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On the Robustness of Stability for Quantum Stochastic Systems

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Abstract— **We investigate the impact of undesired Markovian couplings to external systems on the stabilization towards pure states or subspaces of continuously monitored quantum systems under open-loop and feedback control protocols. In particular, we provide sufficient conditions on perturbations to maintain stability of the target and demonstrate the boundedness in mean of the solutions of perturbed systems under open-loop protocols. Effect on the stability of feedback protocols is also analyzed. This work is a step toward a comprehensive robustness analysis of measurement-based control of quantum systems with respect to model perturbations.**

I. Introduction

Accurate engineering of quantum processes has become a critical step towards realistic applications of quantum information, quantum computation, and quantum chemistry [1]. Measurement-based feedback control methods have garnered significant attention due to their ability to stabilize quantum systems towards a target pure state or subspace. In this framework, a continuously-monitored open quantum system is described by a Quantum Stochastic Master Equation (QSME) [2]–[4], and the control is associated with a (possibly time-dependent) Hamiltonian perturbation or dissipative dynamics [5]–[7].

Despite the growing number of methods for nominal control design, the robustness of stability for QSME remains a critical problem for practice. In recent years, several papers have investigated robust stability and stabilization problems with, e.g., uncertain model operators [8], [9] and control input errors [10] based on robust control techniques. In [11], [12] and related papers, robustness of the filter with respect to initialization has been explored. Moreover, recent works [13], [14] proposed an explicit feedback controller that attains stabilization when the initial state and certain model parameters are unknown. However, these robust control schemes assume perturbations with particular structures that leave the target equilibrium point or subspace invariant. As realistic modeling errors may change or destabilize equilibria completely, a deeper analysis of the impact of more general, undesired modeling errors is needed.

In this work, we study the robustness of QSME stability with respect to undesired Markovian couplings to an external environment, based on stochastic Lyapunov techniques. Markovian couplings are a standard way to model the effect of the environment in open quantum systems [1]. This setting can include input fluctuations and small thermal effects that are often ignored in the control design. A detailed comparison with the literature and the application of similar techniques to general model perturbation will be included in a forthcoming journal version of this work [15]. The paper is organized as follows: In Section II, we introduce the systems of interest, the stability notions and the problems we address. In Section III, we analyze the effects of undesired Markovian couplings on the stability of quantum systems under open-loop protocols. In Section IV, we investigate the stability of perturbed quantum systems under feedback protocols. In Section V, we summarize our findings and discuss the implications of our work for future research.

II. QSMEs and their stability

A. Notations and Models

The imaginary unit is denoted by i. For any finite positive integer n, denote $[n] := \{1, \dots, n\}$. We denote $\mathcal{B}(\mathcal{H})$ the set of all linear operators on a finite-dimensional Hilbert space $\mathcal{H}$. Define $\mathcal{B}_*(\mathcal{H}) := \{X \in \mathcal{B}(\mathcal{H}) | X = X^*\}$, $\mathcal{B}_{\geq 0}(\mathcal{H}) := \{X \in \mathcal{B}(\mathcal{H}) | X \geq 0\}$ and $\mathcal{B}_{>0}(\mathcal{H}) := \{X \in \mathcal{B}(\mathcal{H}) | X > 0\}$. We take $\mathbf{I}$ as the identity operator on $\mathcal{H}$, and take $\mathbb{1}$ as the indicator function. We denote the adjoint $A \in \mathcal{B}(\mathcal{H})$ by A^*. The commutator of two operators $A, B \in \mathcal{B}(\mathcal{H})$ is denoted by $[A, B] := AB - BA$. We denote $\bar{\lambda}(A)$ and $\underline{\lambda}(A)$ the maximum and minimum eigenvalue of the Hermitian matrix A, respectively. The function $\mathrm{Tr}(A)$ corresponds to the trace of $A \in \mathcal{B}(\mathcal{H})$. The Hilbert-Schmidt norm of $A \in \mathcal{B}(\mathcal{H})$ is denoted by $\|A\| := \mathrm{Tr}(AA^*)^{1/2}$. Denote $\mathcal{K}$ as the family of all continuous non-decreasing functions $\mu : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\mu(0) = 0$ and $\mu(r) > 0$ for all $r > 0$. Denote $a \wedge b = \min\{a, b\}$ for any $a, b \in \mathbb{R}$.

We consider quantum systems described on a finite-dimensional Hilbert space $\mathcal{H}$. The state of the system is associated to a density matrix on $\mathcal{H}$,

$$\mathcal{S}(\mathcal{H}) := \{\rho \in \mathcal{B}(\mathcal{H}) | \, \rho = \rho^* \geq 0, \mathrm{Tr}(\rho) = 1\}.$$

Assume the system is monitored continuously via n homodyne/heterodyne detectors, which yield a diffusion observation process. In the quantum filtering regime [2], [3], [11], the measurements record of k-th probe $Y_k(t)$ can be described by a continuous semimartingale with quadratic variation $\langle Y_k(t), Y_k(t) \rangle = t$. Denote the filtration generated by the observations up to time t by $\mathcal{F}_t^Y := \sigma(Y(s), 0 \leq s \leq t)$ where $Y(t) = (Y_k(t))_{k \in [n]}$. Its dynamics satisfies $dY_k(t) = dW_k(t) + \sqrt{\eta_k} \mathrm{Tr}((L_k + L_k^*)\rho_t) dt$, where the innovation process $W_k(t)$ is a one-dimensional Wiener process and

