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Phase locking of linear oscillators with individual parameters

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***Abstract*— A recent paper has shown that linear oscillators can be synchronised only if their parameters are exactly the same. The main reason for this sensitivity lies in the fact that the oscillator network loses energy whenever the oscillators do not follow precisely the same output trajectory. This paper deals with the question how to extend oscillators to make them synchronisable in a practical sense. The oscillators are equipped with an energy source that replaces the energy lost during the synchronisation. It is shown that a power supply rate exists such that oscillators with different eigenfrequencies can phase lock, which means that they follow the same sinusoidal trajectory with some phase gap. For two coupled oscillators explicit relations for the frequency and the phase shift of the synchronous behaviour are derived.**

I. Practical synchronisation

The synchronisation of multi-agent systems has been investigated as an important problem of cooperative control. In the structure of Fig. 1, a networked controller C should make N agents P_i follow a common trajectory $y_{\rm s}(t)$:

$$\lim_{t\to\infty} |y_i(t) - y_{\rm s}(t)| = 0, \quad i = 1, 2, ..., N. \qquad (1)$$

Then the system is said to be *asymptotically synchronised*.

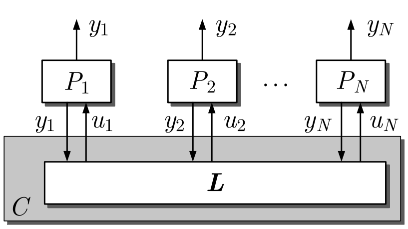

Fig. 1: Autonomous agents with networked controller

The majority of the papers on this subject has considered agents with identical dynamics and derived conditions on the networked controller to satisfy the relation (1) for arbitrary initial states $\boldsymbol{x}_{i0}$ of the agents. However, the recent paper [8] has shown that the synchronisation of linear oscillators by diffusive couplings is not at all robust with respect to parameter deviations. If there are two oscillators with different eigenfrequencies (non-uniform oscillators), the overall system is asymptotically stable and not synchronised for any non-trivial trajectory $y_{\rm s}(t)$.

The main reason for this sensitivity is the fact that the process of synchronisation consumes energy. If the oscillator agents do not have the same eigenfrequency, they cannot follow the same sinusoidal trajectory and lose energy until they have reached their equilibrium state.

This paper proposes a way to extend linear oscillators with different eigenfrequencies to make them synchronisable. The oscillators are equipped with a component that injects energy persistently. The requirement of asymptotic synchronisation is released to *practical synchronisation*, which requires the agent outputs to follow for $t \geq \bar{t}$ a common trajectory $y_{\rm s}(t)$ with some tolerance ε:

$$|y_i(t) - y_{\rm s}(t)| \leq \varepsilon, \quad t \geq \bar{t}, \;\; i = 1, 2, ..., N. \qquad (2)$$

The oscillators considered in the sequel will satisfy this requirement in their phase-locking behaviour. The main result shows that there exists an appropriate power supply rate such that a network of non-uniform oscillators is practically synchronised (Theorem 2). Explicit relations are derived for the frequency and the phase shift of the synchronous behaviour of two phase-locked oscillators.

Three phenomena in the synchronisation of agents with parameter deviations. Multi-agent systems with parameter deviations may exhibit one of the three behaviours of Fig. 2. Asymptotic synchronisation is depicted in the upper subplot. Literature has shown that this behaviour appears only if the parameter deviations do not touch the common dynamics of the agents generating the trajectory $y_{\rm s}(t)$.

Oscillator agents with different eigenfrequencies are not synchronisable. As shown in [8], networks of non-uniform oscillators are asymptotically stable and behave like the middle plot in Fig. 2. This paper investigates how such oscillators can be made practically synchronised to satisfy the weaker form (2) of synchronisation depicted in the lowest plot.

