontinuous example of an $(n + m) \times$

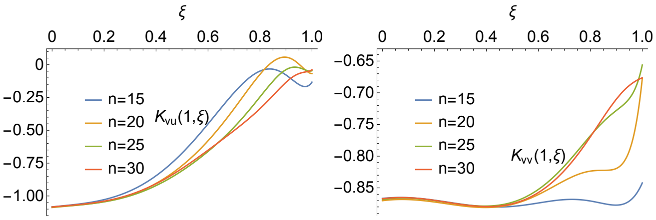

Fig. 6. Hyperbolic 2x2 kernel gains $K_{vu}(1,\xi)$ (left) and $K_{vv}(1,\xi)$ (right) for different orders, showing convergence (Example 4).

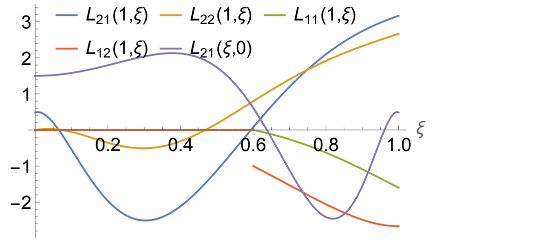

Fig. 7. Gain kernels for Example 5. Note the discontinuous $L_{22}(1,\xi)$ and the piecewise differentiable $L_{11}(1,\xi)$ at the point $\xi = \mu_2/\mu_1 = 0.6$.

$(n+m)$ system[4] , having even a explicit solution for constant coefficients. For simplicity consider only space-varying coefficients for the coupling coefficients but not for the transport speeds (this more challenging case will be considered in future work). Thus, consider the plant

$$v_{1t}(t,x) - \mu_1 v_{1x}(t,x) = \sigma_{12}(x)v_2(t,x), \tag{40}$$

$$v_{2t}(t,x) - \mu_2 v_{2x}(t,x) = \sigma_{21}(x)v_1(t,x), \tag{41}$$

for $x \in (0,1)$, with $\mu_1 > \mu_2 > 0$, with boundary conditions

$$v_1(t,1) = U_1(t), \quad v_2(t,1) = U_2. \tag{42}$$

The objective is to design $U_1(t)$ and $U_2(t)$ so that $v_1(t,0) = \Phi_1(t)$ and $v_2(t,0) = \Phi_2(t)$ for some functions Φ_1, Φ_2 for $t \geq t_M$ where this time is determined from the values of the transport speeds. As argued in [14], the motion planning problem is solved by the inputs

$$U_1 = \Phi_1\left(t + \frac{1}{\mu_1}\right) + \int_0^1 L_{11}(1,\xi)v_1(\xi)d\xi + \int_0^1 L_{12}(1,\xi)v_2(\xi)d\xi, \tag{43}$$

$$U_2 = \Phi_2\left(t + \frac{1}{\mu_2}\right) - \int_0^1 \frac{\mu_1}{\mu_2} L_{21}(\xi,0)\Phi_1\left(t + \frac{1-\xi}{\mu_2}\right)d\xi + \int_0^1 L_{21}(1,\xi)v_1(\xi)d\xi + \int_0^1 L_{22}(1,\xi)v_2(\xi)d\xi, \tag{44}$$

where the kernels L_{11}, L_{12}, L_{21} and L_{22} verify

$$\mu_1\partial_x L_{11}(x,\xi) + \mu_1\partial_\xi L_{11}(x,\xi) = \sigma_{21}(\xi)L_{12}(x,\xi), \tag{45}$$

$$\mu_1\partial_x L_{12}(x,\xi) + \mu_2\partial_\xi L_{12}(x,\xi) = \sigma_{12}(\xi)L_{11}(x,\xi), \tag{46}$$

$$\mu_2\partial_x L_{21}(x,\xi) + \mu_1\partial_\xi L_{21}(x,\xi) = \sigma_{21}(\xi)L_{22}(x,\xi), \tag{47}$$

$$\mu_2\partial_x L_{22}(x,\xi) + \mu_2\partial_\xi L_{22}(x,\xi) = \sigma_{12}(\xi)L_{21}(x,\xi), \tag{48}$$

with boundary conditions

$$L_{11}(x,0) = L_{12}(x,0) = L_{22}(x,0) = 0, \tag{49}$$

$$L_{12}(x,x) = \frac{\sigma_{12}(x)}{\mu_2-\mu_1}, L_{21}(x,x) = \frac{\sigma_{21}(x)}{\mu_1-\mu_2}. \tag{50}$$

