

# Collaborative Coalitions in Multi-Agent Systems: Quantifying the Strong Price of Anarchy for Resource Allocation Games

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**Abstract**—The emergence of new communication technologies allows us to expand our understanding of distributed control and consider collaborative decision-making paradigms. With collaborative algorithms, certain local decision-making entities (or agents) are enabled to communicate with one another and collaborate on their actions to attain better system behavior. By limiting the amount of communication, these algorithms exist somewhere between centralized and fully distributed approaches. To understand the possible benefits of this inter-agent collaboration, we model a multi-agent system as a common-interest game in which groups of agents can collaborate on their action to jointly increase the system welfare. We specifically consider  $k$ -strong Nash equilibria as the emergent behavior of these systems and address how well these states approximate the system optimal, formalized by the  $k$ -strong price of anarchy ratio. Our main contributions are in generating tight bounds on the  $k$ -strong price of anarchy in finite resource allocation games as the solution to a tractable linear program. By varying  $k$ —the maximum size of a collaborative coalition—we observe exactly how much performance is gained from inter-agent collaboration. To investigate further opportunities for improvement, we generate upper bounds on the maximum attainable  $k$ -strong price of anarchy when the agents’ utility function can be designed.

## I. INTRODUCTION

Controlling large-scale systems like transportation services [1], robotic fleets [2], supply chains [3], or machine job schedules [4] is often challenging as managing each component simultaneously can be difficult if not impossible. As a method of reducing control algorithm complexity—or to satisfy various physical and technical constraints—a system operator may opt to distribute decision-making across the system’s components [5]. Doing so forms a multi-agent system, where each agent makes system-relevant decisions locally. Distributed control for multi-agent systems has been relevant and highly studied in many domains and proven effective at providing reasonable system behavior [5]–[7]. However, the idea of fully localizing decision-making is perhaps excessive. New technologies allow for greater communication and coordination between components of large-scale systems [8], [9]. These new technologies open the door to new opportunities to control multi-agent systems with *collaborative decision-making algorithms*, in which

decision-making is no longer fully distributed but can be partially coordinated between the agents of the system. How these control paradigms compare to the more well-studied distributed approach is not well understood. In this work, we seek to answer the following questions:

- (i) How does collaborative decision-making affect the behavior of a multi-agent system?
- (ii) Can performance be improved by intelligently designing how a coalition makes decisions?

We model the multi-agent system as a common-interest game in which the agents’ joint-action leads to some system welfare. In the fully distributed setting, the price of anarchy serves as a quantitative metric to compare the behavior that emerges from distributed decision-making algorithms to the optimal system performance [10]. Additionally, by designing the agents utility function, this performance can be improved upon [11]. We introduce collaborative coalitions to the decision-making problem, where groups of agents can coordinate their actions to jointly maximize the system objective. These collaborations will alter the behavior of the multi-agent system and ideally lead to improved performance. Further, we can similarly design the utilities of a group of agents and quantify the additional gains of using optimal collaborative designs.

The manner in which agents communicate and collaborate is context dependent, e.g., networked communication [12], pair-wise [13], local [14], etc. To gain general insights on the value of collaboration, we consider that each subset of agents up to size  $k$  can collaborate on their action. Though extensive, this subsumes several other types of communication and can help us understand the possible value inter-agent collaboration can provide. Further, by varying  $k$ , we can probe the specific benefit of various scales of connectivity. In these collaborative environments, a stable state of the system is that of the  *$k$ -strong Nash equilibria*, in which no subset of agents up to size  $k$  can revise their actions to receive a higher payoff for each member of the group [15]. Researchers have studied the existence [16] and computation [17] of strong Nash equilibria in settings including congestion games [18], lexicographical games [19], and Markov games [20]. In this work, we study how well  $k$ -strong Nash equilibria approximate the optimal solution, termed the  *$k$ -Strong Price of Anarchy ( $k$ -SPoA)*.