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$\langle W_k(t), W_l(t)\rangle = \delta_{k,l}t$ for $k,l \in [n]$, and satisfies $\mathcal{F}_t^Y = \mathcal{F}_t := \sigma\{W_s : 0 \le s \le t\}$ where $W(t) = (W_k(t))_{k\in[n]}$, $\eta_k \in (0,1]$ describes the efficiency of the k-th detector, and $L_k \in \mathcal{B}(\mathcal{H})$ is the noise operator induced by interactions with the probe. The evolution of the quantum state can be described by the following QSME,

$$d\rho_t = \mathcal{L}_u(\rho_t)dt + \textstyle\sum_{k=1}^n \sqrt{\eta_k}\mathcal{G}_k(\rho_t)dW_k(t), \qquad (1)$$

with $\rho_0 \in \mathcal{S}(\mathcal{H})$, where $\mathcal{G}_k(\rho) := L_k\rho + \rho L_k^* - \mathrm{Tr}((L_k + L_k^*)\rho)\rho$, and $\mathcal{L}_u(\rho) := -i[H_0 + u_tH_1, \rho] + \sum_{k=1}^n \mathcal{D}_{L_k}(\rho)$ with $\mathcal{D}_A(\rho) := A\rho A^* - A^*A\rho/2 - \rho A^*A/2$ where $H_0 \in \mathcal{B}_*(\mathcal{H})$ is the free Hamiltonian, $H_1 \in \mathcal{B}_*(\mathcal{H})$ is the control Hamiltonian corresponding to the action of external forces on the quantum system and $u_t \in \mathbb{R}$ represents a control input which is a bounded process adapted to $\mathcal{F}_t^Y$.

Suppose the dynamics is perturbed by Markovian couplings to external systems: the corresponding dynamics can be described by a sum of finite Lindblad generators $F_\alpha(\rho) := \alpha\sum_{k=1}^m \mathcal{D}_{C_k}(\rho)$ where $C_k \in \mathcal{B}(\mathcal{H})$ are noise operators related. We assume that $\sum_k \|C_k\| \le 1$ and the whole perturbation intensity is modulated through a real parameter $\alpha \ge 0$. In this case, the dynamics of perturbed quantum system can be described by the following QSME,

$$\begin{aligned} d\sigma_t &= (\mathcal{L}_u(\sigma_t) + F_\alpha(\sigma_t))dt \\ &\quad + \textstyle\sum_{k=1}^n \sqrt{\eta_k}\mathcal{G}_k(\sigma_t)dW_k(t),\ \sigma_0 \in \mathcal{S}(\mathcal{H}). \qquad (2)\end{aligned}$$

The existence and uniqueness of the solution of (1) and (2), the strong Markov property of the solutions and the almost sure invariance of $\mathcal{S}(\mathcal{H})$ for the solution can be proved by combining the arguments in [4, Theorem 3.4, Proposition 3.10] and [5, Section 3].

B. Stability notions

Let $\mathcal{H}_S \subset \mathcal{H}$ be the target subspace. Denote by $\Pi_0 \notin \{0, \mathbf{I}\}$ the orthogonal projection on $\mathcal{H}_S \subset \mathcal{H}$. Define the set of density matrices

$$\mathcal{I}(\mathcal{H}_S) := \{\rho \in \mathcal{S}(\mathcal{H}) | \mathrm{Tr}(\Pi_0\rho) = 1\},$$

namely those whose support is contained in $\mathcal{H}_S$.

Definition 2.1: For the system (1), the subspace $\mathcal{H}_S$ is called invariant almost surely if $\rho_0 \in \mathcal{I}(\mathcal{H}_S)$, $\rho_t \in \mathcal{I}(\mathcal{H}_S)$ for all $t > 0$ almost surely.

Based on the stochastic stability defined in [16], [17] and the definition used in [18], [19], we phrase the following definition on the stochastic stability of the invariant subspace.

Definition 2.2: Let $\mathcal{H}_S \subset \mathcal{H}$ be the invariant subspace for the system (1), and denote Π_0 the orthogonal projection on $\mathcal{H}_S$ and $\mathbf{d}_0(\rho) := \|\rho - \Pi_0\rho\Pi_0\|$, then $\mathcal{H}_S$ is said to be

1) *globally exponentially stable (GES) in mean*, if there exist a pair of positive constants λ and c such that

$$\mathbb{E}\big(\mathbf{d}_0(\rho_t)\big) \le c\,\mathbf{d}_0(\rho_0)e^{-\lambda t}, \quad \forall \rho_0 \in \mathcal{S}(\mathcal{H}).$$

2) *stable in probability*, if for every $\varepsilon \in (0,1)$ and for every $r > 0$, there exists $\delta = \delta(\varepsilon, r) > 0$ such that,

$$\mathbb{P}\big(\mathbf{d}_0(\rho_t) < r \text{ for } t \ge 0\big) \ge 1 - \varepsilon,$$

whenever $\mathbf{d}_0(\rho_0) < \delta$.

3) *globally exponentially stable (GES) almost surely*, if

$$\limsup_{t\to\infty} \frac{1}{t}\log\big(\mathbf{d}_0(\rho_t)\big) < 0, \quad \forall \rho_0 \in \mathcal{S}(\mathcal{H}), \quad a.s.$$

The left-hand side of the above inequality is called the *sample Lyapunov exponent*.

Based on the nature of the control input u_t, there are two types of control protocols for the stabilization of the system (1): 1) if u_t is a real bounded deterministic process depending on t, this protocol is called open-loop stabilization [6], [19]. 2) if u_t is a real bounded stochastic process adapted to $\mathcal{F}_t^Y$, this protocol is called feedback stabilization [5], [13].

In this paper, we suppose the target subspace $\mathcal{H}_S \subset \mathcal{H}$ is GES almost surely with respect to the nominal system (1), we will analyze the effect of the perturbation, i.e., noisy Lindbladians, on the stability of the nominal system under open-loop and feedback protocols respectively.

C. Conditions for robust invariance

Let $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$ and $X \in \mathcal{B}(\mathcal{H})$, the matrix representation in an appropriately chosen basis can be written as

$$X = \begin{bmatrix} X_S & X_P \\ X_Q & X_R \end{bmatrix},$$

where X_S, X_R, X_P and X_Q are matrices representing operators from $\mathcal{H}_S$ to $\mathcal{H}_S$, from $\mathcal{H}_R$ to $\mathcal{H}_R$, from $\mathcal{H}_R$ to $\mathcal{H}_S$, from $\mathcal{H}_S$ to $\mathcal{H}_R$, respectively. We denote by Π_R the orthogonal projection on $\mathcal{H}_R \subset \mathcal{H}$. In [18], Ticozzi and Viola showed that the invariance of the subspace enforces a given structure for the associated semi-group generator. It is possible to prove that if $\mathcal{H}_S$ is invariant for the nominal dynamics (1), the following assumption $\mathbf{A}$ on noisy Lindbladians are satisfied, then $\mathcal{H}_S$ is also invariant almost surely with respect to the perturbed system (2) with arbitrary α,

A: $\forall k \in [m]$, $C_{k,Q} = 0$ and $\sum_k C_{k,S}^* C_{k,P} = 0$.