It must be observed that L_{12}, differently from the other kernels, possesses two boundary conditions, namely $L_{12}(x,0) = 0$ and $L_{12}(x,x) = \frac{\sigma_{12}(x)}{\mu_2-\mu_1}$. This is solved by observing the fact that hyperbolic equations can accommodate discontinuities along characteristic lines and still be smooth everywhere else [9]. In particular, the characteristic line of L_{12} is found from its PDE (46) to be $\xi = \mu_2/\mu_1 x$. Thus one needs to define L_{12} piecewise, so that $L_{12}(x,\xi) = L_{12}^1(x,\xi)$ if $\xi < \mu_2/\mu_1 x$ and $L_{12}(x,\xi) = L_{12}^2(x,\xi)$ if $\xi > \mu_2/\mu_1 x$, and its value undefined along the characteristic. Now, since L_{12} appears in the L_{11} equation (45), so similarly $L_{11}(x,\xi) = L_{11}^1(x,\xi)$ if $\xi < \mu_2/\mu_1 x$ and $L_{11}(x,\xi) = L_{11}^2(x,\xi)$ if $\xi > \mu_2/\mu_1 x$, but in this case the line $\xi = \mu_2/\mu_1 x$ is not a characteristic line of (45) and therefore the kernel L_{11} has to be continuous along that line, namely, the condition $L_{11}^1(x,\mu_2/\mu_1 x) = L_{11}^2(x,\mu_2/\mu_1 x)$ needs to be added (however in general the resulting L_{11} kernel will not be differentiable along the line so a kink may be visible). In this fashion, there are as many boundary conditions as analytic kernel pieces and the resulting linear system verified by the power series coefficients is well-posed.

Leveraging the proofs of [14] (which are based on the method of characteristics and the solution of integral equations with successive approximation series) and carefully considering the different regions of the triangular domain where the solution is defined according to the previous discussion, the following theorem is stated.

Theorem 4: If there exists $\delta > 0$ such that σ_{12} and σ_{21} are analytic on $\mathcal{D}_{1+\delta}$, and $\mu_1 > \mu_2 > 0$, then there exists piecewise power series solutions for $L_{11}, L_{12}, L_{21}, L_{22}$ which converge and define analytic functions in the polydisc $\mathcal{D}_{1+\delta/2} \times \mathcal{D}_{1+\delta/2}$, that are the unique solution of (45)–(50).

For the particular case $\mu_1 = 0.5$, $\mu_2 = 0.3$, $\sigma_{12}(x) = 0.2+x/3$, $\sigma_2 1(x) = 0.3+y^2/3$, the resulting gain kernels are shown in Fig. 7 for $n = 25$. Note the discontinuous character of $L_{12}(1,\xi)$ and the piecewise differentiable character of $L_{11}(1,\xi)$ with a visible kink at $\xi = \mu_2/\mu_1 = 0.6$. This example does not require very high-order polynomials, but different values of the parameters may produce oscillatory solutions that require them, resulting in extremely slow computation and/or numerical issues. Indeed this is the case in the explicit solutions shown in [8]; reproducing Figures 1-2 of [8] requires n above 40.

IV. Multilayer Timoshenko Beams

This example is extracted from [7], [23]. The physical model of Multilayer Timoshenko beam is a coupled second-order hyperbolic PDE system, namely

$$v_{1,tt} = \eta_1\left(v_{1,xx} + \theta_{1,x}\right) - k_N s_N, \tag{51}$$

$$\zeta_1\theta_{1,tt} = \theta_{1,xx} - \eta_1\left(v_{1,x} + \theta_1\right) + h_1 k_T s_T, \tag{52}$$

$$\beta v_{2,tt} = \eta_2\left(v_{2,xx} + \theta_{2,x}\right) + k_N s_N, \tag{53}$$

$$\zeta_2\theta_{2,tt} = \alpha\theta_{2,xx} - \eta_2\left(v_{2,x} + \theta_2\right) + h_2 k_T s_T, \tag{54}$$

$$s_T = -h_1\theta_1 - h_2\theta_2, \quad s_N = v_1 - v_2, \tag{55}$$

[4]The notation $(n+m)\times(n+m)$ is usual in the backstepping literature and it refers to a hyperbolic 1-D system having n convecting (typically uncontrolled) and m counterconvecting (typically controlled) states.

