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Controller Design for Systems Governed by Time-Periodic Delay Differential Equations: A Spectrum Optimization Approach

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***Abstract*— Systems governed by delay differential equations (DDEs) are distinguished by having infinite-dimensional dynamics. This key feature in the context of linear time-delayed systems reflects itself in the associated operator eigenvalue problem. Moreover, in numerous applications including power systems and machining, an accurate description leads to time-periodic models. To address the stabilization problem of such systems, we minimize the spectral radius of the discretized monodromy operator. We propose two approaches to impose smoothness on the designed periodic feedback gain. In our first approach, we limit the total variation of the feedback gain and impose approximate periodicity of the controller gain by adding regularization terms to the cost function. Conversely, in the second approach, the feedback gain is constrained to be characterized by a finite number of harmonics of its Fourier series representation; hence, smoothness and periodicity are explicitly encoded in the controller parametrization. We develop computationally tractable formulations for the derivative of the spectral radius with respect to the controller parameters, which are employed by the optimization algorithm. Finally, through a case study involving a milling machine, we validate the efficacy and applicability of the presented work.**

I. Introduction

Time delay is a pivotal component within many accurate system description, such as neural response latency, data processing delay, transmission and reception delay, etc. [1][2]. Time delay elements express that the transfer of material, energy, or information between subsystems mostly occurs with latency. Moreover, the system behavior can vary with time. In this research, we are interested in scenarios where the system model varies periodically. The periodicity can stem from the periodic variation of the system parameters [3], the local stability analysis of periodic solutions of non-linear delay differential equations (DDEs) [4], or directing a time-invariant system with a time-periodic controller such as act-and-wait controllers [5]. The combination of delay and parametric excitation leads us to the study of systems characterized by Time-Periodic Delay Differential Equations (TPDDEs). Systems governed with TPDDEs of retarded type are defined as

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{m} A_i(t)x(t-\tau_i) + B(t)u(t), \\ y(t) = \sum_{j=1}^{n} C_j(t)x(t-\bar{\tau}_j), \end{cases} \tag{1}$$

where vectors $x(t) \in \mathbb{C}^d$, $y(t) \in \mathbb{C}^p$, and $u(t) \in \mathbb{C}^r$ stand for the state, the output, and the control input vectors at time t. The delay variables τ_i and $\bar{\tau}_j$ are nonnegative real values that are associated with the state and output delays, respectively. The matrix-valued functions $A_i : \mathbb{R} \to \mathbb{R}^{d\times d}$ for $i = 1, \ldots, m$, $B : \mathbb{R} \to \mathbb{R}^{d\times r}$ and $C_j : \mathbb{R} \to \mathbb{R}^{p\times d}$, for $j = 1, \ldots, n$ are real-valued piecewise continuous functions with period $T > 0$. It should be noted that the approach presented in this article is adaptable to situations with single input delay in the model, as well. This is feasible as the single delay in the control input can be readily translated to the output.

In this article, we design a static output feedback controller, described by

$$u(t) = \mathcal{K}(t)y(t) = \sum_{j=1}^{n} \mathcal{K}(t)C_j(t)x(t-\bar{\tau}_j). \tag{2}$$

The function $t \mapsto \mathcal{K}(t) \in \mathbb{C}^{r\times p}$ is the time-periodic feedback gain whose fundamental period is assumed to be that of the system.

A. Literature Review

As a key aspect, time-delay systems form a class of infinite-dimensional systems, which leads to rich dynamics and challenges in analysis and design. Within the framework of linear time-invariant (LTI) DDEs, the availability of the characteristic equation in an *explicit* form has spurred the development of novel methods not only for analyzing system stability but also for designing stabilizing controllers to direct the system behavior in the desired manner [6][7][8][9].

In contrast to LTI DDEs, the currently available methods for TPDDEs almost exclusively address analysis problems. Several computational methods have been proposed for stability analysis, grounded upon Floquet theory and spectrum approximation of the monodrmoy operator. The typical discretization techniques include semi-discretization [3] and spectral methods relying on (Galerkin) projection [10] or collocation [11]. In [12], an extension to robust stability analysis of TPDDEs with perturbations on the coefficients via the notion of pseudospectral radius of the monodromy operator has been described. In [13] and [14], the stability analysis of TPDDEs and the computation of $\mathcal{H}_2$ norm via the notion of delay Lyapunov matrix has been conducted, respectively. However, the computation of delay Lyapunov matrices is a challenging task, while existing work is limited to special cases of TPDDEs. To approach the stability analysis, analytic methods based on averaging have also been

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proposed [15] [16]. Averaging, nevertheless, requires that the variation of the periodic terms is sufficiently fast compared to the system dynamics, which is not the case in our targeted applications.

The few work on control of TPDDEs include [17][18], where a time-invariant feedback gain is computed by stability optimization. Next, the so-called act-and-wait controller [5] is a powerful time-periodic control scheme that induces a finite-dimensional monodromy operator. However, its application is limited to LTI systems with input and output delay. Finally, some literature has been devoted to the stabilization of TPDDEs, where controller parameters are tuned with the aid of stability charts [5][19]. Such an approach is insightful but restricts to problems with a small number of controller parameters.

Since DDEs fall within the scope of infinite-dimensional systems, a finite-dimensional approximation, followed by a controller design using techniques for finite-dimensional systems (see, e.g., the LQR controllers synthesized in [19]) may lead to a high-dimensional controller or an approximation of a distributed delay controller. In contrast, this article fits within the *direct optimization framework* towards the design of controllers with a prescribed-order or structure [20], where stability, robustness and performance criteria of the closed-loop system are directly optimized as a function of the available controller parameters.