III. Robustness of open-loop stabilization

In this section, we suppose that $u_t \equiv u$ where u is a real constant and $\mathcal{H}_S$ is GES a.s. with respect to the nominal system (1). We define the following map,

$$\mathcal{L}_{R,\alpha}(\rho_R) := -i[H_R, \rho_R] + \textstyle\sum_k \mathsf{D}_{L_k}(\rho_R) + \alpha\sum_k \mathsf{D}_{C_k}(\rho_R),$$

where $H_R := H_{0,R} + uH_{1,R}$, $\mathsf{D}_A(\rho_R) := A_R\rho_R A_R^* - \frac{1}{2}\{A_P^*A_P + A_R^*A_R, \rho_R\}$, and $\rho_R \in \mathcal{B}_{\ge 0}(\mathcal{H}_R)$. For R-block of any Lindblad generator, we denote $\mathcal{L}_{R,\alpha}^*$ the adjoint of $\mathcal{L}_{R,\alpha}$ with respect to the Hilbert-Schmidt inner product on $\mathcal{B}(\mathcal{H}_R)$. We recall the following results [19] concerning on the spectral property of $\mathcal{L}_{R,\alpha}^*$ and the relation with the GES in mean with respect to the Lindblad generator. Denote by λ_α the spectral abscissa of $\mathcal{L}_{R,\alpha}$, i.e., $\lambda_\alpha := \min\{-\mathbf{Re}(x) | \, x \in \mathrm{sp}(\mathcal{L}_{R,\alpha})\}$.

A. Effect of Perturbations that preserve target invariance

Firstly, we consider the case where $\mathcal{H}_S$ is invariant for the perturbed system (2). Then, we obtain the following results by using similar arguments as in [19, Section 2].

Proposition 3.1: Suppose that $\mathcal{H}_S$ is invariant for $\mathcal{L}_u(\sigma)+F_\alpha(\sigma)$. For any $\alpha \geq 0$ and $\varepsilon > 0$, there exists $K_R \in \mathcal{B}_{>0}(\mathcal{H}_R)$ such that $\mathcal{L}^*_{R,\alpha}(K_R) \leq -(\lambda_\alpha - \varepsilon)K_R$. Moreover, $\mathcal{H}_S$ is GES a.s. for the perturbed system (2) iff $\lambda_\alpha > 0$.

The above proposition requires the full information of the noisy Lindbladian to ensure the GES of target subspace. In the following, we consider the case where $\mathcal{H}_S$ is GES a.s. for the nominal system (1), that is $\lambda_0 > 0$ by Proposition 3.1. Based on K_R and the convergence rate $c > 0$, we show that the target subspace maintains GES when the perturbation is small enough.

Corollary 3.2: Suppose that $\lambda_0 > 0$. Then, there exist $K_R \in \mathcal{B}_{>0}(\mathcal{H}_R)$ and $c > 0$ such that $\mathcal{L}^*_{R,0}(K_R) < -cK_R$. Moreover, if **A** is satisfied and $2\alpha\|K_R\|/\underline{\lambda}(K_R) < c$, then $\mathcal{H}_S$ is GES a.s. for the perturbed system (2).

Proof. The first part can be obtained directly by applying Proposition 3.1. For any $\alpha \geq 0$, we have

$$\begin{aligned}\mathcal{L}^*_{R,\alpha}(K_R) &= \mathcal{L}^*_{R,0}(K_R) + \alpha \textstyle\sum_k \mathsf{D}^*_{C_k}(K_R)\\ &\leq -cK_R + \alpha \textstyle\sum_k \mathsf{D}^*_{C_k}(K_R).\end{aligned}$$

Since $K_R > 0$, for any $k \in [m]$, we have

$$\mathsf{D}^*_{C_k}(K_R) \leq \frac{\bar{\lambda}\big(\mathsf{D}^*_{C_k}(K_R)\big)}{\underline{\lambda}(K_R)}K_R \leq \frac{\|\mathsf{D}^*_{C_k}(K_R)\|}{\underline{\lambda}(K_R)}K_R.$$

By the definition of $\mathsf{D}^*_{C_k}$ and the norm inequality,

$$\begin{aligned}\|\mathsf{D}^*_{C_k}(K_R)\| &\leq (2\|C_{k,R}\|^2 + \|C_{k,P}\|^2)\|K_R\|\\ &\leq 2\|C_k\|^2\|K_R\|. \end{aligned} \tag{3}$$

It implies,

$$\textstyle\sum_k \mathsf{D}^*_{C_k}(K_R) \leq \frac{2\sum_k \|C_k\|^2\|K_R\|}{\underline{\lambda}(K_R)}K_R \leq \frac{2\|K_R\|}{\underline{\lambda}(K_R)}K_R,$$

where we used the assumption $\sum_k \|C_k\| \leq 1$. Therefore, we deduce,

$$\mathcal{L}^*_{R,\alpha}(K_R) \leq -\big(c - 2\alpha\|K_R\|/\underline{\lambda}(K_R)\big)K_R.$$

The results can be concluded by applying the similar arguments as in [19, Section 2]. □

Remark 3.3: Consider the nominal system (1) where $\mathcal{H}_S$ is GES a.s., and **A** is satisfied. By utilizing the *Dissipation-Induced Decomposition* technique [20], we can establish that in fact $\mathcal{H}_S$ remains GES a.s. for the perturbed system (2) for all values of α. Furthermore, we can find a K_R for the perturbed system. Since the proof of this result falls beyond the scope of this paper, we omit it; see [15] for further details.