Quantifying the  $k$ -strong price of anarchy has been studied in network formation games [21], [22] and load balancing games [23]–[25], as well as more general utility maximizing games [26], [27]. In many of these works, the bounds are

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either not tight or only hold in the limit of large numbers of players. In this work we devise tight bounds on the  $k$ -strong price of anarchy in games with finitely many players. Additionally, we study how the strong price of anarchy can be optimized by designing how coalitions of agents make decisions, something previously unstudied in the literature.

We specifically focus our analysis on resource allocation games. In the fully distributed setting ( $k = 1$ ), researchers have been successful in quantifying the price of anarchy for Nash equilibria and devising how agents should make decisions [11], [28]. However, existing techniques do not hold for collaborative decision-making as actions are no longer made unilaterally and simply characterizing an equilibrium becomes more difficult. Despite this, we identify several problem reductions that lead to the following results:

**Theorem 1 (Quantify SPoA):** We generate tight bounds on the  $k$ -strong price of anarchy in resource allocation problems by the value of a tractable linear program.

**Proposition 2 (Optimize SPoA):** We extend this linear program to give an upper bound on  $k$ -strong price of anarchy under the optimal utility design.

In Fig. 1, we plot the  $k$ -strong price of anarchy in maximum coverage problems for  $1 \leq k \leq n$  to illustrate the exact benefit collaborative decision-making and design can provide.

## II. DECISION-MAKING MODEL

Before discussing a specific problem setting, we will review the ideas of distributed decision-making and collaboration in a more general framework for multi-agent systems. For notation,  $x^y = x!/(x-y)!$  when  $x \geq y \geq 0$  and zero otherwise, and  $[n] = \{1, \dots, n\}$ .

### A. Collaborative Decision-Making

Consider a finite set of agents  $N = \{1, \dots, n\}$ . Each agent  $i \in N$  selects an action  $a_i$  from a finite action set  $\mathcal{A}_i$ . When each agent selects an action, we will denote their joint-action by the tuple  $a = (a_1, \dots, a_n) \in \mathcal{A} = \mathcal{A}_1 \times \dots \times \mathcal{A}_n$ . Let  $G = (N, \mathcal{A})$  be a tuple encoding the components of a multi-agent system. The agents' actions ultimately dictate the systems performance, as such, for each joint-action  $a$  we assign a system welfare  $W(a)$  where  $W : \mathcal{A} \rightarrow \mathbb{R}$  is the system designer's objective function. The system designer would like to configure the agents to reach an action that maximizes the system welfare, i.e.,

$$a^{\text{opt}} \in \arg \max_{a \in \mathcal{A}} W(a). \quad (1)$$

Though this system state is ideal, it may be difficult to attain as 1) solving for the optimal allocation can be combinatorial and in some cases (including those from Section III) NP-hard [28], and 2) it requires a centralized authority to control all agents, which may be practically or logistically difficult. To resolve this, we will consider that agents make decisions in a decentralized manner.

Fully distributing the decision-making involves designing each agent to locally update their action and has been widely studied and shown effective at providing reasonable system behavior [5]; however, it need not be necessary as emerging

communication technologies enable *collaborative inter-agent decision-making* [8]. To do so, a system operator must make two decisions: 1) which group of agents can collaborate on their decisions (possibly subject to some operational constraints), and 2) how should the agents collaborate on their decisions. A natural choice for the latter is a group best response. Let  $\Gamma \subseteq N$  be a group of agents endowed with the ability to collaboratively select a *group action*  $a_\Gamma \in \mathcal{A}_\Gamma = \prod_{i \in \Gamma} \mathcal{A}_i$ , which they select by maximizing the system welfare over their possible group actions,

$$a_\Gamma \in \arg \max_{a'_\Gamma \in \mathcal{A}_\Gamma} W(a'_\Gamma, a_{-\Gamma}), \quad (2)$$

where  $a_{-\Gamma}$  denotes the actions of the players not in  $\Gamma$ . If there are multiple elements in the argmax, the group breaks them at random unless they can remain with their current action. In Section III-C, we will consider other ways the group can make a decision. Regardless, one would imagine that the greater the collaborative structure, the greater the impact on emergent behavior.

