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Some relations between different stability notions for discrete-time systems with inputs*

Sergey Dashkovskiy and Andreas Schroll[1]

***Abstract*— In this paper we discuss several ISS-like properties for infinite-dimensional discrete-time systems with inputs. Characterizations of these properties are developed and illustrative examples are provided. Relations between these properties are discussed.**

I. INTRODUCTION

In many applications the data describing the state of the art is updated in a discrete time (think on daily update of Covid-numbers, stock index or opinion of multiple agents). Such dynamics can be modelled by discrete-time systems. Modelling evolution processes of large multi-agent systems, with variable and large number of agents it is sometimes useful to consider (to overapproximate) them as infinite-dimensional systems [1], [2].

It is often of interest to study stability properties of an equilibrium point and in case of external perturbations also to ask for robust stability. The notion of input-to-state stability (ISS) is very fruitful in the latter case. It was introduced originally for continuous-time systems [3] and then also applied to dynamical systems evolving on a discrete-time [4], [5]. Other equivalent characterizations to ISS or weaker properties were introduced for both types of dynamical systems. Different characterizations of these properties were developed during the last two decades. In the case of finite-dimensional systems the ISS-framework is very well-developed and used in applications [6]. For systems with infinite-dimensional states the most of related results appeared during the last decade see, e.g. [7], [8], [9], [2], [10], but many open problems are still under investigation [11], [12].

Several works related to the ISS framework for infinite-dimensional discrete-time systems appeared recently [13], [14], [15], [16]. In this paper we are going to contribute more into this research direction.

The next chapter introduces the considered class of systems and several related notions, see e.g. [17] for the continuous-time case or [16] for the discrete-time case. Some preliminary results are collected in Chapter III. Main results are given in Chapter IV. Several illustrative examples are provided in Chapter V. Finally, Chapter VI concludes the paper.

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[1]Both authors are with Institute of Mathematics, University of Würzburg, Emil-Fischer-Str. 40, 97074 Würzburg, Germany Email: surname.name@uni-wuerzburg.de

II. NOTATION AND BASIC NOTIONS

The nonnegative integers and reals are denoted by $\mathbb{Z}_+$ and $\mathbb{R}_+$, respectively. The open ball of radius r around 0 in a normed vector space X is denoted by $B_r := \{x \in X \colon \|x\|_X < r\}$ (or $B_{r,X}$) and its closure is denoted by $\overline{B_r}$. We denote the sequence spaces of real numbers by

$$\ell^p(\mathbb{N},\mathbb{R}) := \{(x_n)_n \subset \mathbb{R} \colon \|x_n\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p\right)^{\frac{1}{p}} < \infty\}$$

if $1 \leq p < \infty$ and

$$\ell^\infty(\mathbb{N},\mathbb{R}) := \{(x_n)_n \subset \mathbb{R} \colon \|x_n\|_\infty = \sup\{|x_n| \colon n \in \mathbb{N}\} < \infty\}.$$

By $\mathscr{K}, \mathscr{K}_\infty, \mathscr{K}\mathscr{L}$ we denote the classes of comparison functions, see, e.g., [12], [18] or [19] for definitions.

Definition 1: Let $(X, \|\cdot\|_X)$, $(U, \|\cdot\|_U)$ be Banach spaces and $\mathscr{U} \subset \{f \colon \mathbb{Z}_+ \to U\}$ be a vector space equipped with the norm $\|u(\cdot)\|_{\mathscr{U}} := \sup_{k \in \mathbb{Z}_+} \|u(k)\|_U$ which satisfies the axiom of shift invariance: For all $u \in \mathscr{U}$ and all $r \in \mathbb{Z}_+$ we have $u(\cdot + r) \in \mathscr{U}$ with $\|u\|_{\mathscr{U}} \geq \|u(\cdot + r)\|_{\mathscr{U}}$.
Now consider a map $\varphi \colon \mathbb{Z}_+ \times X \times \mathscr{U} \to X$. The triple $\Sigma = (X, \mathscr{U}, \varphi)$ is called a discrete-time control system if the following properties hold:

(Σ1) Identity property: For every $(x,u) \in X \times \mathscr{U}$ it holds that $\varphi(0,x,u) = x$.

(Σ2) Causality: For every $(k,x,u) \in \mathbb{Z}_+ \times X \times \mathscr{U}$, for every $\tilde{u} \in \mathscr{U}$ with $u|_{[0,k]} = \tilde{u}|_{[0,k]}$ it holds that $\varphi(k,x,u) = \varphi(k,x,\tilde{u})$.