B. Contributions

In this article, we propose a design approach to synthesize the feedback controller (2), relying on minimizing the spectral radius of the (discretized) monodromy operator. We propose two approaches to impose smoothness on the designed periodic feedback gain:

1) As our first approach, we limit the total variation of the feedback gain and impose approximate periodicity of the controller gain by adding regularization terms to the cost function.
2) In our second approach, we confine the periodic gain to functions that specifically contain a designated number of harmonics within their Fourier series expansion. Imposing such a structure on the feedback gain induces smoothness and periodicity explicitly.

We introduce computationally tractable formulas for the derivative of modulus of a simple Floquet multiplier with respect to the controller parameters. These derivatives are exploited in solving the formulated nonsmooth optimization problems, grounded upon minimizing the spectral radius of the induced discretized monodromy operator. Finally, we study the efficacy and applicability of our work through a case study of a milling machine.

II. PREMILINARIES

In what follows, we give a brief overview of Floquet theory for TPDDEs. For more detailed information, the reader can consult with the referred monographs [21][22].

A. Definitions

The closed-loop system dynamics formed from directing the system (1) with the control input (2) is expressed as

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{m} A_i(t)x(t-\tau_i) + \\ \qquad \sum_{j=1}^{n} B(t)\mathcal{K}(t)C_j(t)x(t-\bar{\tau}_j), \quad t \geq t_0, \\ x(t) = \phi(t-t_0), \quad t \in [t_0 - \tau_{max}, t_0], \end{cases} \tag{3}$$

where we define t_0 as the start time, τ_{max} as the largest delay component considering delays present both in output and state, and $\phi \in \mathcal{X} = \mathcal{C}([-\tau_{max}, 0], \mathbb{R}^d)$ as the initial condition, where the set $\mathcal{X}$ represents the space of continuous functions from $[-\tau_{max}, 0]$ to $\mathbb{R}^d$. This introduced Cauchy problem has a unique solution in the interval $[t_0 - \tau_{max}, t]$ that we denote by $x(t; t_0, \phi)$. The corresponding state at time $t \geq t_0$ is then given by the function segment $x_t(t_0, \phi) \in \mathcal{X}$, defined by

$$x_t(t_0, \phi)(\sigma) := x(t+\sigma; t_0, \phi), \quad \sigma \in [-\tau_{max}, 0],$$

Definition 1 (Solution operator): We denote the solution operator with $\mathcal{T}(t, t_0) : \mathcal{X} \to \mathcal{X}$ as the strongly continuous semigroup that maps the initial function ϕ at time t_0 onto the state at time $t \geq t_0$, which can be stated as

$$\mathcal{T}(t, t_0)\phi = x_t(t_0, \phi).$$

The spectrum of the solution operator $\mathcal{T}(T, 0)$ is an at most countable compact set in $\mathbb{C}$ possibly with accumulation point zero. This spectrum is equal to the spectrum of the solution operator $\mathcal{T}(t_0 + T, t_0)$, for all $t_0 \in \mathbb{R}$. This leads us to the definition the monodromy operator for TPDDEs.

Definition 2 (Monodromy operator): The monodromy operator is defined as $\mathcal{U} := \mathcal{T}(T, 0)$, where T is the fundamental period. Any nonzero member of the spectrum of the monodromy operator is called a Floquet multiplier.

Let $\{\mu_i\}_{i=1}^{\infty}$ be the Floquet multiplier and let $\{\phi_i\}_{i=1}^{\infty}$ be the corresponding set of (generalized) eigenfucntions. Then, any solution $x(t; 0, \phi)$ can be decomposed as

$$x(t; 0, \phi) = \sum_{i=1}^{\infty} c_i x(t; 0, \phi_i),$$

where $\{c_i\}_{i=1}^{\infty}$ depend on ϕ. Hence, all solutions can be decomposed in elementary solutions. Note that these solutions emanating from an eigenfunction have the form $x(t; 0, \phi) = \rho_i(t)e^{\gamma_i t}$, where $\rho_i(t)$ is T-periodic and $e^{\gamma_i T} = \mu_i$ [23].

Proposition 1: The zero solution of (3) is asymptotically stable if and only if the modulus of all Floquet multipliers of the corresponding monodromy operator is smaller than 1.

B. A Collocation-Based Solver for Computing the Dominant Floquet Multiplier

As discussed previously, TPDDEs are characterized by having infinite-dimensional dynamics. Accordingly, computation of all the infinitely many eigenvalues of the monodromy operator is infeasible. In this subsection, we present the solver introduced in [12] to compute the eigenvalues of the discretized monodromy operator, as it forms the basis

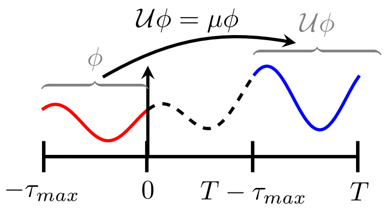

Fig. 1. This figure illustrates how the solution x in blue emanating from an eigenfunction of the monodromy operator ϕ in red is linked via the corresponding Floquet multiplier for the scenario where $T > \tau_{max}$ and $|\mu| > 1$.

for the further developments. Subsequently, in Section III, we solve a reduced-control problem, namely, we design a controller to relocate all the Floquet multipliers within the unit disk.