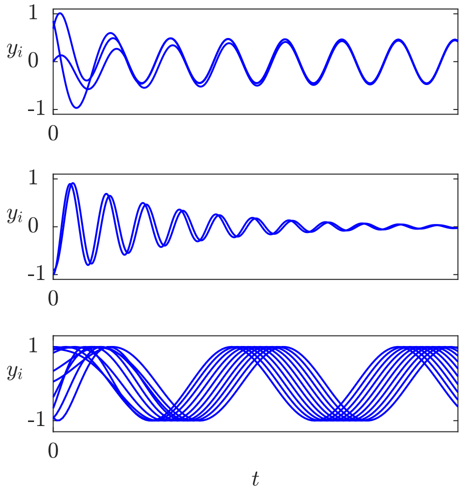

Fig. 2: Three phenomena encountered in the synchronisation of oscillator agents with parameter deviations: asymptotic synchronisation (top), asymptotic stability (middle), and phase locking as a form of practical synchronisation (bottom)

Literature survey. Most of the papers on the synchronisation of multi-agent systems have considered agents with identical dynamics (for a survey cf. [9]). As parameter deviations bring about agents with individual dynamics, asymptotic synchronisation necessitates that the agents satisfy the Internal-Reference Principle derived in [3], [7], [15]. A detailed analysis shows that most of the papers on the robustness of synchronisation have been concerned with agents that satisfy this principle for all parameter deviations. In particular, [13] has proved the robustness for agents with additive perturbations, whereas [12], [18] demonstrated the robustness for double-integrator systems. [16] has considered agents with an uncertain memoryless nonlinearity. In contrast, if the parameter deviations affect the diffusive couplings, the overall system generally reacts with a large robustness [6].

Practical synchronisation [5], [17] allows for a bounded deviation of the agent outputs, for which the requirement (2) is satisfied. [10] has shown that nonlinear agents with individual dynamics can be synchronised with ε arbitrarily small if the networked controller uses a sufficiently high feedback gain. For Kuramoto oscillators, the phase-locking behaviour has been extensively studied in physics as a form of practical synchronisation [1].

To the knowledge of the author, [8] and [12] are the only publications that point to the fact that arbitrarily small parameter deviations of oscillator agents may turn a synchronised system into an asymptotically stable system, which is the motivation of this paper.

Aims and results of this paper. The short summary of the results of [8] on the sensitivity of oscillator networks with respect to parameter variations in Section II shows that non-uniform oscillators do not have any synchronous behaviour at all. Therefore, the paper introduces in Section III a class of oscillators that have an internal energy source. The new freedom can be used to adapt the power supply rate to the power loss due to the synchronising couplings. Even unstable oscillators with parameter deviations may become practically synchronised, as Section IV proves. The investigations concentrate on two coupled oscillators to shorten the models and to get explicit solutions.

II. The Loss of Synchrony of Linear Oscillator Networks due to Parameter Deviations

To summarise results of [8] as the starting point of the later investigations, consider the coupled harmonic oscillators shown in Fig. 3. This network consists of the agents

$$P_i: \begin{cases} \dot{\boldsymbol{x}}_i(t) &= \boldsymbol{A}_i \boldsymbol{x}_i(t) + \boldsymbol{b}_i u_i(t), \quad \boldsymbol{x}_i(0) = \boldsymbol{x}_{i0} \\ y_i(t) &= \boldsymbol{c}_i^{\mathrm{T}} \boldsymbol{x}_i(t) \end{cases} \tag{3}$$

with the state vector $\boldsymbol{x}_i \in \mathrm{IR}^{n_i}$, the scalar input u_i and the scalar output y_i and the parameters

$$\boldsymbol{A}_i = \begin{pmatrix} 0 & -\omega_i^2 \\ 1 & 0 \end{pmatrix}, \ \boldsymbol{b}_i = \begin{pmatrix} 0 \\ b_{i2} \end{pmatrix} \text{ and } \boldsymbol{c}_i^{\mathrm{T}} = (0 \quad 1) \tag{4}$$

$(i = 1, 2, ..., N)$, where $\omega_i = \frac{1}{\sqrt{L_i C_i}}$ denotes the eigenfrequency (natural frequency) of the i-th oscillator and $b_{i2} > 0$ holds. The resistor network introduces the couplings

$$\boldsymbol{u}(t) = -\frac{1}{R} \boldsymbol{L} \boldsymbol{y}(t), \tag{5}$$

which for the two oscillators in Fig. 3 are described by

$$C: \quad \boldsymbol{u}(t) = -\frac{1}{R} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \boldsymbol{y}(t).$$

The conductance $k = \frac{1}{R}$ acts as the coupling strength. For more than two oscillators the resistor network induces a symmetric Laplacian matrix $\boldsymbol{L}$ in eqn. (5).

