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Disturbance Observer with Switched Output Redefinition for Robust Stabilization of Non-Minimum Phase Linear Systems

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***Abstract*— Q-filter-based disturbance observer (DOB) has emerged as a powerful robust control technique renowned for its effectiveness in mitigating disturbances and addressing plant uncertainties. Despite its advantage, a key limitation of the Q-filter-based DOB lies in its requirement for plants to be of minimum phase. In this paper, we introduce an approach allowing the utilization of the Q-filter-based DOB as a stabilizing controller for non-minimum phase linear systems based on switched output redefinition of the systems. By redefining the output of systems to be controlled periodically, the approach stabilizes unstable internal dynamics of the systems as well as the original output. The proposed method is verified by an illustrative example.**

I. Introduction

Q-filter-based disturbance observer (DOB) [1], which was first proposed in [2], is a powerful robust control technique well-known for its capability to reject disturbances and compensate plant uncertainties.

The Q-filter-based DOB has been widely applied to various applications in industry by reason of its outstanding performance and relatively easy design. For instance, [3] used Q-filter-based DOB for mitigating tip-tilt vibrations in astronomical telescope. Authors of [4] proposed a periodic Q-filter-based DOB to compensate periodic disturbances induced by repetitive operations and demonstrated practical performance of the periodic Q-filter-based DOB in experiments on a multi-axis manipulator. More recently, [5] presented a closed-loop sensitivity shaping method for motion control systems with Q-filter-based DOB by employing a general form of Q-filter and verified the approach on a timing-belt setup of a laser marking machine. Also, a method to select parameters in Q-filter-based DOB was provided in [6] to enhance the robustness of the DOB-controlled servo motor system. Q-filter-based DOB was also applied to control powered exoskeletons, which are promising applications of robotic rehabilitation, to guarantee robust tracking performance in [7].

On the other hand, theoretical research on DOB-based control systems has also been extensively conducted. Early works in this area focused on the derivation of robust stability conditions for DOB-based control systems. For example, [8] and [9] utilized singular perturbation theory to analyze DOB-controlled systems with linear and nonlinear plant, respectively. Thereafter, [10] presented an almost necessary and sufficient condition for robust stability of the closed-loop system with Q-filter-based DOB. Recently, the focus has moved to making use of Q-filter-based DOB for various purposes so as to widening the class of systems to which the (possibly modified) Q-filter-based DOB can be applied for achieving desired control goal. To name a few, [11], [12], and [13] proposed design method of Q-filter-based DOB and corresponding robust stability condition for systems with unknown relative degree, systems with large sensor noise, and systems represented by a class of differential algebraic equations, respectively.

However, there is a significant hurdle for expanding the scope of applicable systems for the Q-filter-based DOB: the system is required to be of minimum phase, as pointed out in [8]–[10]. With this issue widely recognized, several research works have aimed to enable the application of the Q-filter-based DOB even in the cases where the target system exhibits non-minimum phase characteristics. Among them, [14] modified the configuration of the conventional Q-filter-based DOB to deal with non-minimum phase linear systems by adding a new filter, which can be systematically designed based on H_∞ methods under multiplicative system uncertainties. On the other hand, [15] proposed an approximation of unstable zeros by using non-causal transfer functions to subsequently design the Q-filter-based DOB. Lastly, [16] solved the issue by imposing a specific condition on the design of both outer-loop controller and Q-filter.

In this paper, we propose an output redefinition of non-minimum phase systems for which the Q-filter-based DOB can be utilized as a stabilizing controller. The output redefinition is based on state measurements of the internal dynamics of the systems to be controlled. Since measuring the internal state variables may be too costly, we also propose a switched output redefinition approach for the case where the internal state variables are assumed to be measured only periodically. It is shown that the Q-filter-based DOB with the switched output redefinition carries out robust stabilization even though the internal state variables are not available the whole time.

The remainder of this paper is organized as follows. Section II provides system description and formally states a stabilization problem of interest. Section III reviews conventional Q-filter-based DOB as a stabilizing controller. The switched output redefinition is presented in Section IV to use Q-filter-based DOB for stabilization of non-minimum phase systems. In Section V, we demonstrate the effectiveness of the proposed approach through an illustrative example.

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Finally, Section VI concludes this paper with a remark on future work.

Notation: The set of all complex numbers, real numbers, nonnegative real numbers, nonnegative integers, and positive integers are denoted by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{Z}_{\geq 0}$ and $\mathbb{N}$, respectively. For a sequence $v_1, \ldots, v_n$ of vectors or scalars, we define $[v_1; \cdots; v_n] := [v_1^\top \cdots v_n^\top]^\top$. For $r \geq 0$ and $x \in \mathbb{C}$, the open ball of radius r centered at x is denoted by $B(x, r) := \{z \in \mathbb{C} \mid \|x - z\| < r\}$. The spectrum of a matrix $A \in \mathbb{R}^{n \times n}$ is written as $\sigma(A) \subset \mathbb{C}$.