Fig. 1 represents a solution emanating from an eigenfunction ϕ of the monodromy operator. We can state that the value μ is the nonzero eigenvalue of the monodromy operator if and only if there exists an eigenfunction ϕ such that $\mathcal{U}\phi = \mu\phi$. This can also be equivalently expressed as

$$x(\sigma + T; 0, \phi) = \mu\phi(\sigma), \quad \text{for all} \quad \sigma \in [-\tau_{max}, 0]. \tag{4}$$

Our approach is based on approximating the solution corresponding to an eigenfunction of the monodromy operator, i.e, we approximate the solution $[-\tau_{max}, T] \ni t \mapsto x(t; 0, \phi)$ such that (4) holds. Note that such a solution corresponds to an elementary solution as was introduced earlier. The solution is approximated with a piecewise polynomial and three constraints are imposed: continuity, collocation constraints for differential equation (3), and finally, expression (4). Writing these constraints together results in a linear finite-dimensional generalized eigenvalue problem.

As sketched in Fig. 2, we subdivide the intervals $[-\tau_{max}, 0]$ and $[0, T]$ into N_τ and N_T subintervals, respectively. Each subinterval is represented by Ω_j. The solution can be approximated via a piecewise polynomial p defined as

$$x(t; 0, \phi) \approx p(t) = \sum_{j=1}^{N} p_j(t)\chi_{\Omega_j}(t),$$

where $N := N_T + N_\tau$ is the total number of subintervals, p_j is the polynomial approximating the solution over jth subinterval, and finally, χ_{Ω_j} represents the corresponding indicator function. Let us define the polynomial p_j with a polynomial of degree $M \in \mathbb{N}$ for $j = 1, \ldots, N$ and $t \in [-\tau_{max}, T]$ as

$$p_j(t) = \sum_{s=0}^{M} v_s^{(j)} T_s^{(j)}(t),$$

where $\{T_s^{(j)}(t)\}_{s=0}^{M}$ is the chosen polynomial basis, and $v_s^{(j)} \in \mathbb{C}^d$ are the associated coefficients corresponding to jth subinterval. In this article, we employ Chebyshev-distributed grid points over the subinterval $\Omega_j = [a_j, b_j]$ and Chebyshev polynomials to approximate the solution [24]. We define a generic mesh as $\alpha_\ell = -\cos(\frac{\pi\ell}{\Gamma})$ over the subinterval Ω_j, whose inverse in the original time coordinate can be calculated as $\vartheta_\ell^j = f_j(\alpha_\ell) = \frac{1}{2}(a_j + b_j) + \frac{\alpha_\ell}{2}(b_j - a_j)$ for $\ell = 1, \ldots, \Gamma$, where

$$\begin{cases} \Gamma = M + 1, & \text{if } \Omega_j \subset [-\tau_{max}, 0], \\ \Gamma = M, & \text{if } \Omega_j \subset [0, T]. \end{cases}$$

Let us define $M_T := N_T M$ and $M_\tau := N_\tau(M + 1)$ that stand for the number of collocation points in the interval $[0, T]$ and $[-\tau_{max}, 0]$, respectively. For simplicity of representation in the later use, let us store all the collocation points in the interval $[0, T]$ from left to right in θ_ℓ for $\ell = 1, \ldots, M_T$ and store all the collocation points in the interval $[-\tau_{max}, 0]$ from left to right in θ_ℓ for $\ell = M_T + 1, \ldots, M_T + M_\tau$. Furthermore, we store the unknown coefficients from left to right in the compact form $V \in \mathbb{C}^{N(M+1)d}$. The solution of the Cauchy problem is approximated by imposing the following three constraints:

- Consider the subinterval $\Omega_j = [a_j, b_j] \subset [0, T]$. We impose continuity conditions on a_j as the breaking points of the subintervals, which read as

$$p_{j-1}(a_j) = p_j(a_j), \quad \text{for} \quad j = 2, \ldots, N_T. \tag{5}$$

  The polynomials p_1 and $p_{N_T+N_\tau}$ have to be continuous at $t = 0$, which can be written as

$$p_{N_T+N_\tau}(0) = p_1(0). \tag{6}$$

- We impose collocation conditions on the DDE (3). Take $\mathcal{I}$ to signify the interpolation operator,

$$\mathcal{I} : [-\tau_{max}, T] \times \mathbb{N} \to \mathbb{R}^{d \times N(M+1)d},$$
$$(t,\kappa) \mapsto \mathcal{I}^{(\kappa)}(t), \text{where} \ \ \mathcal{I}^{(\kappa)}(t)V = p^{(\kappa)}(t),$$

  which returns the value for the κth derivative ($\kappa \geq 0$) of the piecewise polynomial p at time t. We rewrite (3) with the use of interpolation operator for $\ell = 1, \ldots, M_T$ as follows:

$$\Big(\mathcal{I}^{(1)}(\theta_\ell) - \sum_{i=1}^{m} A_i(\theta_\ell)\mathcal{I}^{(0)}(\theta_\ell - \tau_i) - \sum_{j=1}^{n} B(\theta_\ell)\mathcal{K}(\theta_\ell)C_j(\theta_\ell)\mathcal{I}^{(0)}(\theta_\ell - \bar{\tau}_j)\Big)V = 0. \tag{7}$$

- Finally, we require the solution to be an eigenfunction of the discretized monodromy operator for $\ell = M_T + 1, \ldots, M_T + M_\tau$ as

$$\mathcal{I}^{(0)}(\theta_\ell + T)V = \mu\mathcal{I}^{(0)}(\theta_\ell)V.$$

Writing all the constraints together gives rise to the following linear general eigenvalue problem:

$$\mathcal{M}(\mu)V := (\mathcal{A}_M - \mu\mathcal{B}_M)V = 0, \tag{8}$$

where the pair (μ, V) stand for the right eigenpair of the pencil $(\mathcal{A}_M, \mathcal{B}_M)$. In [12], it has been indicated that $\det(\mathcal{A}_M - \mu\mathcal{B}_M) = 0$ can be rearranged as $\det(\mu I - \mathcal{U}_M) = 0$, where $\mathcal{U}_M$ is the induced discretized monodromy operator.