For the system operator's decision over which groups should collaborate, let  $\mathcal{C} \subseteq 2^N$  denote the *collaboration set*, or the set of groups of agents ( $\Gamma \in \mathcal{C}$ ) able to collaborate their decisions. These collaborations can overlap—where agents can partake in multiple, disparate collaborations—and vary in size. For example, if agents send signals through a communication network [12], we will have  $\mathcal{C} = \{(i, j) \in N^2 \mid (i, j) \in E\}$  where  $E$  are the edges in a communication graph. If agents are allowed to communicate with each other one at a time and make pairwise decisions [13], then  $\mathcal{C} = \{(i, j) \in N^2\}$ . If agents can only communicate with others within a local proximity [14], then  $\mathcal{C} = \{\Gamma \subseteq N \mid \rho(i, j) \leq d \forall i, j \in \Gamma\}$  where  $\rho$  measures the distance between two agents and  $d$  is a maximum communication range. Once the system operator decides on the collaborative structure and the group decision-making protocol, the agents' decision-making process forms a collaborative multi-agent system, denoted by the tuple  $(G, W, \mathcal{C})$ . Future work will study the costs of greater collaboration; in this work, we seek to understand what benefit it can provide. We study a particular choice of collaborative structure where agents collaborate in groups up to size  $k$ .

### B. Strong Nash equilibria

Let  $\mathcal{C}_k = \{\Gamma \subseteq N \mid |\Gamma| = k\}$  denote the subsets of exactly  $k$  agents and  $\mathcal{C}_{[k]} = \bigcup_{\zeta \in [k]} \mathcal{C}_\zeta$  be the subsets that contain at most  $k$  agents. As  $k$  increases, so too does the size and number of collaborating groups. Varying  $k$  between 1 and  $n$  will inform us of the added benefit of increased collaboration.

For some  $k$ , the states that are stable under collaborative decision-making processes are those where no  $k$ -lateral deviation increases the system objective  $W$ . These states align with the set of  $k$ -strong Nash equilibria in the common-interest game with payoffs  $W$  and player/actions  $G$ .

**Definition 1.** A joint-action  $a^{k\text{SNE}} \in \mathcal{A}$  is a  $k$ -strong Nash equilibrium for the common-interest game  $(G, W, \mathcal{C}_{[k]})$  if

$$W(a^{k\text{SNE}}) \geq W(a'_\Gamma, a_{-\Gamma}^{k\text{SNE}}), \quad \forall a'_\Gamma \in \mathcal{A}_\Gamma, \Gamma \in \mathcal{C}_{[k]}. \quad (3)$$

Let  $k\text{SNE}(G, W) \subseteq \mathcal{A}$  denote the set of all  $k$ -strong Nash equilibria in the game  $(G, W, \mathcal{C}_{[k]})$ . When  $k = 1$ , this recovers the definition of a Nash equilibrium that is highly studied in the fully distributed setting [10], [11], [28]. When  $k = n$ , agents fully collaborate and the only  $k$ -strong Nash equilibria are those that maximize the system welfare. When  $1 < k < n$ , there is only partial collaboration among the agents, and the resulting system behavior is not yet well understood. This is in part because agents can participate in multiple collaborations; this overlap is valuable because it helps connect the decisions in different parts of the system (even if not directly) and is why we do not simply partition the agents into disjoint groups.

Definition 1 differs slightly from the more general definition in the literature in which groups take actions that are Pareto-optimal over the individual payoffs of the group members [15]. The definitions are equivalent when considering common-interest games. This refinement is natural for our setting as we seek to design multi-agent systems which maximize a central objective. Additionally, in general a  $k$ -strong Nash equilibrium need not exist, however in our setting that is not the case.