B. Effect of general Perturbations

Next, we consider the general case, the noisy Lindbladians are supposed to be unknown. In this case, GES should not be expected since $\mathcal{H}_S$ may not be invariant for the perturbed system. Instead, we study the boundedness of $\mathbb{E}[\mathbf{d}_0(\sigma(t))]$.

Proposition 3.4: Suppose that $\lambda_0 > 0$. Then, there exist $K_R \in \mathcal{B}_{>0}(\mathcal{H}_R)$ and $c > 0$ such that $\mathcal{L}^*_{R,0}(K_R) < -cK_R$. Moreover, for the perturbed system (2) and for all $t \geq 0$,

$$\mathbb{E}[\mathbf{d}_0(\sigma(t))] \leq c_1 e^{-ct/2} + \sqrt{\alpha D}, \tag{4}$$

for some constant $c_1 > 0$, where $D := \frac{2\|K_R\|}{3\underline{\lambda}(K_R)c}$, and

$$\mathbb{P}\big(\mathbf{d}_0(\sigma(t)) < \delta(c_1 e^{-ct/2} + \sqrt{\alpha D})\big) \geq 1 - 1/\delta,\ \forall \delta \geq 1. \tag{5}$$

Proof. Denote the *extension of K_R to $\mathcal{B}(\mathcal{H})$* by $K = \left[\begin{smallmatrix} 0 & 0 \\ 0 & K_R\end{smallmatrix}\right]$. Then $\mathrm{Tr}(K\sigma) = \mathrm{Tr}(K_R\sigma_R)$. Due to the linearity, we have

$$\begin{aligned}\mathscr{L}\mathrm{Tr}(K\sigma) &= \mathrm{Tr}(\mathcal{L}^*_{R,\alpha}(K_R)\sigma_R)\\ &\leq -c\mathrm{Tr}(K\sigma) + 2\alpha\|K_R\|,\end{aligned}$$

where $\mathscr{L}$ is related to the equation (2), and for the second inequality, we used the Cauchy–Schwarz inequality, the relation $\|\sigma_R\| \leq \|\sigma\| \leq 1$ and (3) with the assumption $\sum_k \|C_k\| \leq 1$. By applying the Itô's formula and then taking the expectation, we get $\frac{d}{dt}e^{ct}\mathbb{E}(\mathrm{Tr}(K\sigma_t)) \leq \alpha D e^{ct}$, which implies

$$\mathbb{E}(\mathrm{Tr}(K\sigma_t)) \leq e^{-ct}\big(\mathrm{Tr}(K\sigma_0) - \alpha D/c\big) + \alpha D/c.$$

In addition to the inequalities $\mathbf{d}_0(\sigma)^2 \leq 3\mathrm{Tr}(\Pi_R\sigma) \leq \frac{3}{\underline{\lambda}(K_R)}\mathrm{Tr}(K\sigma)$ (see [21, Lemma 6] for the proof) where $\sqrt{x+y} \leq \sqrt{x} + \sqrt{y}$ for all $x, y \geq 0$, the equation (4) can be obtained. Moreover, for any $\delta \geq 1$, we have

$$\begin{aligned}&\mathbb{E}[\mathbf{d}_0(\sigma_t)\mathbb{1}_{\{\mathbf{d}_0(\sigma_t)\geq\delta(c_1e^{-ct/2}+\sqrt{\alpha D})\}}]\\ \geq&\delta(c_1 e^{-ct/2} + \sqrt{\alpha D})\mathbb{P}\big(\mathbf{d}_0(\sigma_t) \geq \delta(c_1 e^{-ct/2} + \sqrt{\alpha D})\big),\end{aligned}$$

together with the inequality (4), the equation (5) can be concluded. □

Remark 3.5: The property (4) ensures that the solution of the perturbed system (2) is bounded in mean, moreover, when the perturbation vanishes, i.e., $\alpha = 0$, the system (2) is GES in mean, then by following the similar arguments as in [19], we can show that it is also GES a.s. The property (5) is referred to as the *stochastic noise-to-state stability* [22], [23]. In our case stability is also exponential.

IV. Robustness of feedback stabilization

In this section, we consider the case where the initial state ρ_0 and the measurement efficiency η_k of the nominal system (1) are unknown, following the treatments in [14], we construct an estimator $\hat{\rho}_t$ using

$$\begin{aligned}d\hat{\rho}_t =& \mathcal{L}_u(\hat{\rho}_t)dt + \textstyle\sum_{k=1}^n \sqrt{\hat{\eta}_k}\mathcal{G}_k(\hat{\rho}_t)\big(dW_k(t)\\ &+ \mathcal{T}_k\big(\rho_t, \hat{\rho}_t\big)dt\big),\ \ \hat{\rho}_0 \in \mathcal{S}(\mathcal{H})\end{aligned} \tag{6}$$

where $\mathcal{T}_k(\rho, \hat{\rho}) := \sqrt{\eta_k}\mathrm{Tr}((L_k + L_k^*)\rho) - \sqrt{\hat{\eta}_k}\mathrm{Tr}((L_k + L_k^*)\hat{\rho})$. The control input is a function of the estimator, i.e., $u_t = u(\hat{\rho}_t)$, and suppose $u \in \mathcal{C}^1(\mathcal{S}(\mathcal{H}), \mathbb{R})$ to ensure the existence and uniqueness of the solution of the coupled system (1)–(6) (see [5, Proposition 3.5]) and the a.s. invariance of $\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H})$ for (1)–(6). Now, let us consider the system perturbed by noisy Lindbldadians $F_\alpha(\sigma)$. For the perturbed system (2), the noisy Lindbldadians are usually unknown, and sometimes we have only partial information about them. Thus, under feedback protocols, it is natural to continue using (6) to estimate the trajectories of the perturbed system (2).

Under open-loop protocols, the Lindblad generator $\mathcal{L}_u(\rho)$ dominates the stabilization process and the converse (linear) Lyapunov theorem for the nominal system (1) is proved in [19, Theorem 1.2]. However, under feedback protocols, it is very challenging to construct a global Lyapunov function. Thus, we cannot follow the treatments in Section III to characterize the behavior of the perturbed system. Instead, we investigate the dependence of the trajectories on the perturbation magnitude α and the effect of noisy Lindbladians on the stability.