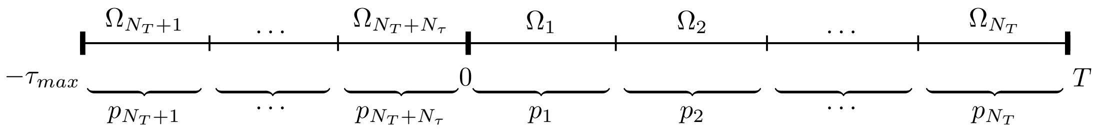

Fig. 2. Division of the domain $[-\tau_{max}, T]$ into N subintervals, where $N := N_\tau + N_T$.

Note that the accuracy of the approximated dominant Floquet multiplier is impacted by the number of collocation points; unnecessarily choosing large M_T, nevertheless, leads to increased computational costs. As we rely on the eigevalue solver in the control design discussed in what follows, we determine the number of collocation point a priori (i.e., for the open loop system) such that the large Floquet multipliers are accurately approximated, and re-evaluate this choice a posteriori (for the synthesized controller).

III. Optimal Time-Periodic Feedback Gain Design

The stabilization problem of a system in the form (3) corresponds to designing the feedback gain $\mathcal{K}$ such that all Floquet multipliers are confined to the open unit disk. The problem of moving the unstable Floquet multipliers into the unit disk, as well as the problem of increasing the decay rate of solutions of a stable system towards the zero equilibrium, leads us to the optimization problem of minimizing the spectral radius of the monodromy operator as a function of the controller variables. Meanwhile, from an implementation perspective, there are restrictions on the extent to which the feedback gain changes throughout the period. This can be imposed in two ways:

1) The first approach entails incorporating a regularization term limiting the total variation of the feedback gain.
2) In the second technique, we impose smoothness explicitly by parametrizing the controller in the form of a truncated Fourier series, considering only the first few harmonics.

A. Controller Design Based on Limiting the Total Variation of the Feedback Gain

The optimization problem is introduced as in (9).

$$\begin{aligned} &\underset{\mathcal{K}}{\text{Minimize}} \;\; \mathcal{P}\big(\mathcal{U}(\mathcal{K})\big) + \lambda_1 \int_0^T w(t) \left\| \dot{\mathcal{K}}(t) \right\|_{Fr}^2 dt, \\ &\text{Subject to} : \mathcal{K}(0) = \mathcal{K}(T), \end{aligned} \tag{9}$$

where $\mathcal{P}\big(\mathcal{U}(\mathcal{K})\big)$ stands for the spectral radius of the monodromy operator, $\|*\|_{Fr}^2$ represents the square Frobenius norm of the matrix $*$, and the parameter λ_1 penalizes the total variation of the feedback gain. Function w represents the time-continuous weight function. Discretizing the eigenvalue problem as outlined in Section II induces a discretization of the controller gain $\mathcal{K}$ into

$$K := [K_1, \ldots, K_\ell, \ldots, K_{M_T}],$$

where $K_\ell = K(\theta_\ell)$ stands for the approximated value of $\mathcal{K}$ at time instants $t = \theta_\ell$. These time instants correspond to the M_T collocation points in the interval $[0, T]$, which remains constant during the optimization. In light of this definition, the discrete form of the infinite-dimensional optimization problem (9) is formulated as

$$\underset{K}{\text{Minimize}} \;\; \mathcal{P}\big(\mathcal{U}_M(K)\big) + \lambda_1 \mathcal{R}_1(K) + \lambda_2 \mathcal{R}_2(K), \tag{10}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are regularization functions defined as

$$\begin{aligned} \mathcal{R}_1(K) &:= \sum_{i=1}^{M_T-1} w_i \frac{\|K_{i+1} - K_i\|_{Fr}^2}{(\theta_{i+1} - \theta_i)}, \\ \mathcal{R}_2(K) &:= \|K_{M_T} - K_1\|_{Fr}^2 . \end{aligned}$$

The constant λ_2 is the penalty parameter for inducing approximate periodicity in the discretized form of the feedback gain. Note that

$$\mathcal{P}\big(\mathcal{U}_M(K)\big) = \max \;\; \{|\mu| : \det\big(\mathcal{M}(\mu; K)\big) = 0\},$$

where we make the dependency of $\mathcal{M}$ on the feedback matrix K explicit, i.e.,

$$\begin{aligned} \mathcal{M} : \mathbb{C} \times \mathbb{R}^{r \times pM_T} &\to \mathbb{C}^{N(M+1)d \times N(M+1)d}, \\ (\mu, K) &\mapsto \mathcal{M}(\mu, K). \end{aligned}$$