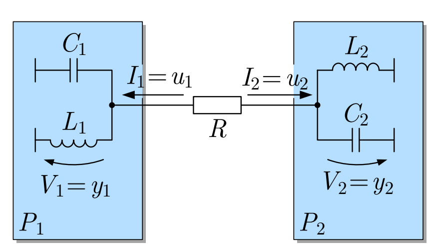

Fig. 3: Two coupled oscillators with individual parameters

There are two important results of [8], which are repeated here without proof.

Theorem 1: **(Properties of linear oscillator networks)** Consider a connected oscillator network (3) – (5).

- If all oscillators have the same eigenfrequency $\omega_1 = \omega_2 = ... = \omega_N = \omega$ they are said to be uniform and the network asymptotically synchronises for arbitrary parameters R and b_{i2}, $(i = 1, 2, ..., N)$.
- If there is a pair of oscillators with $\omega_i \neq \omega_j$, $(i \neq j)$, the network is said to be non-uniform and it is asymptotically stable.

Hence, uniform oscillator networks exhibit the behaviour shown in the top part of Fig. 2, whereas non-uniform oscillators eventually approach a stable equilibrium point as in the middle part of the figure. The reason for this behaviour is the fact that the process of synchronisation consumes the energy that the overall system loses over the resistive coupling network. Therefore, to make practical synchronisation possible, the oscillators have to be equipped with a power source.

III. Oscillators with internal power supply

The aim of this paper is to enable the oscillator network to exhibit some form of synchronisation even if some components have different eigenfrequencies. Since asymptotic synchronisation is impossible due to the violation of the Internal-Reference Principle, practical synchronisation is investigated, which for oscillator networks appears as *phase locking*.

The new idea is to introduce a power source into each of the oscillators, which should replace the power lost for synchronisation. The power is emitted by a voltage-controlled current source symbolised by the circle with the arrow in Fig. 4.

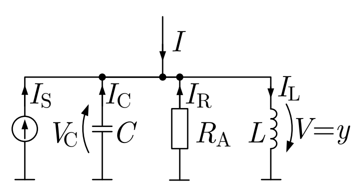

Fig. 4: Oscillator with voltage-controlled current source

If the current source replaces the energy consumed by the internal resistor $R_{\rm A}$, the oscillator behaves like the undamped oscillator considered in Section II. The question to be answered in the next part of the paper is how to parameterise the current source in order not only to replace the energy consumed by $R_{\rm A}$, but also to replace the energy consumed by the resistive couplings to make the oscillators phase lock.

The oscillator depicted in Fig. 4 has the following model (with the agent index i omitted). The capacitor, the inductor and the resistor have the current-voltage relations

$$I_{\rm C}(t) = C\dot{V}_{\rm C}(t) \quad (6)$$
$$V(t) = L\dot{I}_{\rm L}(t) \quad (7)$$
$$I_{\rm R}(t) = \frac{1}{R_{\rm A}}V_{\rm C}(t). \quad (8)$$

The voltage-controlled current source generates the current $I_{\rm S}(t)$ that is proportional to the voltage $V_{\rm C}$

$$I_{\rm S}(t) = \alpha V_{\rm C}(t). \quad (9)$$

Kirchhoff's laws yield

$$V_{\rm C}(t) + V(t) = 0 \quad (10)$$
$$I_{\rm S}(t) + I_{\rm C}(t) + I(t) + I_{\rm R}(t) - I_{\rm L}(t) = 0. \quad (11)$$

The current $I(t)$ is introduced to represent the couplings.