(Σ3) Cocycle property: For all $k,h \in \mathbb{Z}_+$, for all $x \in X$, $u \in \mathscr{U}$ we have

$$\varphi(h, \varphi(k,x,u), u(k+\cdot)) = \varphi(k+h,x,u).$$

The space X is called the state space, $\mathscr{U}$ the input space and φ the transition map.
For a given mapping $f \colon X \times U \to X$ a discrete-time control system can be described by the recursion

$$x(k+1) = f(x(k),u(k)), \quad x(0) = x_0. \tag{1}$$

Σ given by (1) is forward complete, that is for any $x_0 \in X$ and $u \in \mathscr{U}$ we have that $x(k) \in X$ for all $k \in \mathbb{Z}_+$. This is guaranteed by the requirement $f \colon X \times U \to X$. Recall that for continuous-time systems like $\dot{x} = f(x,u)$ the forward completeness is not always satisfied. We always assume that Σ has an equilibrium at the origin that is $f(0,0) = 0$. The forward completeness and well-posedness may not hold without the requirement $f \colon X \times U \to X$ as the next example demonstrates:

Example 1: For $X = \ell^\infty(\mathbb{N},\mathbb{R})$, $U = \ell^\infty(\mathbb{N},\mathbb{R})$ and $n \in \mathbb{N}$ consider

$$x_n(k+1) = \prod_{i=1}^{n}(x_i(k)\cdot u_i(k)), \quad k \in \mathbb{Z}_+ \tag{2}$$

where $x(0) = x_0$. Now, choose for example $x_n(0) = 2$ and $u_n(0) = 1\ \forall n \in \mathbb{N}$. By our choice of $x(0)$ and $u(0)$ we get

$$x_n(1) = \prod_{i=1}^{n}(x_i(0)\cdot u_i(0)) = 2^n$$

and $\|x(1)\|_X = \infty$. Therefore, the system has finite escape time with $x(1) \notin X$.

The following definition describes some of the properties used in this work.

Definition 2: We say that Σ defined by (1) is continuous at the equilibrium point (CEP), if 0 is an equilibrium and $\forall \varepsilon > 0,\ h \geq 1$ there is a $\delta = \delta(\varepsilon, h) > 0$ such that

$$\begin{aligned} k \in [0,h]\cap\mathbb{Z}_+ \wedge x \in \overline{B_\delta} \wedge u \in \overline{B_{\delta,\mathscr{U}}} \\ \implies \|\varphi(k,x,u)\|_X \leq \varepsilon. \end{aligned} \tag{3}$$

Σ has bounded reachability sets (BRS), if $\forall C > 0,\ \tau \in \mathbb{N}$ it holds that

$$\sup\{\|\varphi(k,x,u)\|_X : x \in \overline{B_C}, u \in \overline{B_{C,\mathscr{U}}}, k = 0,\ldots,\tau\} < \infty. \tag{4}$$

Σ is called uniformly locally stable (ULS), if $\exists \sigma,\gamma \in \mathscr{K}_\infty$, $r > 0$ such that for all $x \in \overline{B_r}, u \in \overline{B_{r,\mathscr{U}}}$ it holds that

$$\|\varphi(k,x,u)\|_X \leq \sigma(\|x\|_X) + \gamma(\|u\|_\mathscr{U}) \quad \forall k \in \mathbb{Z}_+. \tag{5}$$

Σ has the uniform limit property (ULIM), if $\exists \gamma \in \mathscr{K}\cup\{0\}$ such that for every $\varepsilon, r > 0$ there exists a $\tau = \tau(\varepsilon, r)$ such that for all $u \in \mathscr{U}$, $x \in X$ with $x \in \overline{B_r}$ there is a $k \leq \tau$ with

$$\|\varphi(k,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathscr{U}). \tag{6}$$

Σ has the uniform asymptotic gain property (UAG) if $\exists \gamma \in \mathscr{K}_\infty \cup \{0\}$ such that $\forall \varepsilon, r > 0\ \exists \tau = \tau(\varepsilon, r) < \infty$ such that $\forall u \in \mathscr{U}$, $x \in B_r$

$$\mathbb{Z}_+ \ni k \geq \tau \quad \implies \|\varphi(k,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_\mathscr{U}). \tag{7}$$

Σ is ISS if $\exists \beta \in \mathscr{KL},\ \gamma \in \mathscr{K}$ such that $\forall x \in X, u \in \mathscr{U}$

$$\|\varphi(k,x,u)\|_X \leq \beta(\|x\|_X, k) + \gamma(\|u\|_\mathscr{U}), \quad k \in \mathbb{Z}_+. \tag{8}$$

Σ is globally asymptotically stable at zero (0-GAS) if

$$\|\varphi(k,x,0)\|_X \leq \sigma(\|x\|_X), \quad x \in X, \quad k \in \mathbb{Z}_+, \tag{9}$$

$$\lim_{k\to\infty}\|\varphi(k,x,0)\|_X = 0, \quad x \in X. \tag{10}$$

Remark 1: Due to the causality of Σ we get an equivalent characterization of the ISS property if we replace $\|u\|_\mathscr{U}$ in (8) with $\sup_{i=0,\ldots,k-1}\|u(i)\|_U$.
Note that ISS implies that for a bounded input any solution is uniformly bounded in time. As well ISS implies the 0-GAS property. We will also see that ISS is equivalent to UAG, CEP and BRS. Hence, the BRS property is equivalent to the boundedness of f in (1) see [16].