The spectral radius of the discretized monodromy operator is in general a nonsmooth function of the gain parameters due to the maximum operator in $\mathcal{P}(\mathcal{U}_M)$. It may even fail to be locally Lipschitz continuous if a dominant Floquet multiplier corresponds to a defective multiple eigenvalue. However, it is almost everywhere differentiable since the dominant Floquet multiplier is almost everywhere simple, which implies differentiability. This considerable merit enables us to solve the optimization problem via a MATLAB software package named HANSO (Hybrid Algorithm for NonSmooth Optimization) [25]. HANSO solves nonsmooth optimization problems by the BFGS algorithm with weak Wolfe line search, possibly combined with the gradient sampling algorithm. Since HANSO only necessitates evaluating the objective function and its gradient almost everywhere, providing computationally tractable formulation for the gradient of the objective function, wherever the dominant Floquet multiplier is simple, is sufficient. If the dominant Floquet multiplier is simple, the gradient of the objective function defined as $F = \mathcal{P}(\mathcal{U}_M) + \lambda_1 \mathcal{R}_1 + \lambda_2 \mathcal{R}_2$ is computed as

$$\begin{aligned} &\nabla_K \mathcal{F}(\tilde{K}) = \\ &\nabla_K |\mu| (\tilde{K}) + \lambda_1 \nabla_K \mathcal{R}_1(\tilde{K}) + \lambda_2 \nabla_K \mathcal{R}_2(\tilde{K}), \end{aligned} \tag{11}$$

where ∇_K denotes the gradient with respect to $\operatorname{vec}(K)$, which represents the vectorized form of K. The only challenging term in the above representation is $\nabla_K |\mu| (\tilde{K})$. The following proposition is introduced to compute the gradient of a simple eigenvalue with respect to parameters of the feedback gain $K_\ell \in \mathbb{R}^{r\times p}$. In what follows, we define the derivative of a simple eigenvalue with respect to feedback gain K_ℓ evaluated at $\tilde{K}$ as the matrix of the same dimensions as K_ℓ such that:

$$\frac{\partial\mu(\tilde{K})}{\partial K_\ell} := \begin{bmatrix} \frac{\partial\mu(\tilde{K})}{\partial K_{1,1}(\theta_\ell)} & \cdots & \frac{\partial\mu(\tilde{K})}{\partial K_{1,p}(\theta_\ell)} \\ \vdots & \ddots & \vdots \\ \frac{\partial\mu(\tilde{K})}{\partial K_{r,1}(\theta_\ell)} & \cdots & \frac{\partial\mu(\tilde{K})}{\partial K_{r,p}(\theta_\ell)} \end{bmatrix} \tag{12}$$

Proposition 2: Consider the linear eigenvalue problem derived in (8). Let $\tilde{K}$ represent a specific feedback gain and $\tilde{\mu}$ be a simple eigenvalue with $\tilde{W}, \tilde{V} \in \mathbb{C}^{N(M+1)d}$ as left and right eigenvectors, respectively, such that $\tilde{W}^*\mathcal{M}(\tilde{\mu};\tilde{K}) = 0$ and $\mathcal{M}(\tilde{\mu};\tilde{K})\tilde{V} = 0$, where $\tilde{W}^*$ stands for the complex conjugate transpose of $\tilde{W}$. Let $\tilde{W}$ be partitioned with block $\tilde{W}_i \in \mathbb{C}^d$ for $i = 1,\dots,N(M+1)$. We store all the blocks $\tilde{W}_i$ for $i = N_\tau(M+1)+N_T+1,\dots,N(M+1)$ in vector $\tilde{Z} \in \mathbb{C}^{M_T d}$. We denote with $\tilde{Z}_\ell \in \mathbb{C}^d$ as ℓth block of the partitioned vector $\tilde{Z}$. It holds for $\ell = 1,\dots,M_T$ that

$$\nabla_{K_\ell}\mu(\tilde{K}) = \frac{\big(B(\theta_\ell)^T\bar{\tilde{Z}}_\ell\big)\big(\tilde{V}^T[\sum_{j=1}^n C_j(\theta_\ell)\mathcal{I}^{(0)}(\theta_\ell-\bar{\tau}_j)]^T\big)}{\tilde{W}^*\mathcal{B}_M(\tilde{K})\tilde{V}}, \tag{13}$$

where $\bar{\tilde{Z}}_\ell$ stands for the complex conjugate of $\tilde{Z}_\ell$. In the scenarios where the feedback gain is expected to remain constant throughout the period, (13) for $\ell = 1,\dots,M_T$ is updated as

$$\nabla_{K_\ell}\mu(\tilde{K}) = \frac{\sum_{\ell=1}^{M_T}\big(B(\theta_\ell)^T\bar{\tilde{Z}}_\ell\big)\big(\tilde{V}^T[\sum_{j=1}^n C_j(\theta_\ell)\mathcal{I}^{(0)}(\theta_\ell-\bar{\tau}_j)]^T\big)}{\tilde{W}^*\mathcal{B}_M(\tilde{K})\tilde{V}}.$$

Proof: Consider feedback gain $K_\ell \in \mathbb{R}^{r\times p}$. The derivative of a simple eigenvalue with respect to a scalar variable $K_{\ell_{z,q}}$, namely zth row and qth column entry of K_ℓ where $z = 1,\dots,r$ and $q = 1,\dots,p$ is calculated via the following well-known formula from [26]:

$$\frac{\partial\mu(\tilde{K})}{\partial K_{\ell_{z,q}}} = -\frac{\tilde{W}^*\frac{\partial\mathcal{M}(\tilde{\mu};\tilde{K})}{\partial K_{\ell_{z,q}}}\tilde{V}}{\tilde{W}^*\frac{\partial\mathcal{M}(\tilde{\mu};\tilde{K})}{\partial\mu}\tilde{V}}.$$