From eqns. (6) and (11) one obtains

$$\dot{V}_{\rm C}(t) = \frac{1}{C}I_{\rm L}(t) - \frac{1}{C}I(t) - \frac{1}{C}I_{\rm R}(t) - \frac{1}{C}I_{\rm S}(t)$$

and with eqns. (8) and (9) the first state equation

$$\dot{V}_{\rm C}(t) = \left(-\frac{\alpha}{C} - \frac{1}{R_{\rm A}C}\right)V_{\rm C}(t) + \frac{1}{C}I_{\rm L}(t) - \frac{1}{C}I(t).$$

Equations (7) and (10) lead to the second state equation

$$\dot{I}_{\rm L}(t) = -\frac{1}{L}V_{\rm C}(t). \quad (12)$$

In summary, the circuit has the model (3) with

$$\boldsymbol{A} = \begin{pmatrix} 2\delta & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} -\frac{1}{C} \\ 0 \end{pmatrix}, \quad \boldsymbol{c}^{\rm T} = (-1 \quad 0) \quad (13)$$
$$\boldsymbol{x} = \begin{pmatrix} V_{\rm C}(t) \\ I_{\rm L}(t) \end{pmatrix}, \quad u(t) = I(t), \quad y(t) = -V_{\rm C}(t)$$

and

$$\delta = -\frac{1}{2}\left(\frac{\alpha}{C} + \frac{1}{R_{\rm A}C}\right). \quad (14)$$

The current source replaces the energy loss of the resistor $R_{\rm A}$ if its parameter is chosen to be $\alpha = -1/R_{\rm A}$. Then the isolated oscillator $(I(t) = 0)$ maintains a steady oscillation. In physics, such circuits are called *self-sustained oscillators* [11].

The following investigations will be made for

$$\alpha < -\frac{1}{R_{\rm A}} \quad (15)$$

for which the oscillator (3), (13) is unstable because the persistent energy input exceeds the energy loss in the resistor R_A and its output has an exponentially increasing magnitude. The characteristic polynomial

$$\det\begin{pmatrix} \lambda - 2\delta & -\frac{1}{C} \\ \frac{1}{L} & \lambda \end{pmatrix} = \lambda^2 - 2\delta\lambda + \frac{1}{LC} = 0$$

leads to the eigenvalues

$$\lambda_{1/2} = \delta \pm \sqrt{\delta^2 - \frac{1}{LC}},$$

which have positive real part δ if the inequality (15) is satisfied. The eigenvalues are complex conjugate for $\delta^2 < \frac{1}{LC}$, i. e. if α satisfies the relation

$$\left(\alpha + \frac{1}{R_\mathrm{A}}\right)^2 < \frac{4C}{L}.$$

The eigenvalues $\lambda_{1/2}$ are depicted in the root locus of Fig. 5 in dependence upon δ for $L = 100\,\mathrm{mH}$ and $C = 10\,\mu\mathrm{F}$. For $\delta = 600\,\frac{1}{\mathrm{s}}$ the eigenvalues are marked by Δ to characterise the dependency. The double eigenvalue $\lambda_{1/2} = 1000\,\frac{\mathrm{rad}}{\mathrm{s}}$ marked by $\square$ is obtained for

$$\delta = \frac{1}{\sqrt{LC}} = 1000\,\frac{1}{\mathrm{s}}.$$

The parameter δ is called the *power supply rate*. It fixes the parameter α of the current source due to eqn. (14).

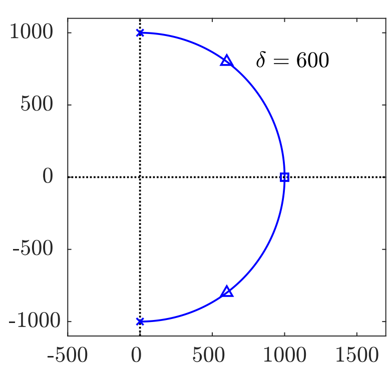

Fig. 5: Eigenvalues of an isolated oscillator in dependence upon δ

IV. Phase locking of unstable oscillators

A. Coupled oscillators

To emphasise the main ideas, this section is restricted to two oscillators. The diffusive couplings