In the following proposition, we bound the trajectories distance as a function of α. Let $(\rho_t, \hat{\rho}_t)$ and $(\sigma_t, \hat{\sigma}_t)$ be the solutions of nominal coupled system (1)–(6) and perturbed coupled system (2)–(6) with $(\rho_0, \hat{\rho}_0) = (\sigma_0, \hat{\sigma}_0)$. It specifies the power rate of convergence, and the rate of getting to infinity of the lengths of the time interval. Both rates depend on the perturbation magnitude α. Denote $\Delta_t^\alpha := \rho_t - \sigma_t$, $\hat{\Delta}_t^\alpha := \hat{\rho}_t - \hat{\sigma}_t$ and $U_t^\alpha := \mathbb{E}(\|\Delta_t^\alpha\|^2 + \|\hat{\Delta}_t^\alpha\|^2)$.

Proposition 4.1: There exist two constants $A, B > 0$ such that for all $t \geq 0$, $\sqrt{U_t^\alpha} \leq \alpha A(e^{Bt} - 1)$. Moreover, for any $\gamma \in (0,1)$ and $\alpha > 0$, $U_t^\alpha \leq \alpha^{2\gamma}$ for all $t \in [0, \frac{1}{B}\log(1 + \frac{1}{A\alpha^{1-\gamma}})]$.
The proof is provided in Appendix.

Corollary 4.2: Suppose that $(\mathcal{H}_S, \mathcal{H}_S)$ is GES a.s. for the nominal coupled system (1)–(6) with the sample Lyapunov exponent less than or equal to $-\lambda$. Then, for any initial state $(\rho_0, \hat{\rho}_0)$, there exists a finite constant $c > 0$ such that, for any $\gamma \in (0,1)$ and $\alpha > 0$, for all $t \in [0, \frac{1}{B}\log(1 + \frac{1}{A\alpha^{1-\gamma}})]$

$$\mathbb{E}(\mathbf{d}_0(\sigma_t)^2) \leq c_1\big(\alpha^\gamma + ce^{-\lambda t}\big),$$

for some constant $c_1 > 0$.

Proof. Due to the a.s. GES of $(\mathcal{H}_S, \mathcal{H}_S)$, for all initial state, there exists a finite random variable $R(\omega) > 0$ such that $\mathbf{d}_0(\rho_t) \leq R(\omega)e^{-\lambda t}$ a.s. Together with the inequalities $\mathrm{Tr}(\Pi_R\sigma)^2/N \leq \mathbf{d}_0(\sigma)^2 \leq 3\mathrm{Tr}(\Pi_R\sigma)$ with $N = \dim(\mathcal{H})$, by applying Proposition 4.1, we have the following estimation, for any $\gamma \in (0,1)$, for all $t \in [0, \frac{1}{B}\log(1 + \frac{1}{A\alpha^{1-\gamma}})]$,

$$\begin{aligned}\mathbb{E}(\mathbf{d}_0(\sigma_t)^2) &\leq 3\mathbb{E}(\mathrm{Tr}(\Pi_R\sigma_t)) \\ &\leq 3\mathbb{E}\big(|\mathrm{Tr}(\Pi_R\sigma_t) - \mathrm{Tr}(\Pi_R\rho_t)| + \mathrm{Tr}(\Pi_R\rho_t)\big) \\ &\leq c_1\big(\alpha^\gamma + \mathbb{E}(R)e^{-\lambda t}\big),\end{aligned}$$

where we used the Lipschitz arguments $|\mathrm{Tr}(\Pi_R\rho) - \mathrm{Tr}(\Pi_R\sigma)| \leq c_1\|\Delta\|$ for some constant $c_1 > 0$. □

A. Impact of perturbations on stability in probability

In the following, we first present a Lyapunov-based approach for analyzing stochastic systems, which allows us to investigate how perturbations affect the stability of the system in probability. Specifically, we consider a classical SDE and introduce a perturbation in the drift term by adding $\alpha f_{\mathrm{dis}}(q)$, where $\alpha \geq 0$,

$$dq_t = f(q_t)dt + \alpha f_{\mathrm{dis}}(q_t)dt + g(q_t)dW_t. \tag{7}$$

Assume f_{dis} is appropriately defined functions so that $\{q_t\}_{t\geq 0}$ becomes a unique strong solution. Let $\bar{S} \subset Q$ be a target subset of a control problem. Define $\tau_r := \inf\{t \geq 0 |\, V(q_t) = r\}$ with $r \geq 0$. Moreover, we make the following assumption:

H: there exists $V \in \mathcal{C}^2(Q, \mathbb{R}_{\geq 0})$ such that $V(q) = 0$ iff $q \in \bar{S}$ and a function $\mu \in \mathcal{K}$ such that, $\mathscr{L}V(q) \leq -\mu(V(q)) + \alpha D$ for all $q \in S_c := \{q \in Q \setminus \bar{S} |\, V(q) < c\}$ for some $c, D > 0$, where $\mathscr{L}$ is related to (7).

The following lemma shows how the unknown α and f_{dis} deteriorate the stability.

Lemma 4.3: Assume that **H** is satisfied. For any $\varepsilon \in (0,1)$ and $r \in (0,c)$, there exists $\delta = \delta(\varepsilon, r) \in (0,r)$ such that for all $q_0 \in S_\delta$,

$$\mathbb{P}(V(q_t) < r, \forall t \geq 0) \geq 1 - \varepsilon - \alpha D\mathbb{E}(\tau_r)/r. \tag{8}$$

Moreover, the stability in probability is restored when α tends to zero.