It directly follows from (8) that $\tilde{W}^*\frac{\partial\mathcal{M}(\tilde{\mu};\tilde{K})}{\partial\mu}\tilde{V} = -\tilde{W}^*\mathcal{B}_M(\tilde{K})\tilde{V}$, and the only contribution to the matrix $\frac{\partial\mathcal{M}(\tilde{\mu};\tilde{K})}{\partial K_{\ell_{z,q}}}$ is via (7). Note that since the only constraint depending on K is (7), if we partition vector $\tilde{W}$ in blocks of size d, only the block vectors from the index $N_\tau(M+1)+N_T+1$ appear in the derivative. We store them in vector $\tilde{Z} \in \mathbb{C}^{M_T d}$. Note that $\tilde{Z}_\ell \in \mathbb{C}^d$ represents the ℓth block of partitioned vector $\tilde{Z}$. It is straightforward to write

$$\begin{aligned}&\tilde{W}^*\frac{\partial\mathcal{M}(\tilde{\mu};\tilde{K})}{\partial K_{\ell_{z,q}}}\tilde{V}\\ &= \tilde{Z}_\ell^* B(\theta_\ell)e_z^p e_q^{rT}\sum_{j=1}^n C_j(\theta_\ell)\mathcal{I}^{(0)}(\theta_\ell-\bar{\tau}_j)\tilde{V}\\ &= \big(e_z^{pT}B(\theta_\ell)^T\bar{\tilde{Z}}_\ell\big)\big(\tilde{V}^T[\sum_{j=1}^n C_j(\theta_\ell)\mathcal{I}^{(0)}(\theta_\ell-\bar{\tau}_j)]^T e_q^r\big).\end{aligned}$$

Putting together the formulas for derivatives with respect to each entry of K_ℓ yields the formula for the derivative with respect to all the elements of the matrix K_ℓ as

$$\begin{aligned}&\tilde{W}^*\frac{\partial\mathcal{M}(\tilde{\mu};\tilde{K})}{\partial K_\ell}\tilde{V} =\\ &\big(B(\theta_\ell)^T\bar{\tilde{Z}}_\ell\big)\big(\tilde{V}^T[\sum_{j=1}^n C_j(\theta_\ell)\mathcal{I}^{(0)}(\theta_\ell-\bar{\tau}_j)]^T\big).\end{aligned}$$

■

Proposition 3: The gradient of the modulus of a simple eigenvalue is calculated as

$$\nabla_K|\mu|(\tilde{K}) = \frac{\mathrm{Re}\big(\mu(\tilde{K})^*\nabla_K\mu(\tilde{K})\big)}{|\mu|(\tilde{K})},$$

where $\mathrm{Re}(*)$ stands for the real part of $*$.

Proof: Consider $|\mu|^2(\tilde{K}) = \mu(\tilde{K})\mu(\tilde{K})^*$. We can write

$$\begin{aligned}\nabla_K|\mu|^2(\tilde{K}) =& \mu(\tilde{K})^*\nabla_K\mu(\tilde{K}) + \mu(\tilde{K})\nabla_K\mu(\tilde{K})^*\\ =& 2\,\mathrm{Re}(\mu(\tilde{K})^*\nabla_K\mu(\tilde{K})).\end{aligned}$$

Following this, we can write:

$$\nabla_K|\mu|(\tilde{K}) = \frac{\nabla_K|\mu|^2(\tilde{K})}{2|\mu|(\tilde{K})} = \frac{\mathrm{Re}\big(\mu(\tilde{K})^*\nabla_K\mu(\tilde{K})\big)}{|\mu|(\tilde{K})}.$$

■

B. Controller Design Based on the Truncated Fourier Series of the Feedback Gain

In this section, we restrict the time-periodic gain matrix to a function having a prescribed number of harmonics in its Fourier series expansion. Consider that the feedback gain can be represented as

$$\mathcal{K}(t) = K_0 + \sum_{h=1}^{\iota}\big(K_h^{\cos}\cos(\frac{2h\pi t}{T}) + K_h^{\sin}\sin(\frac{2h\pi t}{T})\big).$$

In this scenario, the variables of optimization problem are the coefficients in the Fourier series, which can be represented in the compact form

$$K = \begin{bmatrix}K_0 & K_1^{\cos} & K_1^{\sin} & \dots & K_\iota^{\cos} & K_\iota^{\sin}\end{bmatrix}^T.$$

With regard to the above explanation, the optimization problem is stated as

$$\underset{K}{\text{Minimize}}\ \ \mathcal{P}\big(\mathcal{U}_M(K)\big).$$

The gradient of a simple eigenvalue with respect to elements of K evaluated at $\tilde{K}$ reads as

$$\nabla_K|\mu|(\tilde{K}) = [\nabla_{K_0}|\mu|(\tilde{K}) \quad \nabla_{K_1^{\cos}}|\mu|(\tilde{K}) \quad \nabla_{K_1^{\sin}}|\mu|(\tilde{K}) \quad \dots \quad \nabla_{K_\iota^{\cos}}|\mu|(\tilde{K}) \quad \nabla_{K_\iota^{\sin}}|\mu|(\tilde{K})]^T.$$

The benefit of enforcing smoothness and periodicity through parameterization, rather than regularization terms, lies not only in bypassing the challenge of determining the proper penalty parameters but also in dealing with significantly fewer optimization variables. In the following section, a case study on the milling operation is presented to provide a further detailed comparison and to discuss the strengths and weaknesses of each method.

