2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

LMI conditions for k-contraction analysis: a step towards design

Samuele Zoboli[1], Andreu Cecilia[1,2], Ulysse Serres[1], Daniele Astolfi[1], Vincent Andrieu[1]

***Abstract*—Recently, k-contraction has been proposed as a generalization of contraction properties for nonlinear time-variant systems. Existing tools for k-contraction analysis exploit complex mathematical tools known as matrix compounds. This prevented the development of related design methodologies. In this paper, we link k-contraction properties to partial stability analysis tools. This leads to new, design-oriented sufficient conditions for k-contraction analysis which do not involve matrix compounds. We also show that such sufficient conditions are necessary for the linear time-invariant framework. Finally, we compare our results to existing methods and highlight their advantages.**

I. Introduction

Contraction theory is an emerging topic that has been used in numerous applications, such as observer design [1], multi-agent system synchronization [2], [3] and controller design [4]–[8]. Nonetheless, many systems cannot present classical contractivity properties, e.g. multi-stable systems [9]. This motivated the study of suitable generalizations. Some notable examples are horizontal contraction [10, Section VII], transversal exponential stability [11] and p-dominance [12], [13]. Motivated by the results of Muldowney [14], recent works presented the notion of k-contraction [15], which generalizes the classical concept of shrinking distances between system trajectories to contraction of volumes. As such, k-contraction includes classical contraction as the special case $k = 1$. For $k > 1$, this property can be used to analyze asymptotic behavior of systems that are not contractive in the classical sense. For example, for 2-contractive time-invariant systems, every bounded solution converges to an equilibrium point (not necessarily unique).

Existing sufficient conditions for k-contraction are given in terms of a particular matrix compound of the Jacobian of the vector field dynamics [14], [15]. Although these conditions are adequate for system analysis, their application for feedback design is limited. First, matrix compounds rapidly explode in dimension for low value k and systems of large dimension. This fact drastically increases the computational complexity of potential feedback design algorithms. Second, the use of matrix compounds tools prevents transforming the feedback design into a tractable LMI problem. Consequently, a k-contractive design methodology has yet to be developed.

Considering these limitations, this work presents alternative design-oriented conditions for k-contraction that do not rely on matrix compounds but rather on simple matrix inequalities on the given system dynamics. Moreover, the connections between k-contraction, horizontal contraction and p-dominance [12], [13] are discussed.

Notation: $\mathbb{R}_+ := [0,\infty)$ and $\mathbb{N} := \{0,1,2,\ldots\}$. Given $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, we set $(x,y) := (x^\top, y^\top)^\top$. The operation $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ depicts the binomial coefficient, with $n!$ denoting the factorial of $n \in \mathbb{N}$. The inertia of a matrix P [16, Definition 2.1] is defined by the triplet of integers $\mathrm{In}(P) := (\pi_-(P), \pi_0(P), \pi_+(P))$, where $\pi_-(P)$, $\pi_+(P)$ and $\pi_0(P)$ denote the numbers of eigenvalues of P with negative, positive and zero real part, resp., counting multiplicities.

II. Preliminaries on k-contraction

In this work, we consider nonlinear systems of the form

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n \tag{1}$$

where f is sufficiently smooth with respect to its argument. The flow of (1) is denoted by ψ^t, so that $\psi^t(x_0)$ is the trajectory of (1) passing through x_0 at time 0. In this section, we formally define the property of k-contraction studied in this article. Our definition strongly focuses on geometrical interpretation and it is related to the notion presented in the works [14], [15]. Moreover, it directly translates to the definition of contraction presented in [11] when considering objects of dimension 1, i.e. when $k = 1$.

In [11], 1-contraction expresses the fact that the length of any C^1 curve from $[0,1]$ to $\mathbb{R}^n$ decreases with time. To extend such a notion to any positive integer $k \in [1,n]$, with n being the state dimension of (1), we consider a set of sufficiently smooth functions $\mathcal{I}_k$ defined on $[0,1]^k$, namely

$$\mathcal{I}_k := \big\{\Phi : [0,1]^k \to \mathbb{R}^n \mid \ \Phi \text{ is a smooth immersion}\big\}. \tag{2}$$

Let $P \in \mathbb{R}^{n\times n}$ be a positive definite symmetric matrix. For each Φ in $\mathcal{I}_k$, we define the volume $\ell^k(\Phi)$ of Φ as

$$\ell^k(\Phi) := \int_{[0,1]^k} \sqrt{\det\left\{\frac{\partial \Phi}{\partial r}(r)^\top P \frac{\partial \Phi}{\partial r}(r)\right\}}\,\mathrm{d}r\,. \tag{3}$$

Note that, since f in (1) is sufficiently smooth, for each forward invariant set and for each t in $\mathbb{R}_+$ it yields that the corresponding flow ψ^t is also sufficiently smooth in this set.