We will consider other stability properties of the ISS framework as well. Therefore we need a few more definitions, i.e., the definitions of integral ISS (iISS) and later strong iISS. The latter one was so far neglected in the literature for discrete-time systems. Later, we will see that it makes sense to distinguish strong iISS from ISS and iISS.

Definition 3: Σ given by (1) is called iISS if $\exists \alpha,\mu \in \mathscr{K}_\infty$, $\beta \in \mathscr{KL}$ such that $\forall x \in X,\ u \in \mathscr{U},\ k \in \mathbb{Z}_+$

$$\alpha(\|\varphi(k,x,u)\|_X) \leq \beta(\|x\|_X, k) + \sum_{i=0}^{k-1}\mu(\|u(i)\|_U). \tag{11}$$

In finite dimensions we have the equivalence between iISS and 0-GAS if f is continuous, see [20]. It follows immediately from the definition that iISS implies 0-GAS in infinite dimensions. But 0-GAS does not imply iISS anymore. We will see this in the Example 2. For the next chapter we also need

Definition 4: Σ is norm-to-integral ISS if $\exists \alpha,\psi,\sigma \in \mathscr{K}_\infty$ such that $\forall x \in X,\ u \in \mathscr{U},\ k \in \mathbb{Z}_+$ holds

$$\sum_{i=1}^{k}\alpha(\|\varphi(i,x,u)\|_X) \leq \psi(\|x\|_X) + k\cdot\sigma(\|u\|_\mathscr{U}). \tag{12}$$

Furthermore, an iISS and ISS Lyapunov function is defined as

Definition 5: A continuous function $V\colon X \to \mathbb{R}_+$ is called iISS Lyapunov function for (1) if $\exists \psi_1,\psi_2,\sigma \in \mathscr{K}_\infty$ and a positive definite function α such that

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad x \in X \tag{13}$$

$$V(f(x,u)) - V(x) \leq -\alpha(\|x\|_X) + \sigma(\|u\|_U), \quad x \in X, u \in U \tag{14}$$

holds. V is called ISS Lyapunov function if, in addition, $\alpha \in \mathscr{K}_\infty$. V is called a noncoercive ISS Lyapunov function if (13) is replaced with $0 < V(x) \leq \psi_2(\|x\|_X)$, $x \in X\setminus\{0\}$.

III. Preliminary Results

In this section, we will repeat some results from the literature which will be useful for the next chapters. For $\dim(X) < \infty$ the following Theorem was stated in [20]

Theorem 1: Let $\dim(X) < \infty$. Then,

Σ is iISS if and only if it admits an iISS Lyapunov function.

For $\dim(X) = \infty$ the next result was shown in [16].

Theorem 2: Σ is ISS $\iff$ Σ is ULIM, ULS, BRS.

This result will be useful in the proof of Theorem 5. We also know from [16] the following

Proposition 1: Σ is norm-to-integral ISS $\implies$ Σ is CEP and BRS.

and

Theorem 3: Σ is ISS $\iff$ Σ is norm-to-integal ISS.

Combining Proposition 1 and Theorem 3 shows

Proposition 2: Σ is ISS $\implies$ Σ is CEP and BRS.

We have also shown in [16]

Theorem 4: $\exists$ noncoercive ISS Lyapunov function V for $\Sigma \implies \Sigma$ is ISS.

The next characterization will also be useful for the proof of Proposition 3

Lemma 1: Σ is ULS $\quad\Leftrightarrow\quad \forall \varepsilon > 0\ \exists \delta > 0$ such that

$$\|x\|_X \leq \delta,\ \|u\|_\mathscr{U} \leq \delta,\ k \in \mathbb{Z}_+ \Rightarrow \|\varphi(k,x,u)\|_X \leq \varepsilon. \tag{15}$$

This result was already proven in [16] and therefore, the proof is omitted here.

IV. MAIN RESULTS

In this section, we show the equivalence of ISS and iISS if UAG holds. This result is given in Theorem 6. To this aim, we first show that UAG and CEP imply ULIM and ULS in Proposition 3. Then, we are going to show that ISS is equivalent to UAG, CEP and BRS as stated in Theorem 5. Afterwards, we prove in Proposition 4 that ISS implies iISS and in Proposition 5 iISS implies CEP and BRS.

Proposition 3: Σ is UAG and CEP $\implies$ Σ is ULIM and ULS

Proof: Let Σ be UAG and $\gamma \in \mathcal{K}_\infty \cup \{0\}$ the corresponding function from Definition 2. Let $\varepsilon, r > 0$ be arbitrary and choose $\tau = \tau(\varepsilon, r)$ such that (7) holds for all $k \geq \tau$. Then, define $\tau_{ULIM} = \tau + 1$. Thus, $\|\varphi(k,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{\mathcal{U}})$ is valid for $k = \lceil \tau \rceil \leq \tau_{ULIM}$ which shows ULIM.