with boundary conditions

$$v_{1,x}(0,t) = \theta_1(0,t) - \xi_1 v_{1,t}(0,t) - \xi_2 v_1(0,t), \tag{56}$$
$$v_{2,x}(0,t) = \theta_2(0,t) - \xi_3 v_{2,t}(0,t) - \xi_4 v_2(0,t), \tag{57}$$
$$v_{1,x}(1,t) = U_1(t), \quad \theta_{1,x}(0,t) = 0, \tag{58}$$
$$\theta_{1,x}(1,t) = U_2(t), \quad v_{2,x}(1,t) = U_3(t), \tag{59}$$
$$\theta_{2,x}(0,t) = 0, \quad \theta_{2,x}(1,t) = U_4(t), \tag{60}$$

where $U_1(t)$, $U_2(t)$, $U_3(t)$ and $U_4(t)$ being the actuation variable. See [7] for the definition of the coefficients appearing in the plant. Under the assumption that the anti-damping coefficients ξ_1 and ξ_3 appearing in (60) verify $\xi_1 \neq 1/\sqrt{\eta_1}$ and $\xi_3 \neq 1/\sqrt{\eta_2}$, the plant is equivalent, via a Riemann change of coordinates, to a a $(4+4)\times(4+4)$ heterodirectional system of hyperbolic PIDEs, which can be then stabilized by well-established backstepping control design methods. Even if the system has constant coefficients, the stabilizing backstepping design requires the solution of 48 coupled kernels, but the domain should be divided into 7 different areas due to the presence of lines of discontinuities, so potentially up to $16+7\times32 = 240$ power series expansions may need to be computed (the actual number is smaller due to exploiting the specific couplings).

The resulting control law [7] is as follows

$$U_1(t) = \frac{U_p(t)}{\sqrt{\eta_1}} - \frac{v_{1,t}(1,t)}{\sqrt{\eta_1}}, \tag{61}$$
$$U_2(t) = U_r(t) - \sqrt{\zeta_1}\theta_{1,t}(1,t), \tag{62}$$
$$U_3(t) = \frac{U_q(t)}{\sqrt{\eta_2}} - \frac{v_{2,t}(1,t)}{\sqrt{\eta_2}}, \tag{63}$$
$$U_4(t) = \frac{U_s(t)}{\sqrt{\alpha}} - \frac{\sqrt{\zeta_2}\theta_{1,t}(1,t)}{\sqrt{\alpha}}. \tag{64}$$

where $U = [U_p,\, U_r,\, U_q,\, U_s]^T$ is defined as

$$U = \int_0^1 K(1,y)\, Z(y,t) dy + \int_0^1 L(1,y)\, Y(y,t) dy + \Phi(1)X, \tag{65}$$

where the variables Z, Y, X are defined in terms of derivatives and traces of the state (see [7]), and whose gain kernels are the particular values of the 4×4 matrices $K_{ij}, L_{ij}, \Phi_{ij}$ evaluated at $x = 1$. These matrices are found by solving

$$\Sigma K_x + K_y \Sigma = (K - L)\Lambda - \Omega(x)K - F + \int_y^x [K(x,s) - L(x,s)]\, F ds, \tag{66}$$
$$\Sigma L_x - L_y \Sigma = (K - L)\Lambda - \Omega(x)L - F + \int_y^x [K(x,s) - L(x,s)]\, F ds, \tag{67}$$
$$\Phi_x = \Sigma^{-1}\Phi A - \Sigma^{-1}\Pi - \Sigma^{-1}\Omega(x)\Phi + \int_0^x \Sigma^{-1}(K-L)\Pi dy + \Sigma^{-1}L(x,0)\Sigma D, \tag{68}$$

with boundary conditions for K and L,

$$\Sigma L(x,x) + L(x,x)\Sigma = -\Lambda, \tag{69}$$
$$\Sigma K(x,x) - K(x,x)\Sigma = -\Lambda + \Omega(x), \tag{70}$$
$$K(x,0)\Sigma - L(x,0)\Sigma C = \Phi(0)B. \tag{71}$$

See [7] for condition for $\Phi(0)$ required for stabilization.