II. Problem Formulation

Consider a single-input single-output (SISO) linear time-invariant (LTI) system in Byrnes-Isidori normal form [17]:

$$\begin{aligned} y &= x_1, \\ \dot{x}_i &= x_{i+1}, \qquad 1 \leq i \leq \nu - 1, \\ \dot{x}_\nu &= \psi^\top z + \phi^\top x + g(u + d), \\ \dot{z} &= Sz + Gx_1 \in \mathbb{R}^{n-\nu}, \end{aligned} \tag{1}$$

where $[x; z] = [x_1; x_2; \cdots; x_\nu; z] \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input, $y \in \mathbb{R}$ is the output, and $d \in \mathbb{R}$ is the unknown disturbance. The relative degree of system (1) is $\nu \in \mathbb{N}$ which implies that g is a nonzero scalar. In particular, $z \in \mathbb{R}^{n-\nu}$ represents the state of internal dynamics (or zero dynamics). We assume that system parameters $\psi \in \mathbb{R}^{n-\nu}$, $\phi \in \mathbb{R}^n$, $g \in \mathbb{R}$, $S \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$, and $G \in \mathbb{R}^{n-\nu}$ are all unknown, but belong to known compact sets of appropriate dimensions, respectively. The disturbance $d \in \mathbb{R}$ and its derivative are assumed to be bounded.

In this paper, we seek a solution to the following robust stabilization problem.

Problem 1: For a given $\epsilon > 0$, design an output feedback controller such that the closed-loop system (1) with the controller satisfies

$$\limsup_{t\to\infty} \|[x(t); z(t)]\| \leq \epsilon. \tag{2}$$

The explicit answers to Problem 1 based on Q-filter-based disturbance observer (DOB) can be found in Propositions 1 and 2 and Theorem 1 with different settings and requirements on system (1). In particular, Proposition 2 and Theorem 1 deal with the case where system (1) is of non-minimum phase.

III. Review of Q-Filter-Based Disturbance Observer as Stabilizing Controller

In order to solve Problem 1, we first use conventional Q-filter-based disturbance observer (DOB) as a stabilizing controller of system (1). Q-filter-based DOB is a robust control technique well-known for its capability to reject disturbances and convert a given uncertain system into another nominal system in a closed-loop sense under some conditions. Those conditions, presented in [8], [9], will be explained in detail throughout this section.

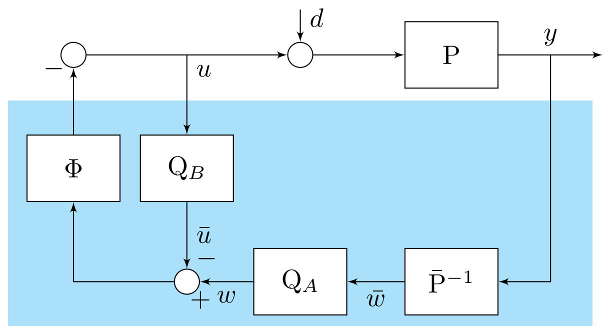

Fig. 1. Closed-loop system with Q-filter-based DOB (cyan block).

Suppose that we are given a (disturbance-free) asymptotically stable nominal model as

$$\begin{aligned} \bar{y} &= \bar{x}_1, \\ \dot{\bar{x}}_i &= \bar{x}_{i+1}, \qquad 1 \leq i \leq \nu - 1, \\ \dot{\bar{x}}_\nu &= \bar{\psi}^\top \bar{z} + \bar{\phi}^\top \bar{x} + \bar{g}u_r, \\ \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{G}\bar{x}_1 \in \mathbb{R}^{n-\nu}, \end{aligned} \tag{3}$$

where all system parameters are known and $u_r \in \mathbb{R}$ is the reference input.

A classical configuration of the closed-loop system with Q-filter-based DOB is depicted in Fig. 1. In the figure, P and $\bar{\text{P}}$ represent real plant (1) and nominal model (3), respectively, and Q_A and Q_B are stable low-pass filters usually called *Q-filter* having a parameter τ that determines their bandwidth. The inverse nominal model $\bar{\text{P}}^{-1}$ is implemented as

$$\begin{aligned} \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{G}y, \\ \bar{w} &= \frac{1}{\bar{g}}(-\bar{\psi}^\top \bar{z} - \bar{\phi}^\top [y, \dot{y}, \cdots, y^{(\nu-1)}]^\top + y^{(\nu)}) \end{aligned} \tag{4}$$

together with realizations of Q_A and Q_B represented as

$$\begin{aligned} \tau\dot{q} &= A_1 q + B_1 \bar{w}, \\ w &= C_1 q \end{aligned} \tag{5}$$

and

$$\begin{aligned} \tau\dot{p} &= A_1 p + B_1 u, \\ \bar{u} &= C_1 p, \end{aligned} \tag{6}$$

respectively, where

$$A_1 := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{l-1} \end{bmatrix}, \quad B_1 := \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