[1] S. Zoboli, A. Cecilia, D. Astolfi, U. Serres and V. Andrieu are with Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 boulevard du 11 novembre 1918, F-69100, Villeurbanne, France (name.surname@univ-lyon1.fr).

[2] A. Cecilia is with Universitat Politècnica de Catalunya, Avinguda Diagonal, 647, 08028 Barcelona, Spain. (andreu.cecilia@upc.edu).

The research leading to these results is partially funded by ANR via project DELICIO, number ANR-18-CE40-0010, and by the Spanish Ministry of Universities funded by the European Union - NextGenerationEU (2022UPC-MSC-93823)

Consequently, for each Φ in $\mathcal{I}_k$ such that $\operatorname{Im}(\Phi)$ is in a forward invariant set, $\psi^t \circ \Phi$ is in $\mathcal{I}_k$. Hence, we can now define k-contraction properties for nonlinear systems of the form (1), which will be used throughout all the article.

From now on and throughout the rest of the paper, we let k be a fixed integer between 1 and n.

Definition 1 (k-contraction)**.** *System* (1) *is said to be k-contractive on a forward invariant set $\mathcal{S} \subseteq \mathbb{R}^n$ if there exist strictly positive constants $\gamma, \eta > 0$ such that*

$$\ell^k(\psi^t \circ \Phi) \leqslant \gamma e^{-\eta t}\, \ell^k(\Phi), \qquad \forall t \in \mathbb{R}_+$$

for all $\Phi \in \mathcal{I}_k$ such that $\operatorname{Im}(\Phi) \subset \mathcal{S}$.

In simple words, we say that a system is k-contractive if, for any parametrized k-dimensional submanifold of $\mathbb{R}^n$ from which trajectories are complete, its volume is exponentially shrinking along the system dynamics. A scheme of this condition is depicted in Fig. 1. When $k = 1$, this means that the length of any sufficiently smooth curve is exponentially decreasing, matching the definition in [11]. Moreover, this definition includes the ones in [14], and [15, Section 3.2].

Remark 1. *When Φ is injective and P is the identity matrix,* (3) *gives the Euclidean k-volume of the submanifold $\Phi([0,1]^k) \subset \mathbb{R}^n$. Note that* 1*-volumes are lengths,* 2*-volumes are areas and* 3*-volumes are standard volumes.*

III. Main result

In this section, we present conditions for k-contraction. For nonlinear systems, we provide sufficient conditions. For linear time-invariant systems, necessary and sufficient conditions are established. The proofs of the theorems are given in Section V.

A. Nonlinear systems

Consider a nonlinear system of the form (1). The following theorem provides sufficient conditions for k-contraction.

Theorem 1. *Let $\mathcal{A} \subset \mathbb{R}^n$ be a compact forward invariant set and assume there exist symmetric matrices $P_0, P_{k-1} \in \mathbb{R}^{n\times n}$ of inertia $\operatorname{In}(P_0) = (0,0,n)$, $\operatorname{In}(P_{k-1}) = (k-1, 0, n-k+1)$ and $\mu_0, \mu_{k-1} \in \mathbb{R}$ such that, for all $x \in \mathcal{A}$*

$$\frac{\partial f}{\partial x}(x)^\top P_0 + P_0 \frac{\partial f}{\partial x}(x) \prec 2\mu_0 P_0, \tag{4a}$$

$$\frac{\partial f}{\partial x}(x)^\top P_{k-1} + P_{k-1} \frac{\partial f}{\partial x}(x) \prec 2\mu_{k-1} P_{k-1} \tag{4b}$$

$$\mu_{k-1} + (k-1)\mu_0 < 0 \tag{4c}$$

Then, system (1) *is k-contractive on $\mathcal{S} := \mathcal{A}$.*

A detailed discussion of Theorem 1 is postponed to Section V-B, along with the relative proof. Intuitively, inequality (4a) bounds the expansion rate of the lifted system by a factor μ_0 (the concept of lifted system will be properly introduced later in (14)). Differently, the second inequality (4b) bounds the contraction rate of a subspace of the tangent bundle by a factor μ_{k-1}. Consequently, inequality (4c) constraints the contraction rate to be faster than the expansion rate. For a deeper insight into the link between condition (4c) and k-contraction, consider the case of surface-contraction, namely $k = 2$. To ease intuition, consider a rectangle with an expanding and a contracting side. If the contracting side shrinks faster than the expanding one, the area of the rectangle goes to zero. Conversely, if the contracting side shrinks at a lower rate, the area diverges in time. A similar intuition relates (4c) to Definition 1.