**Proposition 1.** *In a common-interest game  $(G, W, \mathcal{C}_{[k]})$ , for any  $k \in [n]$ , a  $k$ -strong Nash equilibrium exists and the set  $k\text{SNE}$  is stable and reached in finite time by the deterministic coalition best response dynamics and almost surely by the asynchronous coalition best response dynamics.*

The proof of Proposition 1 is straightforward<sup>1</sup>; a more detailed version appears in an online appendix [29]. The claims follow from  $a^{\text{opt}}$  being a  $k$ -strong Nash equilibrium for any  $k$  and that action revisions strictly increase the welfare, making the set  $k\text{SNE}$  absorbing in both the deterministic or asynchronous dynamics. Certainly, the algorithmic and communication complexity increases with  $k$ ; future work will investigate this cost further. In this work we seek to understand the possible benefit of inter-agent collaboration.

Definition 1 tells us that a  $k$ -strong Nash equilibrium is a local optimum of  $W$  in the neighborhood of  $k$ -lateral deviations. Still, if  $k < n$ , this need not be the global optimal. To quantify the efficacy of  $k$ -strong Nash equilibria as a solution concept, we consider the  $k$ -strong price of anarchy

$$\text{SPoA}_k(G, W) = \frac{\min_{a^{k\text{SNE}} \in k\text{SNE}(G, W)} W(a^{k\text{SNE}})}{\max_{a^{\text{opt}} \in \mathcal{A}} W(a^{\text{opt}})} \in [0, 1]. \quad (4)$$

In the multi-agent system  $(G, W, \mathcal{C}_{[k]})$  any  $k$ -strong Nash equilibrium approximates the optimal solution at least as well as  $\text{SPoA}_k(G, W)$ . Because  $k'\text{SNE} \subseteq k\text{SNE}$  for  $k' > k$ , the  $k$ -strong price of anarchy will be non-decreasing in  $k$ , i.e., the  $k$ -strong Nash equilibrium will do a better job approximating the optimal solution with greater collaboration.

### C. Summary of Contributions

With the possibility of collaboration, an equilibrium becomes more difficult to characterize than in the fully

<sup>1</sup>A more general result holds that there exists an equilibrium for any collaboration set  $\mathcal{C}$ , not just the  $k$ -strong Nash equilibria.

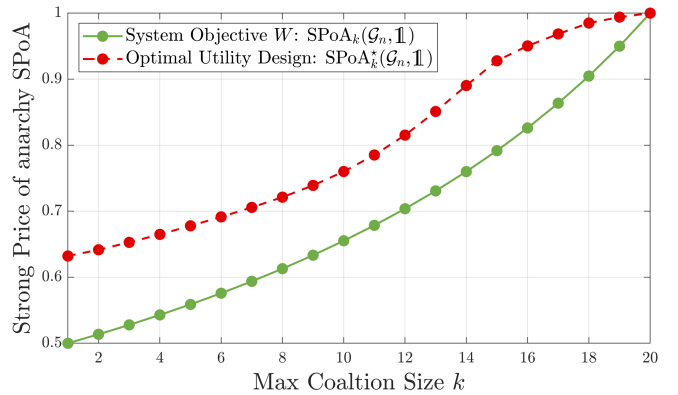


Fig. 1: Strong Price of Anarchy in resource allocation games with  $n = 20$  players and coalitions up to size  $k$  (horizontal axis). As the size of groups that are allowed to collaborate grows, so too does the approximation ratio (i.e., strong price of anarchy) of a  $k$ -strong Nash equilibrium. The efficiency of an equilibrium can be further improved by designing the utility functions agents are set to maximize. The solid green line is the  $k$ -strong price of anarchy when agents maximize the system objective (generated by Theorem 1). The dashed red line is an upper bound on the  $k$ -strong price of anarchy while using an optimal utility design (generated by Proposition 2.)

distributed setting. We circumvent this by introducing a parameterization which allows us to generalize the  $\mathcal{O}\left(\sum_{\zeta=1}^k \binom{n}{\zeta} m^\zeta\right)$  comparisons of (3) (where  $m := \max_{i \in N} |\mathcal{A}_i|$ ) into  $k$  linear inequalities. This reduction is key in Theorem 1, in which we show the tight bound on  $k$ -strong price of anarchy is the solution to a tractable linear program for a broad class of resource allocation problems. In Fig. 1, we plot the bound on the  $k$ -strong price of anarchy in maximum coverage problems for  $1 \leq k \leq n$  to quantify the performance gained by collaborative decision-making.