$$C_1 := \begin{bmatrix} c_0 & c_1 & \cdots & c_k & 0 & \cdots & 0 \end{bmatrix}$$

with system order $l \geq k + \nu$ and $c_0 = a_0$ for unity dc gain. The control input generated by the Q-filter-based DOB is determined as $u = -\Phi(w - \bar{u})$ with a saturation function Φ satisfying

$$\Phi(s) = \begin{cases} \bar{s}, & \text{if } s > \bar{s}, \\ s, & \text{if } |s| \leq \bar{s}, \\ -\bar{s}, & \text{if } s < -\bar{s} \end{cases}$$

for some $\bar{s} \geq 0$. Design of $\bar{s}$ should be made to have the saturation function inactive in the steady-state and this requires the knowledge of the compact sets containing system parameters. One is referred to [9] for more on design of $\bar{s}$.

As long as $|w - \bar{u}| \leq \bar{s}$, namely, the saturation function Φ is inactive, the overall closed-loop system (1), (4), (5), and (6) can be written, as shown in [8], in the standard form of singular perturbation [18, Chapter 11] as

$$\begin{aligned} y &= x_1, \\ \dot{x}_i &= x_{i+1}, \qquad 1 \leq i \leq \nu - 1, \\ \dot{x}_\nu &= \psi^\top z + \phi^\top x + g C_1 (p - q) + g d, \\ \dot{z} &= S z + G x_1, \\ \dot{\bar{z}} &= \bar{S} \bar{z} + \bar{G} x_1, \\ \tau \dot{q} &= A_1 q + \frac{g}{\bar{g}} B_1 C_1 (p - q) \\ &\quad + \frac{1}{\bar{g}} B_1 (-\bar{\psi}^\top \bar{z} + \psi^\top z + (\phi - \bar{\phi}) x) + \frac{g}{\bar{g}} B_1 d, \\ \tau \dot{p} &= A_1 p + B_1 C_1 (p - q). \end{aligned} \tag{7}$$

By putting $\tau = 0$, the quasi-steady-state subsystem of (7) is obtained as

$$\begin{aligned} y &= x_1^{\rm qss}, \\ \dot{x}_i^{\rm qss} &= x_{i+1}^{\rm qss}, \qquad 1 \leq i \leq \nu - 1, \\ \dot{x}_\nu^{\rm qss} &= \bar{\psi}^\top \bar{z}^{\rm qss} + \bar{\phi}^\top x^{\rm qss}, \\ \dot{\bar{z}}^{\rm qss} &= \bar{S} \bar{z}^{\rm qss} + \bar{G} x_1^{\rm qss}, \\ \dot{z}^{\rm qss} &= S z^{\rm qss} + G x_1^{\rm qss} \end{aligned} \tag{8}$$

which consists of the stable nominal model with the zero reference input $u_r \equiv 0$ in addition to the internal dynamics of system (1) standing alone. According to singular perturbation theory, the slow variables x, z, and $\bar{z}$ remain close to the solution of the quasi-steady-state subsystem (8) $x^{\rm qss}$, $z^{\rm qss}$, and $\bar{z}^{\rm qss}$, respectively, under sufficiently small $\tau > 0$ if the boundary-layer subsystem of (7), dynamics of the fast variables p and q during a short transient, is exponentially stable. In fact, the boundary-layer subsystem of (7) is exponentially stable if and only if the matrix

$$A_f := \begin{bmatrix} A_1 - \frac{g}{\bar{g}} B_1 C_1 & \frac{g}{\bar{g}} B_1 C_1 \\ -B_1 C_1 & A_1 + B_1 C_1 \end{bmatrix}$$

is Hurwitz. (See [8], [13] for a formal derivation of the matrix A_f.) Additionally, the fast variables p and q undergo peaking phenomenon during the transient, but the peaking is not propagated into slow variables due to the saturation function Φ [9]. Under these observations, the following proposition, which is a modified version of [9, Theorem 1], reveals that the conventional Q-filter-based DOB directly becomes a solution to Problem 1.

Proposition 1: Suppose that the matrix A_f is Hurwitz. If the matrix S is Hurwitz, then for a given $\epsilon > 0$, there exists a $\tau^* > 0$ such that for all $0 < \tau < \tau^*$, the closed-loop system with the Q-filter-based DOB in Fig 1 solves Problem 1 with

$$\left\| \begin{bmatrix} x(t) \\ \bar{z}(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} x^{\rm qss}(t) \\ \bar{z}^{\rm qss}(t) \\ z^{\rm qss}(t) \end{bmatrix} \right\| \leq \epsilon$$

for all $t \geq 0$, where $[x^{\rm qss}(t); \bar{z}^{\rm qss}(t); z^{\rm qss}(t)]$ is the solution to (8) with $[x^{\rm qss}(0); \bar{z}^{\rm qss}(0); z^{\rm qss}(0)] = [x(0); \bar{z}(0); z(0)]$.