$$u_1(t) = \frac{1}{R}(y_1(t) - y_2(t)) = -u_2(t)$$

lead to the overall system

$$\bar{\Sigma}:\ \left\{\begin{array}{rcl} \dot{\boldsymbol{x}}(t) & = & \left(\begin{array}{cc:cc} 2\delta - \frac{1}{RC_1} & \frac{1}{C_1} & \frac{1}{RC_1} & 0 \\ -\frac{1}{L_1} & 0 & 0 & 0 \\ \hdashline \frac{1}{RC_2} & 0 & 2\delta - \frac{1}{RC_2} & \frac{1}{C_2} \\ 0 & 0 & -\frac{1}{L_2} & 0 \end{array}\right)\boldsymbol{x}(t) \\ \boldsymbol{y}(t) & = & \begin{pmatrix} \boldsymbol{c}^\mathrm{T} & \boldsymbol{0}^\mathrm{T} \\ \boldsymbol{0}^\mathrm{T} & \boldsymbol{c}^\mathrm{T} \end{pmatrix}\boldsymbol{x}(t) \end{array}\right. \tag{16}$$

with $\boldsymbol{x}(t) = (\boldsymbol{x}_1^\mathrm{T}(t), \boldsymbol{x}_2^\mathrm{T}(t))^\mathrm{T}$ and $\boldsymbol{y}(t) = (y_1(t), y_2(t))^\mathrm{T}$. The two oscillators have individual parameters L_i and C_i, but a common power supply rate δ.

B. Power balance of two coupled oscillators

The energy

$$v_i(\boldsymbol{x}_i(t)) = \frac{1}{2}C_i x_{i1}^2(t) + \frac{1}{2}L_i x_{i2}^2(t)$$

of an isolated oscillator ($u_i(t) = 0$) is monotonically increasing if the current source satisfies the condition (15):

$$\begin{aligned} \dot{v}_i(t) &= C_i x_{i1}(t)\dot{x}_{i1}(t) + L_i x_{i2}(t)\dot{x}_{i2}(t) \\ &= 2\delta C_i x_{i1}^2(t) \geq 0. \end{aligned}$$

This result confirms that oscillators with $\delta > 0$ have an ever increasing magnitude of their oscillations.

For two coupled oscillators the energy balance is as follows:

$$\begin{aligned} & v(\boldsymbol{x}(t)) = v_1(\boldsymbol{x}_1(t)) + v_2(\boldsymbol{x}_2(t)) \\ & = \frac{1}{2}C_1 x_{11}^2(t) + \frac{1}{2}L_1 x_{12}^2(t) + \frac{1}{2}C_2 x_{21}^2(t) + \frac{1}{2}L_2 x_{22}^2(t) \end{aligned}$$

$$\begin{aligned} & \dot{v}(t) = \\ & C_1 x_{11}\dot{x}_{11} + L_1 x_{12}\dot{x}_{12} + \frac{1}{2}C_2 x_{21}\dot{x}_{21} + \frac{1}{2}L_2 x_{22} \cot x_{22} \\ & = -\frac{1}{R}(x_{11}(t) - x_{21}(t))^2 + 2\delta C_1 x_{11}^2(t) + 2\delta C_2 x_{21}^2(t). \end{aligned}$$

The first term

$$\dot{v}_\mathrm{neg}(t) = -\frac{1}{R}(y_2(t) - y_1(t))^2 \tag{17}$$

is non-positive and describes the energy consumption for synchronising the oscillators. The second term

$$\dot{v}_\mathrm{pos}(t) = 2\delta C_1 y_1^2(t) + 2\delta C_2 y_2^2(t) \tag{18}$$

is non-negative and represents the power input into the oscillators by the current sources. These terms play against one another. A sinusoidal trajectory with constant magnitude is reached if the power balance over one period of the oscillation is zero. Then the power input precisely compensates the power loss for the synchronisation.