The kernels equation are solved taking into account that $K_{12}, K_{13}, K_{14}, K_{23}, K_{24}, K_{34}$ are all discontinuous due to the fact that (70) and (71) needs to be simultaneously verified. Thus, they possesses "lines of discontinuity" along which they should be split in several analytic parts by dividing the triangular domain $\mathcal{T}$ into several parts. Specifically, we start by solving $K_{11}, K_{12}, K_{13}, K_{14}$ since they are coupled with each other and independent of other kernel functions. six discontinuous kernel functions means the triangular domain should be divided into seven parts. Therefore, 28 coupled kernel functions should be obtained. Next, $K_{21}, K_{22}, K_{23}, K_{24}$ are solved. Since K_{23}, K_{24} are also discontinuous and they are coupled with K_{11}, we also need to divide the areas to 7 which means we equivalently solve 28 coupled kernel functions as well. This procedure is followed until all the kernels are found.

The coefficients of two-layer Timoshenko beams are selected as $\eta_1 = 10, \eta_2 = 14, \zeta_1 = 0.1, \zeta_2 = 0.11, h_1 = 0.04, h_2 = 0.05, k_T = 10, \alpha = 1.1, \beta = 1.15, \xi_1 = \xi_3 = -1, \xi_2 = \xi_4 = 1$ which comes from [23]. We show the solutions of the gain kernels $K_{ij}(1,y), L_{ij}(1,y), 1 \le i \le 4, 1 \le j \le 4$ in Fig. 8, and we do not state any theorem. The boundary values Φ appearing in control law are

$$\Phi(1) = \begin{bmatrix} 59.6112 & 3.9875 & -46.8085 & -2.5694 \\ 516.792 & 37.7977 & -393.15 & -22.9081 \\ -117.036 & -4.86715 & 126.131 & 3.91881 \\ 24.1778 & 0.594879 & -0.206473 & 14.9768 \end{bmatrix}$$

V. Concluding Remarks

The numerical solution of kernel equations is a challenge that every designer applying backstepping must eventually face; in this paper, we have shown that the power series method stands as an alternative to other approaches with several examples, and we have given the necessary condition for the series to converge by leveraging existing proofs and some basic complex analysis results. Thus, normally one would use the method when a kernel solution is known to exist, but it can also be used to show existence of solutions [36]. It must be noted that the provided code, in the form of easily adaptable Mathematica notebooks, can be adapted to many other cases in a matter of minutes quickly providing solution to kernel equations in multiple settings. Other features of the method include precision, and the ability to produce symbolical kernels depending on parameters which may be invaluable for pursuing e.g. adaptive controllers. A complex multi-kernel, multiple-discontinuity example that requires computing dozens of kernels (which are piecewise defined as hundreds of series expansions) for the stabilizing feedback law of a multilayer Timoshenko beam is given, demonstrating the applicability of the method to state-of-the-art designs.

The main limitation of the method comes from the "complexification" of the coefficients. Thus, one needs analyticity of the coefficients not only in the original PDE domain but also in its complex counterpart. Similarly, the positivity of diffusion or transport coefficients needs to be respected in their complex extension, which may restrict the applicability of the method; that the method fails does not mean that