Again, let Σ be UAG and $\gamma \in \mathcal{K}_\infty \cup \{0\}$ the corresponding function from Definition 2. Let $\varepsilon, r > 0$ be arbitrary and $\tau = \tau(\frac{\varepsilon}{2}, r)$. Let δ_1 be arbitrary with $\gamma(\delta_1) \leq \frac{\varepsilon}{2}$. From (7) it follows that $\forall x \in B_r$, $u \in B_{\delta_1}$ we get

$$\begin{aligned} \|\varphi(k,x,u)\|_X &\leq \frac{\varepsilon}{2} + \gamma(\|u\|_{\mathcal{U}}) \\ &\leq \frac{\varepsilon}{2} + \gamma(\delta_1) \\ &\leq \varepsilon \qquad (\star) \end{aligned}$$

for $k \geq \tau$. Now, we will use the CEP property. It follows that $\exists \delta_2 > 0$ such that (3) holds with $h = \tau$ where we set $\delta = \delta_2$. With $(\star)$ and (3) we get $\forall \varepsilon > 0$ $\exists \delta > 0$ with $\delta = \min\{r, \delta_1, \delta_2\}$ such that $\forall x \in \overline{B_\delta}$, $u \in \overline{B_{\delta,\mathcal{U}}}$, $k \in \mathbb{Z}_+$ follows $\|\varphi(k,x,u)\|_X \leq \varepsilon$. This is equivalent to the ULS property due to Lemma 1 which finishes the proof. ∎

Theorem 5: Σ is ISS $\iff$ Σ is UAG, CEP and BRS.

Proof: ($\Rightarrow$) We already know from Proposition 2 that ISS implies CEP and BRS. Therefore, it is only left to show that ISS implies UAG. Suppose there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (8) holds. Let $\varepsilon > 0$, $r > 0$ be arbitrary. Then, we define

$$\tau = \tau(\varepsilon, r) := \min\{\bar{\tau} \in \mathbb{Z}_+ : \beta(r, \bar{\tau}) \leq \varepsilon\}.$$

Now, $\forall u \in \mathcal{U}$, $x \in B_r$, $k \in \mathbb{Z}_+$ with $k \geq \tau$ we get

$$\begin{aligned} \|\varphi(k,x,u)\|_X &\leq \beta(\|x\|_X, k) + \gamma(\|u\|_{\mathcal{U}}) \\ &\leq \beta(r, \tau) + \gamma(\|u\|_{\mathcal{U}}) \\ &\leq \varepsilon + \gamma(\|u\|_{\mathcal{U}}). \end{aligned}$$

This shows UAG.

($\Leftarrow$) Follows from Proposition 3 and Theorem 2. ∎

Theorems 2 and 5 imply the following

Corollary 1: Σ is UAG, CEP and BRS $\iff$ Σ is ULIM, ULS, BRS.

Now, consider the linear continuous-time system

$$\Sigma_{c,lin}: \quad \dot{x}(t) = Ax(t) + Bu(t)$$

where A generates a C_0-semigroup $(T(t))_{t\geq 0}$ on an Banach space X and B is a linear operator from a Banach space U to the extrapolation space X_{-1}. It holds that ISS is equivalent to iISS if B is bounded. iISS implies ISS if B is unbounded. The converse direction is open, see [10]. We recall that if f is Lipschitz continuous on bounded balls of $\mathbb{R}^n \times \mathbb{R}^m$ then the ISS property of $\dot{x} = f(x,u)$, $x(0) = x_0$ implies the iISS property, see [12].

For a discrete-time system (1) we do not need such regularity assumptions to prove this implication even for the general case $\dim(X) \leq \infty$. This result is stated as

Proposition 4: Σ is ISS $\implies$ Σ is iISS.

Proof: Suppose that there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (8) holds. We will use Remark 1 to get our implication. It follows

$$\begin{aligned} \|\varphi(k,x,u)\|_X &\leq \beta(\|x(0)\|_X, k) + \gamma(\sup_{i=0,\ldots,k-1} \|u(i)\|_U) \\ &\leq \beta(\|x(0)\|_X, k) + \sum_{i=0}^{k-1} \gamma(\|u(i)\|_U) \\ &\leq \beta(\|x(0)\|_X, k) + \sum_{i=0}^{k-1} \mu(\|u(i)\|_U), \end{aligned}$$

where $\mu \in \mathcal{K}_\infty$. In the second step we have used that $\sup_{i=0,\ldots,k-1} \|u(i)\|_U \leq \sum_{i=0}^{k-1} \gamma(\|u(i)\|_U)$. Since $\gamma \in \mathcal{K}$ we can find a $\mu \in \mathcal{K}_\infty$ in the last step such that $\gamma(r) \leq \mu(r)$ $\forall r \in \mathbb{R}_+$. This shows iISS with $\alpha(r) = \mathrm{id}(r)$ $\forall r \in \mathbb{R}_+$. ∎

Now, it would be nice to have some kind of backward direction in Proposition 4. For this purpose, we first show the following

Proposition 5: Σ is iISS $\implies$ Σ is CEP and BRS.