It is important to note that we need S to be Hurwitz, meaning that system (1) should be of minimum phase, to use the Q-filter-based DOB as a controller that solves Problem 1.

IV. Q-Filter-Based Disturbance Observer with Switched Output Redefinition

In this section, we propose a Q-filter-based DOB with a switched output redefinition technique which becomes a solution to Problem 1 even for the case where system (1) is of non-minimum phase. The main idea is to replace unstable internal dynamics of system (1) with stable ones by using the switched output redefinition. In order for this, we state the following assumption on the internal dynamics of system (1).

Assumption 1: The internal dynamics of system (1) satisfy the following.

(a) The matrix S has an eigenvalue with a positive real part, but there exists a known vector $\alpha \in \mathbb{R}^{n-\nu}$ such that $S - G\alpha^\top$ is Hurwitz.
(b) There exists a set $\mathcal{T} \subset \mathbb{R}_{\geq 0}$ such that it is able to measure $\alpha^\top \cdot z(t)$ if $t \in \mathcal{T}$.

Remark 1: Since S and G are assumed to be unknown, it is hard to check if Assumption 1 (a) is met. However, let us suppose that there exist (known) matrices $S_0, \ldots, S_r \in \mathbb{R}^{(n-\nu)\times(n-\nu)}$ and $G_0, \ldots, G_{r'} \in \mathbb{R}^{n-\nu}$ such that

$$\begin{aligned} S &= S(\delta) = S_0 + \delta_1 S_1 + \cdots + \delta_r S_r, \\ G &= G(\delta') = G_0 + \delta'_1 G_1 + \cdots + \delta'_{r'} G_{r'}, \end{aligned}$$

where $\delta := [\delta_1, \cdots, \delta_r]^\top \in \Delta \subset \mathbb{R}^r$ and $\delta' := [\delta'_1, \cdots, \delta'_{r'}]^\top \in \Delta' \subset \mathbb{R}^{r'}$ represent uncertainties for polytopic sets Δ and Δ', respectively. In this case, one might use the linear matrix inequality (LMI) test presented in [19, Example 5.5] based on the notion of quadratic stability to determine $\alpha \in \mathbb{R}^{n-\nu}$ that satisfies Assumption 1 (a).

A. Q-filter-based DOB with output redefinition

Let us first consider the case where $\mathcal{T} = \mathbb{R}_{\geq 0}$ in Assumption 1 (b). A configuration of the closed-loop system with the Q-filter-based DOB and (plant's) output redefinition is depicted in Fig. 2. In the figure, the auxiliary output of the real plant is defined as

$$y^{\rm aux} := y + \alpha^\top z, \tag{9}$$

where $\alpha \in \mathbb{R}^{n-\nu}$ is the known vector that satisfies Assumption 1 (a). Defining $x_1^{\rm aux} := y^{\rm aux}$ and $x_{i+1}^{\rm aux} := \dot{x}_i^{\rm aux}$ for $i = 1, \ldots, \nu - 1$ yields

$$x^{\rm aux} := [x_1^{\rm aux}, \cdots, x_\nu^{\rm aux}]^\top = T_x x + T_z z, \tag{10}$$

where

$$T_x := \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \alpha^\top G & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^\top S^{\nu-3} G & \alpha^\top S^{\nu-4} G & \cdots & 1 & 0 \\ \alpha^\top S^{\nu-2} G & \alpha^\top S^{\nu-3} G & \cdots & \alpha^\top G & 1 \end{bmatrix},$$

$$T_z := \left[\alpha^\top; \alpha^\top S; \cdots; \alpha^\top S^{\nu-1}\right].$$

It is obvious that T_x is nonsingular so that we have

$$x = T_x^{-1}(x^{\text{aux}} - T_z z). \tag{11}$$

Then, it follows from system (1) that

$$\begin{aligned}\dot{x}_\nu^{\text{aux}} &= \dot{x}_\nu + \left(\sum_{k=0}^{\nu-2} \alpha^\top S^k G \dot{x}_{\nu-1-k}\right) + \alpha^\top S^{\nu-1}\dot{z} \\ &= \psi^\top z + \phi^\top x + g(u+d) \\ &\quad + \left(\sum_{k=0}^{\nu-2} \alpha^\top S^k G x_{\nu-k}\right) + \alpha^\top S^{\nu-1}(Sz + Gx_1) \\ &= \psi^\top z + \phi^\top T_x^{-1}(x^{\text{aux}} - T_z z) + g(u+d) + \alpha^\top S^\nu z \\ &\quad + \left(\sum_{k=0}^{\nu-1} \alpha^\top S^k G x_{\nu-k}\right) \\ &= \psi^{\text{aux}\top} z + \phi^{\text{aux}\top} x^{\text{aux}} + g(u+d),\end{aligned}$$

where

$$\begin{aligned}\psi^{\text{aux}\top} &:= \psi^\top + \alpha^\top S^\nu \\ &\quad - \left(\phi^\top + \begin{bmatrix}\alpha^\top S^{\nu-1}G & \cdots & \alpha^\top G\end{bmatrix}\right) T_x^{-1} T_z, \\ \phi^{\text{aux}\top} &:= \left(\phi^\top + \begin{bmatrix}\alpha^\top S^{\nu-1}G & \cdots & \alpha^\top G\end{bmatrix}\right) T_x^{-1}.\end{aligned}$$