Proof. The proof basically follows the arguments of [17, Theorem 4.2.2]. For any $\varepsilon \in (0,1)$ and $r \in (0,c)$, we can find $\delta = \delta(\varepsilon, r) > 0$ such that $\sup_{q\in S_\delta} V(q) < \varepsilon r$. Then, Itô's formula gives

$$\mathbb{E}[V(q_{\tau_r\wedge t})] \leq V(q_0) - \mathbb{E}\int_0^{\tau_r\wedge t}\mu(V(q_s))ds + \alpha D\mathbb{E}(\tau_r \wedge t).$$

Using non-negativity of V and the definition of μ, $\mathbb{E}[V(q_{\tau_r\wedge t})] \geq \mathbb{E}[\mathbb{1}_{\{\tau_r\leq t\}}V(q_{\tau_r})] \geq \mathbb{P}(\tau_r \leq t)r$. Since $\sup_{q_0\in S_\delta} V(q_0) < \varepsilon r$ and $r > 0$, $\mathbb{P}(\tau_r \leq t) \leq \varepsilon + \alpha D\mathbb{E}(\tau_r \wedge t)/r$. By monotone convergence theorem, $\mathbb{P}(\tau_r < \infty) \leq \varepsilon + \alpha D\mathbb{E}(\tau_r)/r$.

Then, for the inequality (IV-A), let α tend to zero, we have $\mathbb{P}(\tau_r \leq t) \leq \varepsilon$, which implies that $\bar{S}$ is stable in probability by letting $t \to \infty$. □

Next, by using the above lemma, we investigate how noisy Lindbladians affect the stability of the nominal system (1)–(6) in probability.

Proposition 4.4: Suppose that there exist a function $\mathsf{V} \in \mathcal{C}^2(\mathcal{S}(\mathcal{H}) \times \mathcal{S}(\mathcal{H}), \mathbb{R})$, a constant $l > 0$ and $\mu \in \mathcal{K}$ such that $\mathscr{L}\mathsf{V} \leq -\mu(\mathsf{V})$ whenever $\mathsf{V} < l$, where $\mathscr{L}$ relates to the nominal system (1)–(6). Then, for the perturbed system (2)–(6), for all $\varepsilon > 0$, there exist $\delta \in (0,r)$ and $c > 0$ such that

$$\mathbb{P}\big(\mathsf{V}(\sigma_t, \hat{\sigma}_t) < l, \forall t \geq 0\big) \geq 1 - \varepsilon - \alpha c\mathbb{E}(\tau_l)/l$$

whenever $\mathsf{V}(\sigma_0, \hat{\sigma}_0) < \delta$, where τ_l denotes the first exiting time of $(\sigma_t, \hat{\sigma}_t)$ from $\{\mathsf{V} < l\}$. Moreover, the stability in probability is restored when α tends to zero.

Proof. Due to the continuity of $\partial\mathsf{V}/\partial\sigma$ and the compactness of $\mathcal{S}(\mathcal{H})$, there exists a constant $D > 0$ such that $\bar{\mathscr{L}}\mathsf{V} \leq -\mu(\mathsf{V}) + \alpha D$ where $\bar{\mathscr{L}}$ relates to (2)–(6). The result can be concluded by applying Lemma 4.3. □

B. N-level spin systems

In the following, we consider a concrete scenario of N-level spin systems undergoing non-demolition homodyne detection [13]. In this situation, some further analysis on the effect of invariance-preserving perturbation can be carried out adapting existing Lyapunov-based methods. Let $\Phi := \{e_1, \ldots, e_N\}$ correspond to an orthonormal basis of $\mathcal{H}$, assume the target subspace $\mathcal{H}_S \in \{ae_1, be_N\}$ for some non-zero $a, b \in \mathbb{C}$, i.e., $\dim(\mathcal{H}_S) = 1$ and we are seeking the stabilization toward a target pure state $\Pi_0 \in \Gamma := \{e_1e_1^*, e_Ne_N^*\}$. Set $H_0 = \omega J_z$ with $\omega > 0$, $H_1 = J_y$, $n = 1$ and $L = \sqrt{\gamma}J_z$ with $\gamma > 0$. J_z and J_y are the angular momentum operators along z-axis and y-axis respectively, the matrix form with respect to Φ are given by

$$J_z = \mathrm{diag}[J, \dots, -J], \;\; J_y = \begin{bmatrix} 0 & -ic_1 & & & \\ ic_1 & 0 & -ic_2 & & \\ & \ddots & \ddots & \ddots & \\ & & ic_{2J-1} & 0 & -ic_{2J} \\ & & & ic_{2J} & 0 \end{bmatrix},$$

where $J := (N-1)/2$ represents the fixed angular momentum and $c_m = \sqrt{(N-m)m}/2$.

We impose the following assumptions on feedback controller and parameters:

A1: $u \in \mathcal{C}^1(\mathcal{S}(\mathcal{H}), \mathbb{R})$ such that $u(\hat{\sigma}) \neq 0$ for all $\hat{\sigma} \in \{e_1 e_1^*, \dots, e_N e_N^*\} \setminus \Pi_0$ and $u(\hat{\sigma}) = 0$ whenever $\mathbf{d}_0(\hat{\sigma}) < r$ for some $r > 0$.

A2: $\frac{2N-2}{2N-1} < \sqrt{\frac{\hat{\eta}}{\eta}} < \frac{1}{2} + \frac{1}{2}\sqrt{\frac{N+1}{N-1}}$.

Suppose that **A1** and **A2** are satisfied. [13, Theorem 4.14] shows that, for the nominal system (1)–(6) with $(\rho_0, \hat{\rho}_0) \in \mathcal{S}(\mathcal{H}) \times \mathrm{int}(\mathcal{S}(\mathcal{H}))$ where $\mathrm{int}(\mathcal{S}(\mathcal{H})) := \{\rho \in \mathcal{S}(\mathcal{H}) \,|\, \rho > 0\}$, $(\mathcal{H}_S, \mathcal{H}_S)$ is GES a.s. with the sample Lyapunov exponent less than or equal to $\bar{c} := -(\eta \wedge \hat{\eta} + (N-1)\sqrt{\hat{\eta}}|\sqrt{\eta} - \sqrt{\hat{\eta}}|)\gamma$. Then, Corollary 4.2 can provide an estimation of the trajectories of perturbed spin systems under feedback protocols.

Next, we consider the case where $\mathcal{H}_S$ is invariant for the noisy Lindbladians. By following the similar arguments as in [13], the a.s. GES of target subspace is robust with respect to noisy Lindbldadians if **A** holds.