Finally, we investigate whether further improvement is possible with collaborative decision-making by designing how a group of agents makes decisions. Specifically, we consider that the agents' need not maximize the system welfare as in (2), but could instead maximize a separately designed utility function. Unlike the fully distributed setting, we may desire that groups of different sizes maximize different objectives. In Section III-C, we show that the aforementioned linear program can be extended to provide an upper bound on the maximum attainable  $k$ -strong price of anarchy by utility design in resource allocation problems. Fig. 1 plots these values and demonstrates the possible improvements from intelligently designing the collaborative decision-making.

## III. RESOURCE ALLOCATION PROBLEMS

### A. Definition

To provide more context to the value of inter-agent collaboration, we will consider the  $k$ -strong price of anarchy in a class of resource allocation problems. Consider a set of resources or tasks  $\mathcal{R} = \{1, \dots, R\}$ , to which agents are assigned, i.e., agent  $i \in N$  selects a subset of these resources as its action  $a_i \subseteq \mathcal{R}$  from a constrained set of subsets  $\mathcal{A}_i \subseteq 2^{\mathcal{R}}$ . Each resource  $r \in \mathcal{R}$  has a value  $v_r \geq 0$ ; the welfare contributed by a resource is  $v_r w(|a|_r)$ , where  $w : \{0, \dots, n\} \rightarrow \mathbb{R}_{\geq 0}$  captures the added benefit of having



$$\begin{aligned}
Q^* = & \min_{\mu, \{u_\zeta \in \mathbb{R}_{\geq 0}^n\}_{\zeta \in [k]}} \mu \\
\text{s.t. } & w(o+x) - \mu w(e+x) + \sum_{\zeta \in [k]} \left( \frac{n!}{(n-\zeta)!} u_\zeta(e+x) - \sum_{\substack{0 \leq \alpha \leq \zeta \\ 0 \leq \beta \leq \zeta - \alpha}} \binom{\zeta}{\alpha} \binom{\zeta - \alpha}{\beta} e^\alpha o^\beta (n-e-o)^{\zeta - \alpha - \beta} u_\zeta(e+x + \beta - \alpha) \right) \leq 0 \\
& \forall (e, x, o) \in \mathcal{I} \quad (\text{Q})
\end{aligned}$$

As a method of further improving the benefit of inter-agent collaboration, we will consider designing the agents utility functions as a method to alter the emergent equilibria. Consider a common-interest utility function  $U : \mathcal{A} \rightarrow \mathbb{R}$ . Following the coalitional best response dynamics, if a group  $\Gamma \subseteq N$  is selected to update their action, they now update according to

$$a_\Gamma \in \arg \max_{a'_\Gamma \in \mathcal{A}_\Gamma} U(a'_\Gamma, a_{-\Gamma}). \quad (8)$$

As the multi-agent decision model remains a common-interest game (now with payoff  $U(a)$ ), the existence and convergence results of Proposition 1 hold. With the new objective for the agents, the set of  $k$ -strong Nash equilibria is altered to  $k\text{SNE}(G, U)$ . Changing the utility function of the agents has no impact on the original system objective  $W$ ; our goal is to select the function  $U$  in a way such that the new equilibria offer a better approximation of the optimal system objective  $W$ . To quantify how well this is done, we extend the definition of  $k$ -strong price of anarchy to depend on the utility rule, i.e.,

$$\text{SPoA}_k(G, W, U) = \frac{\min_{a \in k\text{SNE} \in k\text{SNE}(G, U)} W(a^{k\text{SNE}})}{\max_{a^{\text{opt}} \in \mathcal{A}} W(a^{\text{opt}})}. \quad (9)$$

Again, we focus our attention to the class of resource allocation problems as described in Section III-A. As these problems are large and combinatorial, rather than designing a global utility function for each joint action, we will consider local utility rules of the form  $u : \{0, \dots, n\} \rightarrow \mathbb{R}$ . With this, the agents' common-interest objective becomes,

$$U(a) = \sum_{r \in \mathcal{R}} v_r u(|a|_r). \quad (10)$$

Recall that this does not change the overall system objective (5);  $u$  simply alters agents' decision-making. We set  $u(0) = 0$  else the  $k$ -strong price of anarchy will be zero.