IV. Case Study

The milling operation is typically subject to self-excited vibrations, which can result in poor surface finish and dimensional inaccuracies. Moreover, high levels of vibration or operating at resonant frequencies accelerate tool wear and even lead to tool breakage. The reason for such regenerative chatter can be attributed to the instability of the equilibrium of the corresponding nonlinear model. The behavior of the motion of the workpiece around the periodic solution results in a system of TPDDEs. By employing the two methods proposed in this article, we aim to reduce such vibrations through designing a locally stabilizing feedback.

In what follows, we give a brief overview of a milling model with a rigid tool and a flexible workpiece. The reader can refer to [3][12][27] for more details. Consider a TPDDE of the form

$$m_x\ddot{\zeta}(t) + c_x\dot{\zeta}(t) + k_x\zeta(t) = G(t)(\zeta(t-\tau) - \zeta(t)) + k_a u(t), \quad (14)$$

where the coordinate ζ describes the motion of the workpiece along the x axis around a periodic orbit. Parameters m_x, c_x, and k_x stand for the modal mass, damping coefficient, and spring stiffness, respectively. The τ-periodic function G represents the directional cutting coefficient. Equation (14) in its open-loop mode represents a specific case of TPDDEs where the delay and period are identical, defined by $\tau = \frac{60}{z\Omega}$ with z denoting the total number of cutting teeth and Ω indicating the spindle speed. The constant k_a is an approximation of the actuator transfer function, which converts the input voltage to the output force. The control law

$$u(t) = \mathcal{K}(t)y(t) = \begin{bmatrix}\mathcal{K}_1(t) & \mathcal{K}_2(t)\end{bmatrix}\begin{bmatrix}\zeta(t-\tau)\\ \dot{\zeta}(t-\tau)\end{bmatrix}$$

characterizes the input voltage of the actuator, which is strategically positioned on the workpiece holder to compensate the vibrations. We employ the numerical dataset of the milling model provided in [12] as $m_x = 1\ kg$, $c_x = 40\ \frac{Ns}{m}$, $k_x = 10^6\ \frac{N}{m}$, and $z = 2$. Moreover, based on the provided stability charts, we set the axial depth of the cut and the spindle speed equal to $1.5\ mm$ and $5000\ rpm$, respectively. Under such operating conditions, the milling machine experiences unstable regenerative chatter.

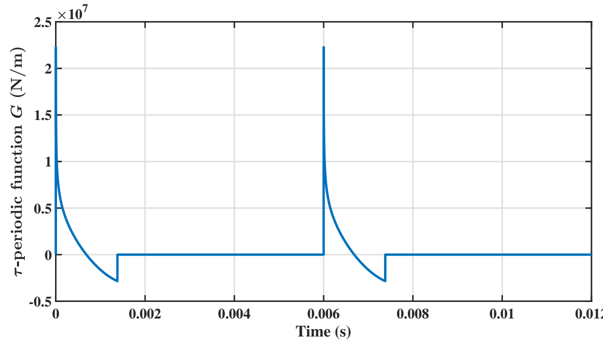

Fig. 3. Temporal behavior of the τ-periodic function G across two periods.

We approximate the transfer function between the output force and the input voltage with $k_a \approx 23.3588$ [27]. The temporal behavior of the function G over two time periods is sketched in Fig. 3. We take $M = 20$, and we choose the length of the subintervals such that the first discontinuity point is identical to the first break point. This gives us five subintervals, four of which are of equal length. The spectral radius of the open-loop system is 1.3847, indicating instability within the system.

First, we attempt to suppress regenerative chatter through stabilization of (14) with a time-constant feedback gain. The dominant Floquet multiplier with such controller is located outside of the unit disk at $1.2976 + 0.4789i$. Therefore, obtaining stability within the TPDDE necessitates a time-varying feedback gain.

To have a well-designed time-periodic feedback gain through solving the optimization problem (9), proper values for the penalty parameters have to be chosen. To this aim, we study the trade-off between the spectral radius of the discretized monodromy operator and the smoothness of the feedback gain. Considering the objective function (10), the regularization term $\mathcal{R}_2$ can be viewed as constraining the variation between the endpoint and the start point of each period. Accordingly, we can write $\mathcal{F} = \mathcal{P}(\mathcal{U}_M) + \lambda_1(\mathcal{R}_1 + \beta\mathcal{R}_2)$. Therefore, we assess the impact of the selected values of λ_1 on the balance between the smoothness indicator term, denoted as $\bar{\mathcal{R}} = (\mathcal{R}_1 + \beta\mathcal{R}_2)$ and $\mathcal{P}(\mathcal{U}_M)$, where β is held constant at 10^2. Fig. 4 illustrates that small values of λ_1 decrease $\mathcal{P}(\mathcal{U}_M)$ at the cost of increasing $\bar{\mathcal{R}}$, resulting in significant variations in the designed gain. Conversely, for sufficiently large values of λ_1, this approach mimics the design of a time-constant feedback gain. Based on the provided explanation, we set $\lambda_1 = 1e-10$ and $\lambda_2 = 1e-8$. Note that solving the optimization problem involves solving the generalized eigenvalue problem (8) iteratively. Since we are only interested in the dominant eigenvalue, we exploit eigenvalue solvers suitable for large-scale matrices [28]. The dominant Floquet multiplier with such a controller is positioned at 0.4508.