Proposition 4.5: Suppose that **A**, **A1** and **A2** are satisfied. Then, for all $(\sigma_0, \hat{\sigma}_0) \in \mathcal{S}(\mathcal{H}) \times \mathrm{int}(\mathcal{S}(\mathcal{H}))$, $(\mathcal{H}_S, \mathcal{H}_S)$ is GES a.s. for the perturbed coupled system (2)–(6) and the sample Lyapunov exponent is less than or equal to $\bar{c}$.

Compared with Proposition 3.1, under feedback protocols, the back-action of the non-demolition measurements, i.e., the diffusion term of (1), plays an important role in the stabilization in Proposition 4.5. It ensures the recurrence property of the trajectories related to any neighbourhood of the target subspace and (local) stability in probability (see [13], [15], [24] for more discussion). In the following, we examine how the perturbations deteriorate the stability in probability. Define $\mathsf{V}(\sigma, \hat{\sigma}) := \mathrm{Tr}(\sigma \Pi_R) + \sqrt{\mathrm{Tr}(\hat{\sigma}\Pi_R)}$ and denote by τ_l the first exiting time of $(\sigma_t, \hat{\sigma}_t)$ from $\{\mathsf{V} < l\}$.

Proposition 4.6: Suppose that **A1** and **A2** are satisfied. For all $\varepsilon > 0$, there exist $\delta \in (0, r)$ and $l > 0$ such that

$$\mathbb{P}\big(\mathsf{V}(\sigma_t, \hat{\sigma}_t) < l, \forall t \geq 0\big) \geq 1 - \varepsilon - \alpha 2(N-1)\mathbb{E}(\tau_l)/l$$

whenever $\mathsf{V}(\sigma_0, \hat{\sigma}_0) < \delta$ and $\hat{\sigma}_0 \in \mathrm{int}(\mathcal{S}(\mathcal{H}))$. Moreover, the stability in probability is restored when α tends to zero.

Proof. By a straightforward calculation, we have

$$\bar{\mathscr{L}}\mathsf{V}(\sigma, \hat{\sigma}) \leq \mathrm{Tr}(\Pi_R F_\alpha(\sigma)) - \mathsf{C}(\sigma, \hat{\rho})\sqrt{\mathrm{Tr}(\hat{\sigma}\Pi_R)},$$
$$\mathsf{C}(\sigma, \hat{\rho}) = \hat{\eta}\mathrm{Tr}(\hat{\sigma}\Pi_R)^2/2 - 2\sqrt{\hat{\eta}}\mathrm{Tr}(\hat{\sigma}\Pi_R)|\mathcal{T}(\sigma, \hat{\sigma})|.$$

Moreover, we have

$$\liminf_{(\sigma,\hat{\sigma}) \to \mathcal{I}(\mathcal{H}_S) \times \mathcal{I}(\mathcal{H}_S)} \mathsf{C}(\sigma, \hat{\rho}) \geq c > 0$$

where $c := (\hat{\eta}/2 - 2J\sqrt{\hat{\eta}}|\sqrt{\eta} - \sqrt{\hat{\eta}}|)\gamma$ and the positivity is guaranteed by **A2**. Due to the continuity, there exist constants $l > 0$ and $\mu \in \mathcal{K}$ such that

$$\bar{\mathscr{L}}\mathsf{V} \leq -\mu(\mathsf{V}) + \alpha \textstyle\sum_k \|\mathcal{D}^*_{C_k}(\Pi_R)\|$$
$$\leq -\mu(\mathsf{V}) + 2\alpha(N-1), \quad \text{whenever } \mathsf{V} < l,$$

where we used the fact $\|\Pi_R\| = N-1$ and $\sum_k \|C_k\| \leq 1$. Note that, V is not $\mathcal{C}^2$ due to the square root, however, by using the similar arguments as in [24, Lemma 4.3], we can show that for all $\hat{\sigma}_0 \in \mathrm{int}(\mathcal{S}(\mathcal{H}))$, $\mathbb{P}(\hat{\sigma}_t \in \mathrm{int}(\mathcal{S}(\mathcal{H})), \forall t \geq 0) = 1$. Then, the result can be concluded by applying the similar arguments as in the proof of Lemma 4.3. $\square$

V. Conclusion

In this paper, we have investigated the effects of undesired Markovian couplings on the stability of open quantum systems, both under open-loop and feedback protocols. We have provided sufficient conditions on perturbations to maintain stability of the target and demonstrated the boundedness in mean of the solutions of perturbed systems under open-loop protocols. Additionally, we have analyzed the dependence of the solution of feedback SME on the perturbation and investigated how the the latter deteriorates the stability in probability. Our findings provide a solid foundation for a more comprehensive study of robust stabilization of QSME, which is essential for practical applications of quantum control. Further research efforts focus on characterizing the behavior of perturbed systems under feedback protocols and considering more general and realistic perturbations to expand the applicability of our findings.

Appendix

A. Itô's formula

Given a stochastic differential equation $dq_t = f(q_t)dt + g(q_t)dW_t$, where q_t takes values in $Q \subset \mathbb{R}^p$, the related infinitesimal generator is the operator $\mathscr{L}$ acting on twice continuously differentiable functions $V : Q \times \mathbb{R}_+ \to \mathbb{R}$ as below

$$\mathscr{L}V(q,t) := \frac{\partial V(q,t)}{\partial t} + \textstyle\sum_{i=1}^p \frac{\partial V(q,t)}{\partial q_i} f_i(q)$$
$$+ \frac{1}{2}\textstyle\sum_{i,j=1}^p \frac{\partial^2 V(q,t)}{\partial q_i \partial q_j} g_i(q) g_j(q).$$

Itô's formula describes the variation of the function V along solutions and is given as follows

$$dV(q,t) = \mathscr{L}V(q,t)dt + \textstyle\sum_{i=1}^p \frac{\partial V(q,t)}{\partial q_i} g_i(q) dW_t.$$