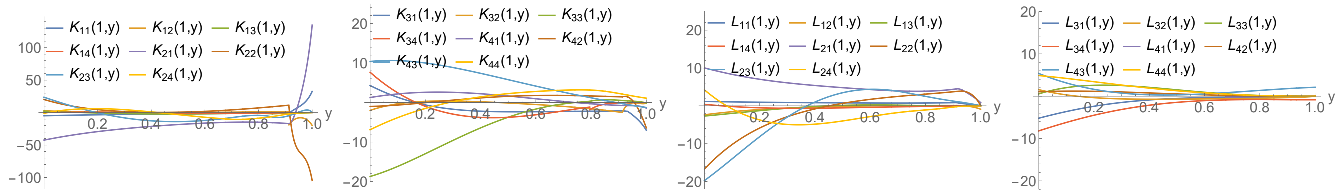

Fig. 8. Solutions of Timoshenko gain kernels $K_{ij}(1, y), L_{ij}(1, y), 1 \le i \le 4, 1 \le j \le 4$ (from left to right).

there is not a solution to the kernel equations, only that it cannot be represented by a power series centered at zero. A second limitation arises from the fact that a power series is not the best possible way to represent some functions, e.g., highly oscillatory ones, and therefore very high orders may be required, considerably slowing down kernel computation.

Future work includes the solution of kernel equations for coupled system with spatially-varying transport speed/diffusion coefficients, which possess lines of discontinuity defined by an analytic differential equation.

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REFERENCES

[1] G. Andrade, R. Vazquez and D. Pagano, "Backstepping stabilization of a linearized ODE-PDE Rijke tube model," *Automatica*, vol. 96, 98-109, 2018.

[2] H. Anfinsen and O.M. Aamo, *Adaptive control of hyperbolic PDEs*, New York: Springer, 2019.

[3] P. Ascencio, A. Astolfi and T. Parisini, "Backstepping PDE design: A convex optimization approach," *IEEE Transactions on Automatic Control*, vol. 63, pp. 1943–1958, 2018.

[4] J. Auriol and D. Bresch-Pietri, "Robust state-feedback stabilization of an underactuated network of interconnected n+ m hyperbolic PDE systems," *Automatica*, vol. 136, 110040, 2022.

[5] L. Camacho-Solorio, R. Vazquez, M. Krstic, "Boundary observers for coupled diffusion-reaction systems with prescribed convergence rate," *Systems and Control Letters*, vol. 134, art. no. 104586, 2020.

[6] Chen, G., Vazquez, R., and Krstic, M.,"Rapid Stabilization of Timoshenko Beam by PDE Backstepping." Preprint available at `https://arxiv.org/abs/2207.04746`.

[7] Chen, G., Vazquez, R., and Krstic, M.,"Backstepping-based Rapid Stabilization of Two-layer Timoshenko Composite Beams," accepted in *IFAC World Congress*, 2023. Preprint available at `http://aero.us.es/rvazquez/2layerTimoshenko.pdf`.

[8] J.-M. Coron, R. Vazquez, M. Krstic, and G. Bastin, "Local Exponential H^2 Stabilization of a 2×2 Quasilinear Hyperbolic System using Backstepping," *SIAM J. Control Optim.*, vol. 51, pp. 2005–2035, 2013.

[9] R. Courant and P. D. Lax, "The propagation of discontinuities in wave motion," *Proceedings of the National Academy of Sciences*, vol. 42, pp.872–876, , 1956.

[10] J.T. Day, "A Runge-Kutta method for the numerical solution of the Goursat problem in hyperbolic partial differential equations," *The Computer Journal*, vol. 9(1), pp. 81–83, 1966.

[11] J. Deutscher and J. Gabriel, "A backstepping approach to output regulation for coupled linear wave-ODE systems," *Automatica*, vol. 123, 109338, 2021.

[12] F. Di Meglio, R. Vazquez, and M. Krstic, "Stabilization of a system of n+1 coupled first-order hyperbolic linear PDEs with a single boundary input," *IEEE Trans. Aut. Contr.*, vol. 58, pp. 3097–3111, 2013.

[13] R.P. Holten, "Generalized Goursat problem," *Pacific Journal of Mathematics*, vol. 12, pp. 207–224, 1962.

[14] L. Hu, F. Di Meglio, R. Vazquez, and M. Krstic,"Control of Homodirectional and General Heterodirectional Linear Coupled Hyperbolic PDEs," *IEEE Transactions on Automatic Control*, vol. 61, No. 10, pp. 3301–3314, 2016.