As our second proposed approach, we employ the first five terms to describe the Fourier series of the feedback gain (i.e., the constant term, the two terms corresponding to

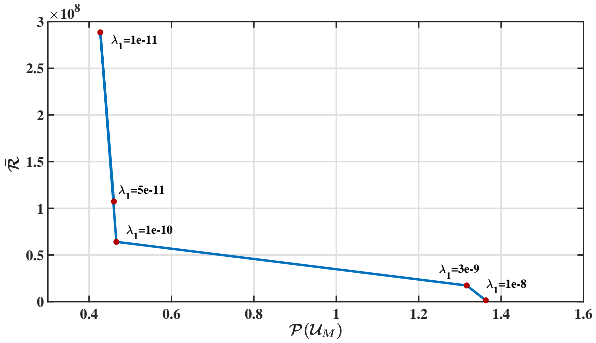

Fig. 4. Assessment of the balance between the spectral radius of the discretized monodromy operator and the smoothness indicator term.

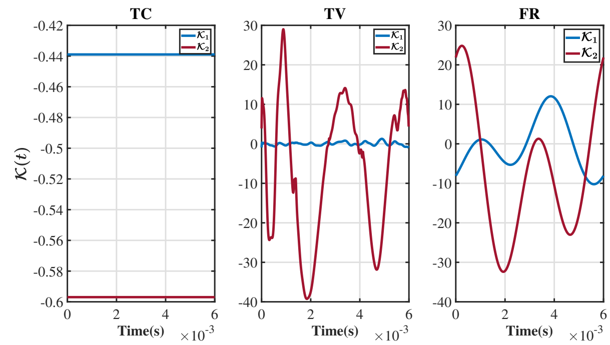

Fig. 5. Temporal behavior of the optimal feedback gain over a time period.

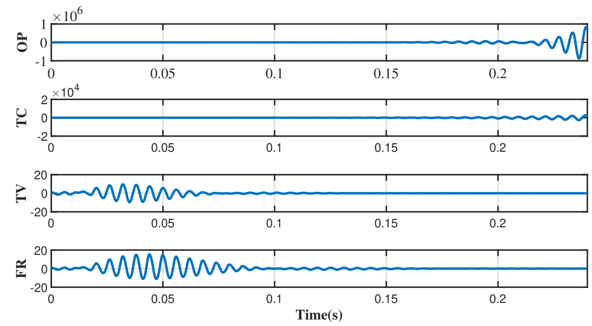

Fig. 6. Temporal behavior of the displacement of the workpiece tool in the milling operation simulated with $dde23$ for different control scenarios.

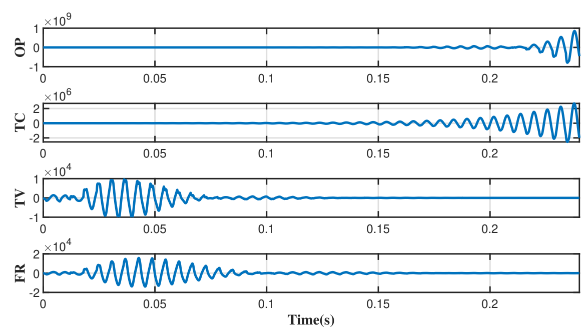

Fig. 7. Temporal behavior of the velocity of the workpiece tool in the milling operation simulated with $dde23$ for different control scenarios.

the fundamental harmonic and the two terms corresponding to the second harmonic). This adjustment is necessary as the system could not be stabilized using only the first three terms. The dominant Floquet multiplier of the closed-loop system with such a controller design is positioned at $0.6102 + 0.1175i$. Table I outlines the spectral radius values corresponding to different optimal feedback gain design scenarios. Note that abbreviations TC, TV, and FR stand for the time-constant feedback gain, feedback gain design based on limiting the total variation, and feedback gain design based on the truncated Fourier series, respectively. The temporal behavior of the feedback gain for the three different design scenarios is sketched in Fig. 5. The FR method exhibits smooth behavior, whereas the feedback gain designed using the TV method experiences minor fluctuations, particularly at the breakpoints of the subintervals. On the other hand, the flexibility of the TV method allows us to have a smaller spectral radius, which results in more rapid vibration suppression.

To validate the obtained results, we investigate the solutions of the TPDDE using $dde23$, a solver in MATLAB for DDEs with constant delays. The temporal behavior of the displacement and velocity is sketched in Fig. 6 and 7, respectively. The four different scenarios include OP, TC, TV, and FR, where OP stands for open-loop. The simulation results reveal that the regenerative chatter cannot be mitigated using a time-constant feedback gain, whereas the other two design methods can provide the system with asymptotic stability. Furthermore, the TV method, due to its flexibility in the controller design, leads to quicker vibration suppression in comparison to the FR method.

V. Conclusions and Future Work

In this article, we discussed that the stabilization problem of TPDDEs demands further attention. Moreover, with the aid of time-periodic feedback, better performance and adaptability are expected in various applications, compared to time-invariant one. We proposed two methods to design a stabilizing time-periodic feedback gain, rooted in minimizing the spectral radius of the monodromy operator. The proposed FR method is more straightforward, since it deals with fewer optimization variables and does not require determining penalty parameters. The TV method, nevertheless, owing to its liberty in the controller design, can provide the system a smaller spectral radius, as long as suitable penalty parameters are chosen. As part of our forthcoming endeavors, we aim to expand our research towards more general structured periodic feedback controllers and to encompass TPDDEs subjected to perturbations in system parameters.

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Method	Spectral Radius	Average Solver Time(s)	Number of Variables
TC	1.3831	7.0098	2
TV	0.4508	277.9851	190
FR	0.6214	339.4299	10

TABLE I

OPTIMIZED SPECTRAL RADIUS, AVERAGE SOLVER TIME, AND NUMBER OF OPTIMIZATION VARIABLES ACROSS DIFFERENT FEEDBACK GAIN DESIGN SCENARIOS FOR 10 STARTING POINTS USING HANSO [25].

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