It is observed that output redefinition (9) does not change the relative degree of system (1), but does reshape the internal dynamics to be

$$\begin{aligned}\dot{z} &= Sz + Gx_1 = Sz + Gy \\ &= (S - G\alpha^\top)z + Gy^{\text{aux}}.\end{aligned}$$

As a result, the closed-loop system with the Q-filter-based DOB and output redefinition (9), if $|w - \bar{u}| \le \bar{s}$, is written as

$$\begin{aligned}y^{\text{aux}} &= x_1^{\text{aux}}, \\ \dot{x}_i^{\text{aux}} &= x_{i+1}^{\text{aux}}, \qquad 1 \le i \le \nu - 1, \\ \dot{x}_\nu^{\text{aux}} &= \psi^{\text{aux}\top} z + \phi^{\text{aux}\top} x^{\text{aux}} + gC_1(p-q) + gd, \\ \dot{z} &= (S - G\alpha^\top)z + Gx_1^{\text{aux}}, \\ \dot{\bar{z}} &= \bar{S}\bar{z} + \bar{G}x_1^{\text{aux}}, \\ \tau\dot{q} &= A_1 q + \frac{g}{\bar{g}} B_1 C_1(p-q) \\ &\quad + \frac{1}{\bar{g}} B_1(-\bar{\psi}^\top \bar{z} + \psi^\top z + (\phi - \bar{\phi})x^{\text{aux}}) + \frac{g}{\bar{g}} B_1 d, \\ \tau\dot{p} &= A_1 p + B_1 C_1(p-q)\end{aligned} \tag{12}$$

which is also in the standard form of singular perturbation. As we have conducted in Section III, we again obtain the quasi-steady-state subsystem of closed-loop system (12) by setting $\tau = 0$ as

$$\begin{aligned}y^{\text{aux}} &= x_1^{\text{aux, qss}}, \\ \dot{x}_i^{\text{aux, qss}} &= x_{i+1}^{\text{aux, qss}}, \qquad 1 \le i \le \nu - 1, \\ \dot{x}_\nu^{\text{aux, qss}} &= \bar{\psi}^\top \bar{z}^{\text{aux, qss}} + \bar{\phi}^\top x^{\text{aux, qss}}, \\ \dot{\bar{z}}^{\text{aux, qss}} &= \bar{S}\bar{z}^{\text{aux, qss}} + \bar{G}x_1^{\text{aux, qss}}, \\ \dot{z}^{\text{aux, qss}} &= (S - G\alpha^\top)z^{\text{aux, qss}} + Gx_1^{\text{aux, qss}}.\end{aligned} \tag{13}$$

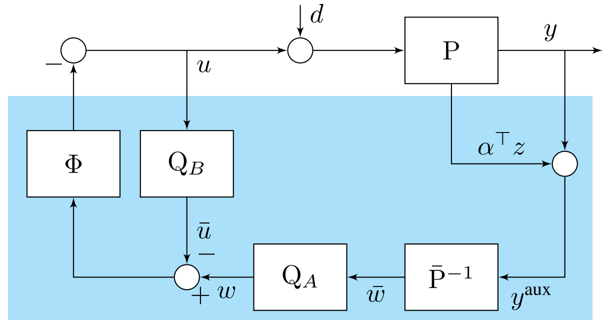

Fig. 2. Closed-loop system with Q-filter-based DOB (cyan block) to which output redefinition (9) is applied.

It should be emphasized that even though S was not Hurwitz in the first place, the $z^{\text{aux, qss}}$-dynamics standing alone in (13) are now exponentially stable according to Assumption 1 (a). Therefore, if the boundary-layer subsystem with respect to (12) is exponentially stable, it holds that for a given $\epsilon > 0$,

$$\begin{aligned}\|x^{\text{aux}}(t) - x^{\text{aux, qss}}(t)\| &\le \epsilon, \\ \|z(t) - z^{\text{aux, qss}}(t)\| &\le \epsilon, \quad \forall t \ge 0\end{aligned}$$

for sufficiently small $\tau > 0$ thanks to the saturation function Φ. Furthermore, both $x^{\text{aux, qss}}(t)$ and $z^{\text{aux, qss}}(t)$ approach to zero as $t \to \infty$ owing to the stability of the nominal model and Hurwitzness of $S - G\alpha^\top$. This finally leads us to attain condition (2) by using (11). In fact, the stability of the boundary-layer subsystem of (12) is solely determined again by A_f since the system matrix of fast dynamics (dynamics of p and q) remains unchanged comparing (7) and (12). Putting this all together, we state the second answer to Problem 1 as follows.