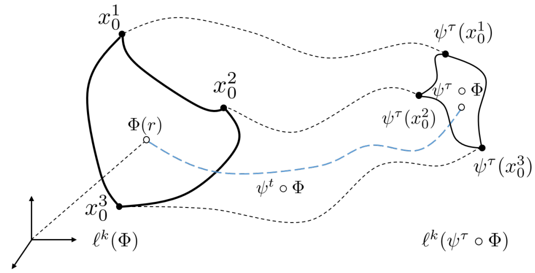

Fig. 1. Scheme of a 2-contractive system. The initial submanifold, described by Φ, is some surface with vertices at x_0^1, x_0^2 and x_0^3. The volume of this submanifold $\ell^k(\cdot)$ decreases exponentially along the trajectories of the system.

Theorem 1 provides sufficient conditions for k-contraction. For linear time-invariant dynamics, sufficient and necessary conditions can be established.

B. Linear systems

Consider now a linear system of the form

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^n. \tag{5}$$

We provide now a set of sufficient and necessary conditions to establish the k-contractivity property of (5). First, we consider the case of a matrix having only distinct real eigenvalues. This simplifies the statement and its readability. The general case will follow, and its proof will be omitted for space reasons.

Proposition 1. *Assume that A has only distinct real eigenvalues. Then, system* (5) *is k-contractive on $\mathcal{S} := \mathbb{R}^n$ if and only if there exist a set of symmetric matrices $P_i \in \mathbb{R}^{n\times n}$, $i = 0, \dots, k-1$, with $\operatorname{In}(P_i) = (i, 0, n-i)$, and a set of real numbers $\mu_i \in \mathbb{R}$, $i = 0, \dots, k-1$, such that*

$$A^\top P_i + P_i A \prec 2\mu_i P_i \qquad \forall i = 0, \dots, k-1\,, \tag{6a}$$

$$\sum_{i=0}^{k-1} \mu_i < 0\,. \tag{6b}$$

The proof of Proposition 1 is postponed to Section V-C. In the linear case, the previous interpretation of inequalities (4) bounding expansion and contraction rates directly maps to μ_i bounding the k largest eigenvalues of matrix A. Then, condition (6b) states that the sum of the k largest eigenvalues of A is negative. As discussed in Section IV-C, this condition is necessary and sufficient for k-contraction in linear time-invariant systems. Inequalities in Theorem 1 are in general conservative, since we ask the slowest stable dynamics to dominate $k-1$ times the fastest unstable one (4c). For linear systems, this condition is relaxed with (6b).

For the general case of arbitrary eigenvalues the previous result needs to be modified. To this end, consider the matrix

A in (5) and let $\Pi : \mathbb{C} \to \mathbb{R}$ denote the canonical projection onto the real axis. Let $\sigma(A)$ be the spectrum of A and suppose $\Pi(\sigma(A)) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ ($m \leqslant n$) with $\alpha_1 > \alpha_2 > \cdots > \alpha_m$. Set $h_i = \text{card}\left(\Pi^{-1}(\alpha_i) \bigcap \sigma(A)\right)$, where eigenvalues have been counted with their algebraic multiplicities (so that $h_1 + h_2 + \cdots + h_m = n$). Finally, let $d_0 = 0$, $d_i = \sum_{j=1}^{i} h_j$, $i \in \{1, \ldots, m\}$ and define $p_k := \max\left(\{d_0, d_1, \ldots, d_{m-1}\} \bigcap [0, k-1]\right)$, $c_k := \text{card}\left(\{d_0, d_1, \ldots, d_{m-1}\} \bigcap [0, k-1]\right)$.

Theorem 2. *System* (5) *is k-contractive if and only if there exist symmetric matrices $P_i \in \mathbb{R}^{n \times n}$ of respective inertia $(d_i, 0, n-d_i)$ and positive constants μ_i such that*

$$\begin{aligned} &A^\top P_i + P_i A \prec 2\mu_i P_i, \qquad \forall i \in \{0, \ldots, c_k - 1\}, \\ &(k - p_k - h_{c_k})\mu_{c_k - 1} + \sum_{i=0}^{c_k - 1} h_{i+1}\, \mu_i < 0\,. \end{aligned} \tag{7}$$