B. Proof of Proposition 4.1

By Itô's formula, we have

$$\mathbb{E}(\|\Delta_t^\alpha\|^2) = \mathbb{E}\int_0^t 2\mathrm{Tr}\Big[\Delta_s^\alpha\Big(S(\rho_s, \hat{\rho}_s) - S(\sigma_s, \hat{\sigma}_s)$$
$$+ \textstyle\sum_k \big(\mathcal{D}_{L_k}(\rho_s) - \mathcal{D}_{L_k}(\sigma_s)\big) - F_\alpha(\sigma_s)\Big)\Big]ds$$
$$+ \mathbb{E}\textstyle\sum_k \eta_k \displaystyle\int_0^t \|\mathcal{G}_k(\rho_s) - \mathcal{G}_k(\sigma_s)\|^2 ds,$$

where $S(\rho,\hat\rho) := [-i(H_0+u(\hat\rho)H_1),\rho]$, and

$$
\begin{aligned}
&\mathbb{E}(\|\hat\Delta^\alpha_t\|^2)\\
&=\mathbb{E}\int_0^t 2\mathrm{Tr}\Big[\hat\Delta^\alpha_s\Big(S(\hat\rho_s,\hat\rho_s)-S(\hat\sigma_s,\hat\sigma_s)\\
&\quad+\textstyle\sum_k\big(\mathcal{D}_{L_k}(\hat\rho_s)-\mathcal{D}_{L_k}(\hat\sigma_s)\big)\\
&\quad+\textstyle\sum_k\sqrt{\hat\eta_k}\big(\mathcal{G}_k(\hat\rho_s)\mathcal{T}_k(\rho_t,\hat\rho_t)-\mathcal{G}_k(\hat\sigma_s)\mathcal{T}_k(\sigma_t,\hat\sigma_t)\big)\Big)\Big]ds\\
&\quad+\mathbb{E}\textstyle\sum_k\hat\eta_k\displaystyle\int_0^t\|\mathcal{G}_k(\hat\rho_s)-\mathcal{G}_k(\hat\sigma_s)\|^2ds.
\end{aligned}
$$

Due to the Lipschitz continuity of $S(\rho,\hat\rho)$, there exists $c_1>0$ such that $\|S(\rho,\hat\rho)-S(\hat\sigma,\sigma)\|\le c_1(\|\Delta^\alpha\|+\|\hat\Delta^\alpha\|)$. By Cauchy-Schwarz inequality, there exists $c_2>0$ such that $\mathrm{Tr}\big[\Delta^\alpha\big(S(\rho,\hat\rho)-S(\hat\sigma,\sigma)\big)\big]\le c_2(\|\Delta^\alpha\|^2+\|\Delta^\alpha\|\|\hat\Delta^\alpha\|)$. By similar arguments, there exist $c_3,c_4,c_5>0$ such that $\mathrm{Tr}\big[\Delta^\alpha\big(\mathcal{D}_{L_k}(\rho_s)-\mathcal{D}_{L_k}(\sigma_s)\big)\big]\le c_3\|\Delta^\alpha\|^2$, $\mathrm{Tr}\big(\Delta^\alpha F_\alpha(\sigma)\big)\le\alpha c_4\|\Delta^\alpha\|$ and $\|\mathcal{G}_k(\rho_s)-\mathcal{G}_k(\sigma_s)\|^2\le c_5\|\Delta^\alpha\|^2$. Thus, there are three constants $c_6,c_7,c_8>0$ such that

$$
\begin{aligned}
\mathbb{E}(\|\Delta^\alpha_t\|^2)\le\int_0^t c_6\mathbb{E}(\|\Delta^\alpha_s\|^2)+\alpha c_7\mathbb{E}(\|\Delta^\alpha_s\|)\\
+c_8\mathbb{E}(\|\Delta^\alpha_s\|\|\hat\Delta^\alpha_s\|)ds.
\end{aligned}
$$

Similarly, we can obtain the following estimation for $\hat\Delta_t$,

$$
\mathbb{E}(\|\hat\Delta^\alpha_t\|^2)\le\int_0^t\hat c_1\mathbb{E}(\|\hat\Delta^\alpha_s\|^2)+\hat c_2\mathbb{E}(\|\Delta^\alpha_s\|\|\hat\Delta^\alpha_s\|)ds
$$

for some $\hat c_1,\hat c_2>0$. Moreover, by Jensen's inequality,

$$
\begin{aligned}
\mathbb{E}(\|\Delta^\alpha\|)&\le\mathbb{E}(\|\Delta^\alpha\|)+\mathbb{E}(\|\hat\Delta^\alpha\|)\\
&\le\sqrt{\mathbb{E}(\|\Delta^\alpha\|^2)}+\sqrt{\mathbb{E}(\|\hat\Delta^\alpha\|^2)}\le\sqrt{2U^\alpha},
\end{aligned}
$$

where we used the fact $\sqrt{x}+\sqrt{y}\le\sqrt{2(x+y)}$ for $x,y>0$ in the last inequality. Due to $xy\le(x^2+y^2)/2$, there exist two constants $a,b>0$ such that

$$
U^\alpha_t\le\int_0^t aU^\alpha_s+\alpha b\sqrt{U^\alpha_s}ds,
$$

by applying the generalized Grönwall inequality [25, pp. 360-361], we have

$$
U^\alpha_t\le\Big(\frac{\alpha b}{2}\int_0^t e^{a(t-s)/2}ds\Big)^2,
$$

which implies $\sqrt{U^\alpha_t}\le\alpha A(e^{Bt}-1)$ for some constants $A,B>0$. Therefore, for any finite $t\ge0$, $\lim_{\alpha\to0}U^\alpha_t=0$. Furthermore, we can use above estimation and make one step further by extending the time interval which depends on α. For any $\gamma\in(0,1)$ and $\alpha>0$, we have

$$
\alpha^{1-\gamma}A(e^{Bt}-1)\le1,\quad t\in[0,T(\alpha)],
$$

where $T(\alpha):=\frac{1}{B}\log(1+\frac{1}{A\alpha^{1-\gamma}})$, $T(\alpha)$ goes to infinity when α tends to zero. Hence, we have $\sqrt{U^\alpha_t}\le\alpha^\gamma$ for all $t\in[0,T(\alpha)]$ with $\alpha>0$.

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