[15] L. Jadachowsi, T. Meurer, and A. Kugi, "An efficient implementation of backstepping observers for time-varying parabolic PDEs," *IFAC Proceedings Volumes*, vol. 45, pp.798–803, 2012.

[16] A. Jeffrey and T. Taniuti, Nonlinear Wave Propagation, Academic Press, New York, 1964.

[17] S. Kerschbaum and J. Deutscher, "Backstepping control of coupled linear parabolic PDEs with space and time dependent coefficients," *IEEE Trans. Aut. Contr.*, vol. 65(7), pp. 3060–3067, 2019.

[18] M. Krstic and A. Smyshlyaev, *Boundary Control of PDEs*, SIAM, 2008

[19] M. Krstic, *Delay Compensation for nonlinear, Adaptive, and PDE Systems*, Birkhauser, 2009.

[20] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Contr. Lett.*, vol. 57, pp. 750–758, 2008.

[21] J. Lebl, *Tasty Bits of Several Complex Variables: A whirlwind tour of the subject*, available at `https://www.jirka.org/scv/scv.pdf`, 2020.

[22] J. E. Marsden, M. J. Hoffman and T. Marsden, *Basic complex analysis,* Macmillan, 1999.

[23] Lenci, S., and Rega, G., "A Limit Model for the Linear Dynamics of a Two-Layer Beam," *In International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, vol. 55997, 2013, pp V008T13A067.

[24] J. Qi, R. Vazquez and M. Krstic, "Multi-Agent Deployment in 3-D via PDE Control," *IEEE Transactions on Automatic Control*, vol. 60, pp. 891–906, 2015.

[25] A. Smyshlyaev and M. Krstic, "On control design for PDEs with space-dependent diffusivity or time-dependent reactivity," *Automatica*, vol. 41, pp. 1601–1608, 2005.

[26] A. Smyshlyaev, E. Cerpa, and M. Krstic, "Boundary stabilization of a 1-D wave equation with in-domain antidamping," *SIAM J. Control Optim.*, vol. 48, pp. 4014–4031, 2010.

[27] A. Smyshlyaev and M. Krstic, *Adaptive Control of Parabolic PDEs*, Princeton University Press, 2010.

[28] A.G. Sveshnikov and A.N. Tikhonov,*Theory of Functions of a Complex Variable*, MIR, 2010.

[29] R. Vazquez and M. Krstic, *Control of Turbulent and Magnetohydrodynamic Channel Flow*. Birkhauser, 2008.

[30] R. Vazquez and M. Krstic, "Control of 1-D parabolic PDEs with Volterra nonlinearities — Part I: Design," *Automatica*, vol. 44, pp. 2778–2790, 2008.

[31] R. Vazquez and M. Krstic, "Boundary observer for output-feedback stabilization of thermal convection loop," *IEEE Trans. Control Syst. Technol.*, vol.18, pp. 789–797, 2010.

[32] R. Vazquez and M. Krstic. Marcum Q-functions and explicit kernels for stabilization of linear hyperbolic systems with constant coefficients. *Systems & Control Letters* , 68:33–42, 2014

[33] R. Vazquez and M. Krstic, "Boundary control of reaction-diffusion PDEs on balls in spaces of arbitrary dimensions," *ESAIM:Control Optim. Calc. Var.*, vol. 22, No. 4, pp. 1078–1096, 2016.

[34] R. Vazquez, and M. Krstic, "Boundary control of coupled reaction-advection-diffusion systems with spatially-varying coefficients," *IEEE Transactions on Automatic Control*, vol. 62, pp. 2026–2033, 2017.

[35] R. Vazquez, E. Trelat and J.-M. Coron, "Control for fast and stable laminar-to-high-Reynolds-numbers transfer in a 2D navier-Stokes channel flow," *Disc. Cont. Dyn. Syst. Ser. B*, vol. 10, pp. 925–956, 2008.

[36] R. Vazquez, M. Krstic, J. Zhang and J. Qi, "Kernel well-posedness and computation by power series in backstepping output feedback for radially-dependent reaction-diffusion PDEs on multidimensional balls," *Systems & Control Letters*, Vol. 177, pp. 105538, 2023,

[37] Wolfram Research, Inc., *Mathematica*, Version 13.2, 2022.