Proof: Suppose that Σ is iISS, i.e. there are $\beta \in \mathcal{KL}$ and $\alpha, \mu \in \mathcal{K}_\infty$ such that (11) holds. Since $\alpha \in \mathcal{K}_\infty$ we know that the function α is invertible and $\alpha^{-1} \in \mathcal{K}_\infty$. Thus, we use α^{-1} on both sides of (11), $\|u(i)\|_U \leq \|u\|_{\mathcal{U}}$, define $\psi(\|x_0\|_X) := \beta(\|x_0\|_X, 0)$ and get

$$\begin{aligned} \|\varphi(k,x,u)\|_X &= \alpha^{-1}(\alpha(\|x(k)\|_X)) \\ &\leq \alpha^{-1}\left(\beta(\|x_0\|_X, k) + \sum_{i=0}^{k-1} \mu(\|u(i)\|_U)\right) \\ &\leq \alpha^{-1}\left(\beta(\|x_0\|_X, 0) + \sum_{i=0}^{k-1} \mu(\|u\|_{\mathcal{U}})\right) \\ &= \alpha^{-1}\left(\psi(\|x_0\|_X + k \cdot \mu(\|u\|_{\mathcal{U}})\right) \\ &\leq \alpha^{-1}(2\psi(\|x_0\|_X) + \alpha^{-1}(2k \cdot \mu(\|u\|_{\mathcal{U}})) \leq \varepsilon. \end{aligned}$$

We have used that $\alpha(a+b) \leq \alpha(2a) + \alpha(2b)$ holds for all $\alpha \in \mathcal{K}_\infty$, $a, b \geq 0$ in the fifth step. Since $\alpha^{-1}, \psi, \mu \in \mathcal{K}_\infty$ we can choose $\delta = \delta(\varepsilon, k) > 0$ small enough such that for $\|x\|_X \leq \delta$ and $\|u\|_{\mathcal{U}} \leq \delta$, we get $\|x(k)\|_X \leq \varepsilon$.

As this bound holds for each k we can choose $\tilde{\delta} = \min_{k \in [0,h] \cap \mathbb{Z}_+} \delta(\varepsilon, k)$ such that Σ is CEP. BRS follows as each $x(k)$ from the upper computation is bounded. ∎

Theorem 6: Σ is ISS $\iff$ Σ is iISS and UAG.

Proof: ($\Rightarrow$) Follows from Theorem 5 and Proposition 4.

($\Leftarrow$) Follows from Proposition 5 and Theorem 5. ∎

The results of Theorem 2, 5, 6, Corollary 1 and Proposition 4 are summarized in figure 1.

Now, we use Proposition 5 to show that 0-GAS does not imply iISS in infinite dimensions.

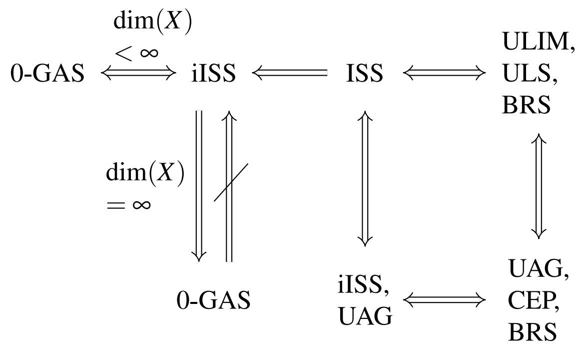


Fig. 1. Some equivalences and implications

Example 2: Let $X = \ell^\infty(\mathbb{N},\mathbb{R})$, $U=\mathbb{R}$. Consider

$$x_n(k+1) = \sqrt[n]{x_n(k)\cdot u(k)} \quad \forall k\in\mathbb{Z}_+, n\in\mathbb{N}, \quad \text{i.e.}$$
$$x_n(k+1) = (x_n(0))^{\frac{1}{n^k}} \cdot \prod_{\ell=0}^{k} (u(\ell))^{\frac{1}{n^{k-l}}}$$

with $x(0)=x_0\in X$. This system is obviously 0-GAS.
Now, take $\varepsilon = \frac{1}{2}$ and any $\delta > 0$. Choose $x_n(0)=\delta$ for all $n\in\mathbb{N}$ and $u(0)=\delta$. For the first time step we get

$$x_n(1) = (x_n(0))^{\frac{1}{n}}\cdot(u(0))^{\frac{1}{n}} = \delta^{\frac{2}{n}} \quad \forall n\in\mathbb{N}.$$

It follows $\|x(1)\|_X = 1$ if $\delta\le 1$ and $\|x(1)\|_X=\delta^2$ if $\delta>1$. In both cases we have $\|x(1)\|_X > \varepsilon$. Therefore, the system is not CEP and hence it is not iISS by Proposition 5.
This is also summarized in figure 1.