As before, we are interested in the performance guaranteed by a collaborative decision-making system. As such, we will consider the efficiency guarantee across the class of instances  $\mathcal{G}_n$  with a welfare function  $w$ . When using utility rule  $u$ , the  $k$ -strong price of anarchy will be denoted  $\text{SPoA}_k(\mathcal{G}_n, w, u) = \min_{G \in \mathcal{G}_n} \text{SPoA}_k(G, w, u)$ . We are particularly interested in designing  $u$  to provide us the best guarantee; let

$$\text{SPoA}_k^*(\mathcal{G}_n, w) = \max_{u: [n] \rightarrow \mathbb{R}} \text{SPoA}_k(\mathcal{G}_n, w, u), \quad (11)$$

denote the optimal  $k$ -strong price of anarchy, or the approximation ratio of a  $k$ -strong Nash equilibrium under the optimal utility design.

In Proposition 2, we provide an upper bound on the efficiency an optimally designed utility rule can provide.

**Proposition 2.** *For the class of resource allocation problems  $\mathcal{G}_n$  with welfare function  $w$ , when agents maximize the optimal utility design objective  $u^*$ ,*

$$\text{SPoA}_k^*(\mathcal{G}_n, w) \leq 1/Q^*(n, w, k), \quad (12)$$

where  $Q^*(n, w, k)$  is the value of the linear program (Q).

The proof appears in an online appendix [29]. The decision variables  $\{u_\zeta\}_{\zeta \in [k]}$  of (Q) are the utility function for groups of size  $\zeta \in [k]$ . If  $u_\zeta = u_{\zeta'}$  for all  $\zeta, \zeta' \in [k]$ , the value of (Q) is the exact solution to (11). However, this need not be the case in general, hence Proposition 2 is an upper bound. The authors conjecture that the class of homogeneous local utility rules can attain a  $\text{SPoA}_k^*$  of  $1/Q^*(n, w, k)$ , however proving this is the subject of ongoing work.

To bring greater context to this, we again consider the case of covering problems with objective  $w(x) = \mathbb{1}[x > 0]$ . When  $k = n$ , it is clear the optimal utility design is to let the group of all agents maximize the common-interest objective  $w$ , leading to an  $n$ -strong price of anarchy of  $\text{SPoA}_n^*(\mathcal{G}_n, \mathbb{1}) = 1$ . When  $k = 1$ , the authors of [28] show that the optimal utility rule gives price of anarchy  $\text{SPoA}_1^*(\mathcal{G}_n, \mathbb{1}) = 1 - \frac{1}{\binom{n-1}{(n-1)(n-1)!} + \sum_{j=0}^{n-1} \frac{1}{j!}}$  which approximately equals 0.632 when  $n = 20$ ; this increase from 1/2 when agents maximize the system welfare motivates the fact that designing agents' utilities can provide performance improvements. In Fig. 1, we use Proposition 2 to show the possible benefit of utility design for groups of size  $1 \leq k \leq n$ . By comparing the naive approach of setting agents to maximize the system objective and the results of Proposition 2, it appears that significant benefit can occur from designing how groups make decisions. Future work will seek to quantify the optimal utility rule in closed form.

## CONCLUSION

In this work, we provide a tractable linear program whose solution is a tight bound on the  $k$ -strong price of anarchy in resource allocation games. Additionally, we exploit this program to provide an upper bound on the largest attainable performance when agents' utility function can be designed. Future work will study the added computational complexity of reaching these equilibria and deriving bounds in closed form when the number of agents is arbitrary but the size of coalitions is a constant fraction of the number of agents.

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