Proposition 2: Suppose that the matrix A_f is Hurwitz and Assumption 1 is satisfied with some $\alpha \in \mathbb{R}^{n-\nu}$ and $\mathcal{T} = \mathbb{R}_{\ge 0}$. Then for a given $\epsilon > 0$, there exists a $\tau^* > 0$ such that for all $0 < \tau < \tau^*$, the closed-loop system with the Q-filter-based DOB and output redefinition (9) in Fig. 2 solves Problem 1 with

$$\left\| \begin{bmatrix} x(t) \\ \bar{z}(t) \\ z(t) \end{bmatrix} - \begin{bmatrix} T_x^{-1}(x^{\text{aux, qss}}(t) - T_z z^{\text{aux, qss}}(t)) \\ \bar{z}^{\text{aux, qss}}(t) \\ z^{\text{aux, qss}}(t) \end{bmatrix} \right\| \le \epsilon$$

for all $t \ge 0$, where $[x^{\text{aux, qss}}(t); \bar{z}^{\text{aux, qss}}(t); z^{\text{aux, qss}}(t)]$ is the solution to (13) with $[x^{\text{aux, qss}}(0); \bar{z}^{\text{aux, qss}}(0); z^{\text{aux, qss}}(0)] = [T_x x(0) + T_z z(0); \bar{z}(0); z(0)]$.

B. *Q-filter-based DOB with switched output redefinition*

Measuring the internal variable z (or at least $\alpha^\top z$) as in Assumption 1 (b) might be expensive and therefore, the usage of this measurement should be minimized. From this perspective, let us now consider $\mathcal{T}$ in Assumption 1 (b) given as

$$\mathcal{T} = \big\{t \ge 0 \big| \exists j \in \mathbb{Z}_{\ge 0} \text{ such that } jT + T_1 \le t < (j+1)T\big\} \subset \mathbb{R}_{\ge 0} \tag{14}$$

for some $T_1 < T$ with a fixed $T > 0$. This means that we are able to measure $\alpha^\top z$ at $jT + T_1 \le t < (j+1)T$ for all $j \in \mathbb{Z}_{\geq 0}$. In this case, we define the auxiliary output of system (1) as

$$y^{\text{aux}}(t) := \begin{cases} y(t), & \text{if } t \notin \mathcal{T}, \\ y(t) + \alpha^\top z(t), & \text{if } t \in \mathcal{T} \end{cases} \tag{15}$$

which is also the input of the Q-filter-based DOB as depicted in Fig. 3. We present the main theorem which shows that the Q-filter-based DOB with switched output redefinition (15) solves Problem 1.

Theorem 1: Suppose that the matrix A_f is Hurwitz and Assumption 1 is satisfied with some $\alpha \in \mathbb{R}^{n-\nu}$ and $\mathcal{T}$ in (14). Then for a given $\epsilon > 0$, there exist $\tau^* > 0$ and $T^* > 0$ such that for all $0 < \tau < \tau^*$ and $0 < T_1 < T^*$, the closed-loop system with the Q-filter-based DOB and switched output redefinition (15) in Fig. 3 solves Problem 1.

Proof: Let us denote the solution to the closed-loop system with the Q-filter-based DOB and switched output redefinition (15) in Fig. 3 as $\chi(t) := [x(t); \bar{z}(t); z(t)]$. We consider the first mode where the output $y^{\text{aux}}(t) = y(t)$ is not redefined and thus, system (7) is active. Despite the unstable matrix S, we can apply the same analysis conducted in [9, Section 3.2] with Tikhonov's theorem for finite time interval. Then, given a $\delta > 0$, it follows that there exists a $\tau_1^* > 0$ such that for all $0 < \tau < \tau_1^*$ and $j \in \mathbb{Z}_{\geq 0}$,

$$\|\zeta(t)\| \leq \delta, \quad \forall jT \leq t < jT + T_1, \tag{16}$$

where $\zeta(t) := \chi(t) - \chi^{\text{qss}}(t)$ and

$$\chi^{\text{qss}}(t) := [x^{\text{qss}}(t); \bar{z}^{\text{qss}}(t); z^{\text{qss}}(t)]$$

is the solution to (8) with $\chi^{\text{qss}}(jT) = \chi(jT)$. Therefore, we have

$$\begin{aligned} \chi(jT + T_1) &= e^{A^{\text{qss}} T_1} \cdot \chi^{\text{qss}}(jT) + \zeta(jT + T_1) \\ &= e^{A^{\text{qss}} T_1} \cdot \chi(jT) + \zeta(jT + T_1), \end{aligned} \tag{17}$$

where A^{qss} is the system matrix of (8).