V. Strong integral ISS

In this section, we will give the definition of strong iISS for discrete-time systems which was introduced for continuous-time systems, see [21]. Furthermore, we show by examples that the class of ISS systems is a strict subset of the class of systems which are strongly iISS, that the class of systems which are strongly iISS is a strict subset of iISS systems and that this classes can not be equal. We couldn't find any reference in the literature which shows this for discrete-time systems.

Definition 6: Σ is strongly iISS if $\exists\, \alpha,\mu\in\mathscr{K}_\infty$, $\gamma\in\mathscr{K}$, $\beta\in\mathscr{KL}$, $R>0$ such that $\forall u\in\mathscr{U}$, $x\in X$, $k\in\mathbb{Z}_+$ the following properties hold:

1) (11) holds,
2) $\|u\|_{\mathscr{U}} < R \implies$ (8) holds.

By Proposition 4 follows that the class of systems which are ISS is a subset of the class of systems which are iISS.
Now, let $X=\mathbb{R}$ and $U=\mathbb{R}$. Then, consider the system

$$x(k+1) = \frac{1}{2}\cdot x(k)\cdot u(k) \quad \forall k\in\mathbb{Z}_+, \quad \text{i.e.}$$
$$x(k+1) = \frac{1}{2^{k+1}} x_0 \cdot \prod_{l=0}^{k} u(l), \tag{16}$$

where $x(0)=x_0$.
First, choose $u(l)=4\ \forall\, l\in\mathbb{Z}_+$. It follows

$$x(k) = \frac{1}{2^k}x_0\cdot\prod_{l=0}^{k-1} 4 = 2^k\cdot x_0,$$

where $\lim_{k\to\infty}|x(k)|=\infty$. This shows that (16) is not ISS. Next, we show that (16) is iISS. For this purpose we need the following result

Lemma 2: For all $a,b\in\mathbb{R}_+$, $r\ge 1$ holds

1) $\tanh(a\cdot b)\le\tanh(a^2)+\tanh(b^2)$,
2) $\tanh(a^r)\le a$.

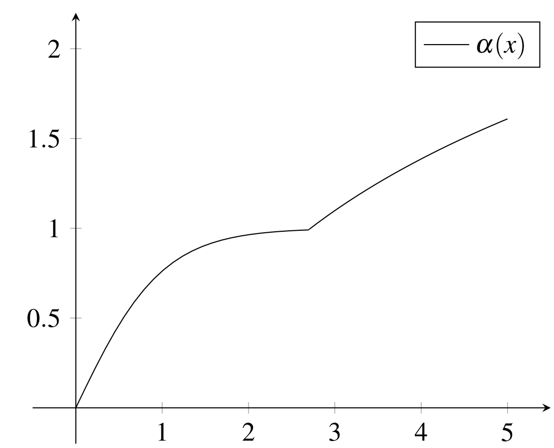
Proof: The first estimate follows by

$$\tanh(ab)\le\tanh(\max\{a^2,b^2\})\le\tanh(a^2)+\tanh(b^2).$$

The second estimate follows by noticing that $\tanh(a^r)\le a$ if $a>1$ and $\tanh(a^r)\le\tanh(a)\le a$ if $a\le 1$. ■

Now, we can define $\alpha\in\mathscr{K}_\infty$ by

$$\alpha(x) := \max\{\tanh(x), \log(x)\}.$$



Thus, if $\alpha(x)=\tanh(x)$ and by using Lemma 2 and $|u|_\infty := \sup_{l=0,\dots,k-1}|u(l)|$, we get

$$\begin{aligned}
\alpha(|x(k)|) &= \alpha\Big(\Big|\frac{1}{2^k}\cdot x_0\cdot\prod_{l=0}^{k-1}u(l)\Big|\Big)\\
&= \tanh\Big(\frac{1}{2^k}\cdot|x_0|\cdot\prod_{l=0}^{k-1}|u(l)|\Big)\\
&\le \tanh\Big(\Big[\frac{1}{2^k}\cdot|x_0|\Big]^2\Big)+\tanh\Big(\Big[\prod_{l=0}^{k-1}|u(l)|\Big]^2\Big)\\
&\le \tanh\Big(\Big[\frac{1}{2^k}\cdot|x_0|\Big]^2\Big)+\tanh(|u|_\infty^{2k})\\
&\le \frac{1}{2^k}\cdot x_0 + |u|_\infty\\
&\le \frac{1}{2^k}\cdot|x_0| + \sum_{l=0}^{k-1}|u(l)|,
\end{aligned}$$

If $\alpha(x)=\log(x)$ we compute

$$\begin{aligned}\alpha(|x(k)|) &= \alpha(|\frac{1}{2^k}\cdot x_0\cdot\prod_{l=0}^{k-1}u(l)|)\\ &= \log(\frac{1}{2^k}\cdot|x_0|\cdot\prod_{l=0}^{k-1}|u(l)|)\\ &= \log(\frac{1}{2^k}\cdot|x_0|)+\log(\prod_{l=0}^{k-1}|u(l)|)\\ &= \log(\frac{1}{2^k}\cdot|x_0|)+\sum_{l=0}^{k-1}\log(|u(l)|)\\ &\le \frac{1}{2^k}\cdot|x_0|+\sum_{l=0}^{k-1}|u(l)|.\end{aligned}$$

Thus, we get the same estimation in both cases. By defining $\beta(|x_0|,k):=\frac{1}{2^k}\cdot|x_0|$ and $\mu(|u(l)|):=|u(l)|$ we get that (16) fulfills (11). Therefore, it is iISS.