Figure 6 shows an example. Two oscillators with $C_1 = C_2 = 10\,\mu\mathrm{F}$, $L_1 = 100\,\mathrm{mH}$, $L_2 = 120\,\mathrm{mH}$ and $\delta = 0.5$

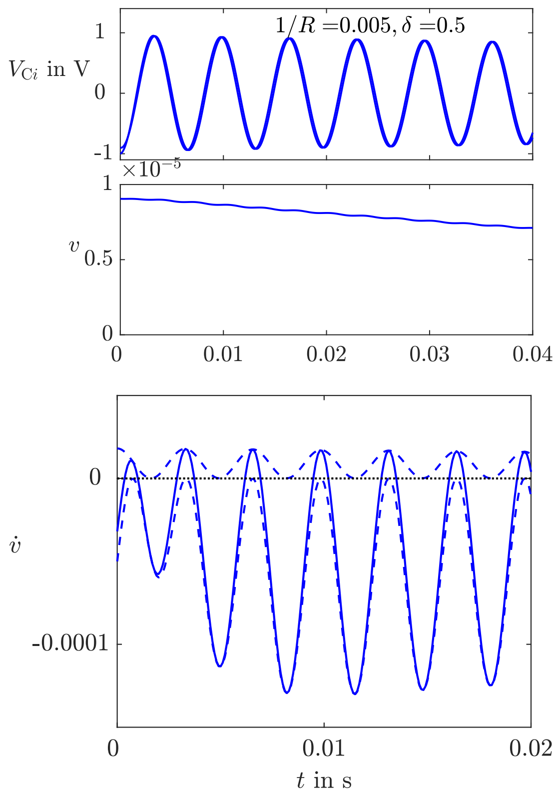

Fig. 6: Behaviour of the coupled oscillators (top) and change of the internal energy (bottom)

are coupled and start their movement with similar initial states $\boldsymbol{x}_{10} = (1, 0)^{\mathrm{T}}$ and $\boldsymbol{x}_{20} = (0.9, 0)^{\mathrm{T}}$. They move along nearly the same trajectory, but lose energy as the curve $v(t)$ in the second subplot shows. The energy change is depicted on the bottom part of the figure. The solid line $\dot{v}$ is the sum of the two dashed lines, where the curve below the time axis is $\dot{v}_{\mathrm{neg}}(t)$ and the curve above the axis $\dot{v}_{\mathrm{pos}}(t)$. On the long run, the internal energy $v(t)$ decreases, which shows that for $\delta = 0.5$ the power input into the two oscillators is too weak to get phase locking. The coupled oscillators are asymptotically stable.

C. Determination of the required power input

This section develops a method to determine the power supply rate δ such that the coupled oscillators receive precisely that amount of power that is necessary to force them to follow a common trajectory with some phase shift. The idea is explained by using Fig. 7. Two coupled oscillators have two eigenvalue pairs, one that should lie on the imaginary axis of the complex plane to define a sinusoidal synchronous trajectory and another one with negative real part that characterises the transient behaviour. The following analysis concerns the first eigenvalue pair.

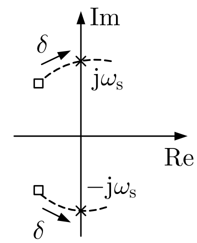

Fig. 7: Eigenvalues of the coupled oscillators in dependence upon δ

For $\delta = 0$ the eigenvalues depicted by the small squares in Fig. 7 have negative real part which lead to the asymptotically stable behaviour shown in the middle of Fig. 2. By increasing δ these eigenvalues are moved towards the imaginary axis, where they should be placed as the two crosses at the points $\pm\mathrm{j}\omega_{\mathrm{s}}$. The problem is how to get the corresponding value of δ, which is denoted by δ_{c}.

Determination of the critical power supply rate δ_{c}. To find an analytical expression for δ_{c}, the assumption

$$C_1 = C_2 = C \tag{19}$$

is made and the abbreviation $\beta = \frac{1}{L_1 C} + \frac{1}{L_2 C}$ introduced. The model (16) has the characteristic polynomial

$$p(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

with the coefficients

$$\begin{aligned} a_0 &= \frac{1}{L_1 L_2 C^2} \\ a_1 &= -2\delta\beta + \frac{\beta}{RC} \qquad (20) \\ a_2 &= 4\delta^2 - \frac{4\delta}{RC} + \beta \\ a_3 &= -4\delta\frac{1}{RC} + \frac{1}{RC}. \qquad (21) \end{aligned}$$