On the other hand, let us consider the second mode where $y^{\text{aux}}(t) = y(t) + \alpha^\top z(t)$ and system (12) is now active. Let us define $\chi^{\text{aux}}(t) := [x^{\text{aux}}(t); \bar{z}(t); z(t)]$ as the solution to (12) with $\chi^{\text{aux}}(jT+T_1) = [T_x x(jT+T_1) + T_z z(jT+T_1); \bar{z}(jT+T_1); z(jT+T_1)]$ defined from $\chi(jT+T_1)$. In the same way, there exists a $\tau_2^* > 0$ such that for all $0 < \tau < \tau_2^*$ and $j \in \mathbb{Z}_{\geq 0}$,

$$\|\xi(t)\| \leq \delta, \quad \forall jT + T_1 \leq t < (j+1)T, \tag{18}$$

where $\xi(t) := \chi^{\text{aux}}(t) - \chi^{\text{aux, qss}}(t)$ and

$$\chi^{\text{aux, qss}}(t) := [x^{\text{aux, qss}}(t); \bar{z}^{\text{aux, qss}}(t); z^{\text{aux, qss}}(t)]$$

is the solution to (13) with $\chi^{\text{aux, qss}}(jT+T_1) = \chi^{\text{aux}}(jT+T_1)$. Accordingly, it holds that

$$\begin{aligned} &\chi^{\text{aux}}((j+1)T) \\ &= e^{A^{\text{aux, qss}}(T-T_1)} \cdot \chi^{\text{aux, qss}}(jT + T_1) + \xi((j+1)T) \\ &= e^{A^{\text{aux, qss}}(T-T_1)} \cdot \chi^{\text{aux}}(jT + T_1) + \xi((j+1)T), \end{aligned} \tag{19}$$

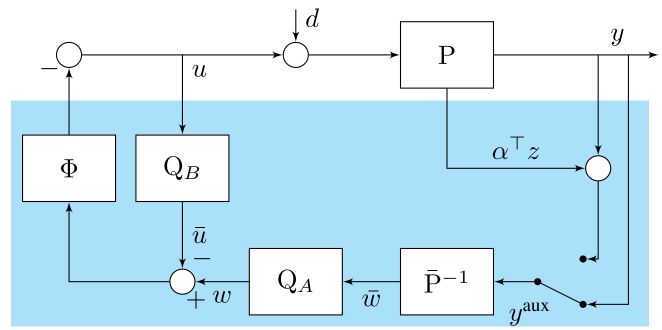

Fig. 3. Closed-loop system with Q-filter-based DOB (cyan block) to which switched output redefinition (15) is applied.

where $A^{\text{aux, qss}}$ is the system matrix of (13). For convenience, we define $T_\chi \in \mathbb{R}^{(2n-\nu)\times(2n-\nu)}$ as the one satisfying

$$\chi^{\text{aux}}(t) = \begin{bmatrix} x^{\text{aux}}(t) \\ \bar{z}(t) \\ z(t) \end{bmatrix} = \begin{bmatrix} T_x & 0 & T_z \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \chi(t) =: T_\chi \cdot \chi(t) \tag{20}$$

from (10), which is nonsingular by definition.

Combining (17), (19), and (20), we have that there exists a $\tau^* \leq \min(\tau_1^*, \tau_2^*)$ such that for all $0 < \tau < \tau^*$ and $j \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} &\chi((j+1)T) = T_\chi^{-1} \cdot \chi^{\text{aux}}((j+1)T) \\ &= T_\chi^{-1}\Big(e^{A^{\text{aux, qss}}(T-T_1)} \cdot T_\chi \cdot \chi(jT+T_1) + \xi((j+1)T)\Big) \\ &= T_\chi^{-1}\Big(e^{A^{\text{aux, qss}}(T-T_1)} \cdot T_\chi\big(e^{A^{\text{qss}}T_1} \cdot \chi(jT) + \zeta(jT+T_1)\big) \\ &\qquad + \xi((j+1)T)\Big) \\ &= \mathrm{A} \cdot \chi(jT) + \mathrm{U}(jT), \end{aligned}$$

where $\mathrm{A} := T_\chi^{-1} \cdot e^{A^{\text{aux, qss}}(T-T_1)} \cdot T_\chi \cdot e^{A^{\text{qss}}T_1} \in \mathbb{R}^{(2n-\nu)\times(2n-\nu)}$ and $\mathrm{U}(jT) := T_\chi^{-1} \cdot e^{A^{\text{aux, qss}}(T-T_1)} \cdot T_\chi \cdot \zeta(jT+T_1) + T_\chi^{-1} \cdot \xi((j+1)T) \in \mathbb{R}^{2n-\nu}$. Note that there exists a $\mu \geq 0$ (regardless of $T_1 > 0$) such that $\|\mathrm{U}(jT)\| \leq \mu\delta$ for all $j \in \mathbb{Z}_{\geq 0}$ by (16) and (18). Also, $A^{\text{aux, qss}}$ is the same with A^{qss}, in which the block matrix S having an eigenvalue with a positive real part is replaced by $S - G\alpha^\top$, and thus is Hurwitz. Then, there exists a $T^* > 0$ such that for all $0 < T_1 < T^*$,