Now, we show that (16) is strongly iISS. Therefore, choose $|u(l)|<R<2$ and it follows

$$x(k)=\frac{1}{2^k}\cdot x_0\cdot\prod_{l=0}^{k-1}u(l)<\left(\frac{R}{2}\right)^k\cdot x_0.$$

By defining $\beta(|x_0|,k):=\left(\frac{R}{2}\right)^k\cdot|x_0|$ and $\gamma(|u|)=|u|$ with $|u|=\max_{k\in\mathbb{Z}_+}|u(k)|$, we get

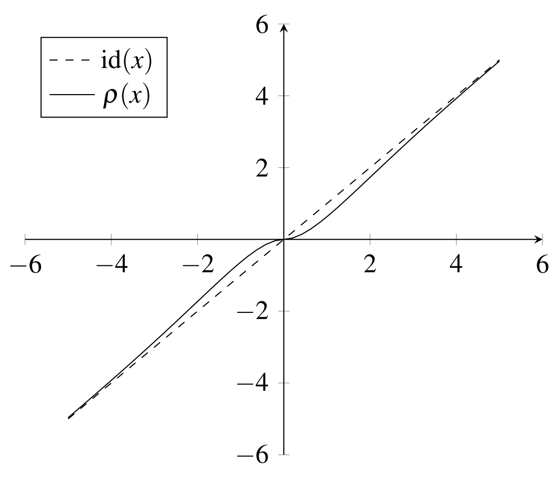
$$\begin{aligned}|x(k)| &< \left(\frac{R}{2}\right)^k\cdot x_0\\ &\le \left(\frac{R}{2}\right)^k\cdot|x_0|+|u|\\ &\le \beta(|x_0|,k)+\gamma(|u|),\end{aligned}$$

which shows (8). Therefore, (16) is strongly iISS.

Now, let $X=\mathbb{R}$ and $U=\mathbb{R}$. Consider the system

$$x(k+1)=\rho(x(k))+u(k),\quad \forall k\in\mathbb{Z}_+, \tag{17}$$

where $\rho(r):=r\cdot(1-\exp(-|r|))$ is continuous and strictly increasing function and the initial condition is $x(0)=x_0$.



First, we show again that (17) is not ISS. We use that $g\colon\mathbb{R}\to\mathbb{R}$, $g(x):=|x|\cdot\exp(-|x|)$ has a local maximum for $x=\pm1$ and $g(\pm1)=\exp(-1)$.

Furthermore, we choose $u(k)=1\ \forall k\in\mathbb{Z}_+$. Thus, we compute

$$\begin{aligned}|x(k)| &= |x(k-1)|\cdot(1-\exp(-|x(k-1)|))+1\\ &= |x(k-1)|-|x(k-1)|\cdot\exp(-|x(k-1)|)+1\\ &\ge |x(k-1)|-\exp(-1)+1\\ &= |x(k-1)|+\frac{\exp(1)-1}{\exp(1)}\\ &\ge\cdots\ge|x_0|+k\cdot\frac{\exp(1)-1}{\exp(1)}.\end{aligned}$$

It follows $\lim_{k\to\infty}|x(k)|\ge\lim_{k\to\infty}|x_0|+k\cdot\frac{\exp(1)-1}{\exp(1)}=\infty$. Therefore, (17) is not ISS.

But (17) is iISS. This can be shown through the existence of an iISS - Lyapunov function, see [20].

We choose $V(x)=|x|$. By choosing $\psi_1(|x|)=|x|=\psi_2(|x|)$ we get the bounds for $V(x)$. Then, we compute

$$\begin{aligned}&V(f(x(k),u(k)))-V(x(k))=|f(x(k),u(k))|-|x(k)|\\ &=|\rho(x(k))+u(k)|-|x(k)|\\ &\le|\rho(x(k))|+|u(k)|-|x(k)|\\ &=|x(k)\cdot(1-\exp(-|x(k)|))|+|u(k)|-|x(k)|\\ &=|x(k)|\cdot(1-\exp(-|x(k)|))+|u(k)|-|x(k)|\\ &=|x(k)|-|x(k)|\cdot\exp(-|x(k)|)+|u(k)|-|x(k)|\\ &=-|x(k)|\cdot\exp(-|x(k)|)+|u(k)|\\ &=-\alpha(|x(k)|)+\sigma(|u(k)|),\end{aligned}$$

where we choose $\alpha(r):=r\cdot\exp(-r)$ as positive definite function and $\sigma(r):=r$, $r\ge0$, $\sigma\in\mathcal{K}_\infty$. This shows that $V(x)=|x|$ is an iISS Lyapunov function and by Theorem 1 we get that (17) is iISS.