When applying the Routh-Hurwitz stability criterion, the principal minor

$$D_3 = a_1 a_2 a_3 - a_1^2 a_4 - a_3^2 a_0 \overset{!}{=} 0$$

has to be zero for the required eigenvalue pair on the imaginary axis. One gets $a_2 a_3 - a_1 a_4 = -2\beta\delta + \frac{\beta}{RC}$ and $D_3 = 0$ if and only if

$$\begin{aligned} &-16RC\delta^3 + 24\delta^2 - \left(\frac{8}{RC} + 2RC\beta - \frac{8R}{\beta L_1 L_2 C}\right)\delta \\ &+ \beta - \frac{4}{\beta L_1 L_2 C^2} = 0 \end{aligned} \tag{22}$$

holds. The critical power supply rate δ_{c} is a positive zero of this polynomial.

> *Theorem 2:* **(Phase-locking behaviour)**
> For the power supply rate δ_c satisfying eqn. (22) the two coupled oscillators have an eigenvalue pair on the imaginary axis and exhibit phase-locking behaviour.

For $C = 10\,\mu\mathrm{F}$, $L_1 = 100\,\mathrm{mH}$, $L_2 = 120\,\mathrm{mH}$ and $R = 200\,\Omega$ the polynomial

$$-0.03\delta^3 + 24\delta^2 - 4060\delta + 15152 = 0$$

with $\beta = 1{,}833{,}333$ leads to three roots of the polynomial including the real positive solution $\delta_\mathrm{c} = 3.817$. Figure 8 depicts the behaviour of the two coupled oscillators for this power supply rate. The power required to bring the two oscillator outputs near together is precisely replaced by the power input from the voltage-controlled current source. The lower part of the figure illustrates that the area under the positive and under the negative parts $\dot{v}_\mathrm{neg}(t)$ or $\dot{v}_\mathrm{pos}(t)$, respectively, are identical. The two oscillators phase-lock.

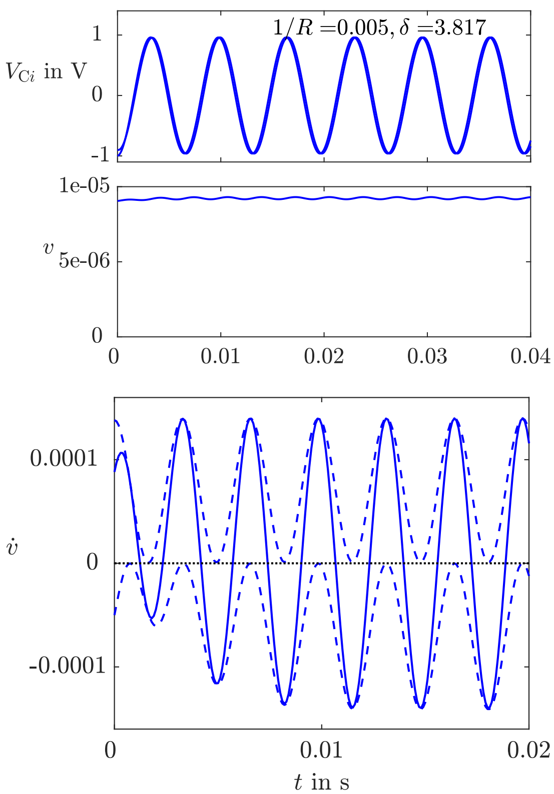

Fig. 8: Behaviour of the coupled oscillators (top) and change of the internal energy (bottom) for $\delta_\mathrm{c} = 3.817$

For the given parameters, the coupled oscillators have the eigenvalues

$$\lambda_{1/2} = \pm \mathrm{j}957.4 \quad \text{and} \quad \lambda_{3/4} = -492.32 \pm \mathrm{j}816.5. \tag{23}$$

That is, the trajectory around which the two oscillators phase lock has the frequency $957.4\,\frac{\mathrm{rad}}{\mathrm{s}}$. For comparison, the eigenfrequencies of the two oscillators are

$$\omega_1 = \frac{1}{\sqrt{L_1 C_1}} = 1000 \quad \text{and} \quad \omega_2 = \frac{1}{\sqrt{L_2 C_2}} = 912.9.$$

To shift these eigenfrequencies of the isolated oscillators towards the common frequency of the phase-locking trajectory requires a persistent activity of the coupling network, which is responsible for the energy consumption.