$$\sigma(\mathrm{A}) = \sigma(T_\chi^{-1} \cdot e^{A^{\text{aux, qss}}(T-T_1)} \cdot T_\chi \cdot e^{A^{\text{qss}}T_1}) \subset B(0,1)$$

by continuity of eigenvalues. As a result, there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that

$$\|\chi(jT)\| \leq \beta(\|\chi(0)\|, jT) + \gamma(\mu\delta) \tag{21}$$

for all $j \in \mathbb{Z}_{\geq 0}$. In addition, let $\kappa > 0$ be the largest real part of the eigenvalues of A^{qss}. Then, there exist $K_1, K_2 \geq 0$ such that for all $j \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} \sup_{jT \leq t < jT+T_1} \|\chi(t)\| &\leq K_1 e^{\kappa T_1} \cdot \|\chi(jT)\| + K_2\delta \\ &\leq K_1 e^{\kappa T_1} \cdot (\beta(\|\chi(0)\|, jT) + \gamma(\mu\delta)) + K_2\delta \end{aligned} \tag{22}$$

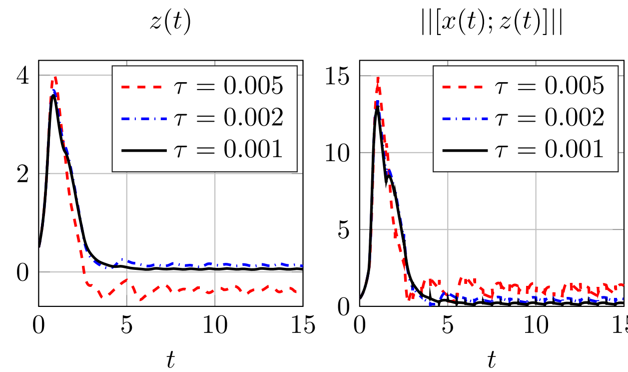

Fig. 4. Response of the closed-loop system with Q-filter-based DOB to which switched output redefinition (15) with $T = 1$ and $T_1 = 0.5$ in (14) is applied. The upper limit of $\|[x(t); z(t)]\|$ decreases as τ is decreased.

by (21). We can also derive an inequality similar to (22) for $jT + T_1 \le t < (j+1)T$ and hence, $\limsup_{t\to\infty} \|\chi(t)\| \le \epsilon$ if $\delta > 0$ has been chosen to be sufficiently small, which concludes the proof. ∎

Roughly speaking, the state of system (1) follows the behavior of (8) and (13) alternately according to the switching period in (14). Specifically, Assumption 1 (a), which asserts the Hurwitzness of $S - G\alpha^\top$, and sufficiently small $T_1 > 0$ imply that stable (13) becomes more dominant in determining stability of system (1), resulting in the state of system (1) satisfying condition (2). On the other hand, we want to have $T - T^* > 0$ as small as possible to minimize the (high) cost of using $y^{\text{aux}} = y + \alpha^\top z$. To this end, we can choose $\alpha \in \mathbb{R}^{n-\nu}$ in Assumption 1 (a) to maximize $T^* > 0$, but a way to design such α is yet unexplored.

V. Illustrative Example

In this section, an illustrative example is presented to describe the utility of the solution to Problem 1 proposed in Theorem 1. System (1) with $n = 2$ and $\nu = 1$ has parameters chosen as $\psi = 1$, $\phi = 2$, $g = 0.5$, $S = 3$, and $G = 1$. The nominal parameters are set to be $\bar{\psi} = -1$, $\bar{\phi} = -1$, $\bar{g} = 0.7$, $\bar{S} = -2$, and $\bar{G} = 1$, which ensure the nominal model $\bar{\mathrm{P}}$ to be asymptotically stable. The Q-filters Q_A and Q_B are designed to be second-order low-pass filters with $a_0 = 1$, $a_1 = 1$, and $c_0 = 1$. It is easy to check that these components make A_f Hurwitz. For switched output redefinition (15), $\alpha = 7$ is taken so that $S - G\alpha = -4$. The saturation level of Φ is set to be $\bar{s} = 65$. The initial condition of the real plant P is $[x(0); z(0)] = [0; 0.5]$ and all the other initial conditions are given by zeros. Fig. 4 shows the simulation result of the closed-loop system with Q-filter-based DOB and switched output redefinition having $T = 1$ and $T_1 = 0.5$ in (14) under disturbance $d(t) = 2\sin(2t)$ for three different values of τ. Clearly, $\|[x(t); z(t)]\|$ remains bounded and its upper limit decays as τ is decreased, which demonstrates that the proposed Q-filter-based DOB with switched output redefinition is a solution to Problem 1. It is also seen that $\|z(t)\|$ occasionally increases because of the unstable original internal dynamics.

VI. Concluding Remarks

The main contribution of this paper is to propose the switched output redefinition of non-minimum phase systems and rigorously analyze the DOB-controlled system when the switched output redefinition is in use. Since this paper is only about stabilization, further study is needed to observe if tracking problems for non-minimum phase systems can also be addressed by the proposed output redefinition approach.

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