Furthermore, we demonstrate that (17) is not strongly iISS. We assume that $\exists 0<R<1$ such that (17) would be strongly iISS. Then, we can choose $u(k)=\frac{R}{2}\ \forall k\in\mathbb{Z}_+$. Notice that $\frac{R}{4}<\frac{1}{4}$. Now, we compute the distance $\mathrm{d}(x_0,\rho(x_0))$ with $|x_0|>1$. It follows

$$\mathrm{d}(x_0,\rho(x_0))=|x_0-\rho(x_0)|=|x_0\cdot\exp(-|x_0|)|.$$

We also know that $\min_{x\in\mathbb{R}}|x\cdot\exp(-|x|)|=0$ and $\max_{x\in\mathbb{R}}|x\cdot\exp(-|x|)|=\exp(-1)$. Because of the mean value theorem we know that there is a x_0 with $\frac{R}{4}=\mathrm{d}(x_0,\rho(x_0))$. With this x_0 we get

$$x(1)=x_0-x_0\cdot\exp(-|x_0|)+\frac{R}{2}=x_0+\frac{R}{4}>x_0.$$

It follows $\mathrm{d}(x(1),\rho(x(1)))<\mathrm{d}(x_0,\rho(x_0))=\frac{R}{4}$ and therefore,

$$\begin{aligned}x(2) &= x(1)-x(1)\cdot\exp(-|x(1)|)+\frac{R}{2}\\ &\ge x(1)-\frac{R}{4}+\frac{R}{2}\\ &= x_0+2\cdot\frac{R}{4}.\end{aligned}$$

Following this approach we get $x(k)\ge x_0+k\cdot\frac{R}{4}$ which means $\lim_{k\to\infty}|x(k)|=\infty$. This shows that there can not exist a $R>0$ such that (17) is strongly iISS.

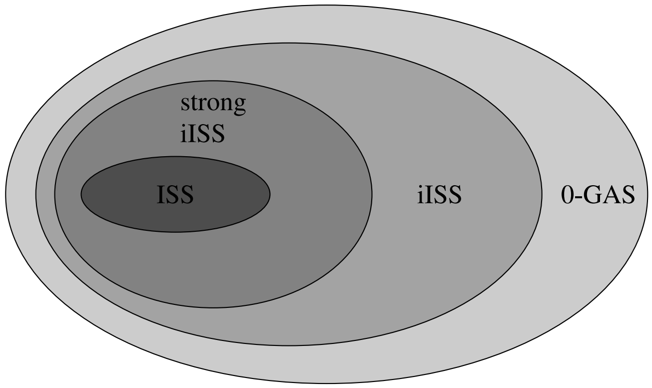


Fig. 2. Relationship between ISS, strong iISS, iISS and 0-GAS

These two examples and Proposition 4 show that the class of systems that are ISS is a strict subset of the class of systems that are strong iISS and the same holds for the class of systems that are strong iISS and iISS. A graphical illustration is given in figure 2. Furthermore, these two examples can be easily extended to infinite dimensions and the relationship can be seen in a similar way.

VI. CONCLUSIONS

We have introduced the iISS and strong iISS properties for infinite-dimensional discrete-time systems. Then we have shown that ISS of a system Σ implies iISS for discrete-time systems if $\dim(X) \leq \infty$. We could not find any results about this in the literature in the case of infinite-dimensional continuous-time systems if the operator f is nonlinear.

Furthermore, we have shown some discrete-time counterparts of the results from [17], e.g. Proposition 3 and Theorem 5 with the same approach. Hence, we have shown that ISS is equivalent to iISS and UAG.

We have seen that the class of systems that are ISS is a strict subset of the class of systems that are strongly iISS. We have also seen that the class of systems that are strongly iISS is a strict subset of the class of systems that are iISS. Our examples are for finite dimensions but they can be extended into infinite dimensions as well by using $x_n(k+1) = \frac{1}{2} \cdot x_n(k) \cdot u(k)$ for all $n \in \mathbb{N}$ in (16) and analogously in (17).

In our future work we plan to study, if 0-UGAS is equivalent to the iISS for $\dim(X) = \infty$ like in the continuous-time case for bounded bilinear and non-autonomous bilinear systems, see [22] and [23].

Recall that ISS and iISS are equivalent for bounded linear systems in the continuous-time case, see [23] and [10] and we are going to study such systems in the discrete-time case. We also plan to derive Lyapunov characterizations of strong iISS and other ISS-like properties, similarly to [21] and [9] for the continuous-time case and we will investigate further characterizations of ISS-like properties like iISS, see [24].

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