D. Frequency of the synchronous trajectory

Although the synchronous trajectory

$$y_\mathrm{s}(t) = \bar{V} \sin(\omega_\mathrm{s} t + \phi) \tag{24}$$

is not asymptotically followed by both oscillators, the question arises what is its frequency ω_s. For the critical power supply rate δ_c, the purely imaginary pair $\pm \mathrm{j}\omega_\mathrm{s}$ of eigenvalues can be obtained from the coefficients a_1 and a_3 of the characteristic polynomial as $\omega_\mathrm{s}^2 = a_1/a_3$. With eqns. (20) and (21) and the assumption (19) one gets

$$\omega_\mathrm{s}^2 \quad = \quad \frac{-2\delta\beta + \frac{1}{RC^2}\beta}{-4\delta + \frac{2}{RC}} = \frac{1}{2}\beta$$

and

$$\omega_\mathrm{s} = \sqrt{\frac{1}{2}\left(\omega_1^2 + \omega_2^2\right)}. \tag{25}$$

For the example oscillators, this result coincides with the first eigenvalue pair in eqn. (23).

Lemma 1: **(Synchronous frequency)** Two oscillators with different parameters and the power supply rate δ_c phase-lock on a sinusoidal trajectory (24) with the frequency ω_s given in eqn. (25).

It is interesting to see that the common frequency is not the arithmetic mean of the individual frequencies.

E. Phase shift in the phase-locking behaviour

Phase-locking means that the two oscillator outputs have the same magnitude and frequency, but a phase shift $\Delta\phi = \phi_1 - \phi_2$. After the transient behaviour has vanished asymptotically, the outputs are represented by

$$y_1(t) \quad = \quad Y \sin(\omega_\mathrm{s} t + \phi_1) \tag{26}$$
$$y_2(t) \quad = \quad Y \sin(\omega_\mathrm{s} t + \phi_2). \tag{27}$$

The power loss and the power input over one period $T = \frac{2\pi}{\omega_\mathrm{s}}$ of the oscillation are equal

$$\int_{t_1}^{t_1+T} \dot{v}_\mathrm{neg}(t)\,\mathrm{d}t + \int_{t_1}^{t_1+T} \dot{v}_\mathrm{pos}(t)\,\mathrm{d}t = 0. \tag{28}$$

With eqns. (17) and (18) this balance can be used to get

$$\phi_1 - \phi_2 = 2 \arcsin \sqrt{\delta RC}. \tag{29}$$

Lemma 2: **(Phase shift)** Phase-locking oscillators have the property of practical synchronisation with the phase shift (29).

For the example with $\delta = 3.817$, $C = 10\,\mu\mathrm{F}$ and $R = 200\,\Omega$ shown in Fig. 8 the phase shift is

$$\begin{aligned} \Delta\phi &= 2 \arcsin \sqrt{3.817 \cdot 0.00001 \cdot 200} \\ &= 0.176\,\mathrm{rad} \quad = 10^o. \end{aligned}$$

V. Conclusions

The idea of this paper was to extend linear oscillators by internal power sources to overcome the impossibility to synchronise linear oscillator networks with parameter variations. Theorem 2 shows that for a specific power supply rate unstable oscillators with different parameters are made to phase-lock, which is a behaviour of practical synchronisation.

From a more general viewpoint, there are several fundamental differences in the synchronisation of identical agents and the synchronisation of agents with parameter deviations. Whereas the interactions among the agents deceases for identical agents, they remain active for non-identical agents. This activity is associated with energy loss in the system that has to be compensated for by persistent energy inputs.

The presented results are restricted to two coupled oscillators, but they pave the way to the analysis of oscillator networks with more agents.

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