Intuitively, Corollary 2 is a rephrasing of Theorem 1 where inequalities corresponding to eigenvalues having the same real part are merged. This is a necessary step since the constant μ_i in (6a) cannot be used to separate eigenvalues of A that have overlapping real parts. As it will be shown in Lemma 5, the matrices P_i have inertia that is opposite to the one of $A - \mu_i I$ (i.e. the number of eigenvalues of $A - \mu_i I$ with negative real part is the number of positive eigenvalues of P_i). If some eigenvalues of A have identical real parts (e.g. $\mathfrak{Re}(\lambda_i) = \mathfrak{Re}(\lambda_{i+1})$), we cannot find a constant μ_i which separates each eigenvalue individually. Consequently, instead of varying by one eigenvalue at a time, the inertia of $A - \mu_i I$ may jump as we change the value of μ_i.

To improve clarity, we propose an example. Consider system (5) with $n = 7$ and eigenvalues satisfying $\mathfrak{Re}(\lambda_1) = \mathfrak{Re}(\lambda_2) > \mathfrak{Re}(\lambda_3) > \mathfrak{Re}(\lambda_4) = \mathfrak{Re}(\lambda_5) > \mathfrak{Re}(\lambda_6) = \mathfrak{Re}(\lambda_7)$. For this case, we have $\Pi(\sigma(A)) = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ with $\alpha_1 = \mathfrak{Re}(\lambda_1), \alpha_2 = \mathfrak{Re}(\lambda_3), \alpha_3 = \mathfrak{Re}(\lambda_4), \alpha_4 = \mathfrak{Re}(\lambda_6)$, and $h_1 = 2, h_2 = 1, h_3 = 2, h_4 = 2$. Moreover, $d_0 = 0, d_1 = 2, d_2 = 3, d_3 = 5, d_4 = 7$. Consider now the conditions for 6-contraction. We have $p_6 = 5$ and $c_6 = 4$. Consequently, conditions in Theorem 2 are evaluated for $i = 0, 1, 2, 3$, with matrices P_0, P_1, P_2, P_3 having inertia $(0, 0, 6)$, $(2, 0, 4)$, $(3, 0, 3)$ and $(5, 0, 1)$ respectively, and the sum $2\mu_0 + \mu_1 + 2\mu_2 + \mu_3 < 0$. Notice that the term $(k - p_k - h_{c_k})\mu_{c_k - 1}$ in (7) reads as $(6 - 5 - 2)\mu_3$, which removes one of the two identical eigenvalues with smallest real part (i.e., λ_6, λ_7). This accounts for the fact that 6-contraction requires any sum of 6 eigenvalues to be negative. Consider now the conditions for 2-contraction. Then, $p_k = 0$ and we obtain $A^\top P_0 + AP \prec 2\mu_0 P_0$, $\mu_0 < 0$. These conditions are identical to the ones obtained when $k = 1$. As a matter of fact, a sufficient condition for k-contractivity is $(k-1)$-contractivity. However, given the eigenvalues of A, the system cannot be 2-contractive without being 1-contractive. Hence, the condition becomes necessary.

IV. Relationship with Existing Results

In this section, we propose a comparison with existing works in k-contraction analysis. First, we clarify the differences between k-contraction and the notion of p-dominance [12], [13]. Then, we compare our results to existing works that exploit matrix compounds [14], [15], [17]. In particular, we highlight the main advantages of our result when compared to matrix compound methods, e.g. [15] and [18]. Finally, we compare our definition of k-contraction provided in Definition 1 to the one used in [19]. We refer to the latter as *infinitesimal k-contraction.*

A. Relation to p-dominance

In what follows, we link our main result to recent developments in p-dominance analysis [12], [13]. We start by recalling the definition of p-dominance.

Definition 2 (p-dominance)**.** *System* (1) *is said to be strictly p-dominant on $\mathcal{S} \subseteq \mathbb{R}^n$ if there exist a real number $\mu > 0$ and a symmetric matrix $P \in \mathbb{R}^{n \times n}$ with inertia $\text{In}(P) = (p, 0, n-p)$ such that*

$$P\frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P \prec -2\mu P\,, \quad \forall x \in \mathcal{S}. \tag{8}$$

It is natural to see the similarities between matrix inequalities (8) and (4b). The property of p-dominance has been related to various differential properties [13, Section V], such as differential positiveness [20] and monotonicity [21]. Condition (4b) sheds light on the relationship between k-contraction and p-dominance. To the best of the authors' knowledge, this link is not found in the literature.

To better understand this relation, consider the variational system of (14). Then, the p-dominance condition (8) splits the tangent space in a vertical subspace of dimension p and a horizontal subspace of dimension $n - p$. More precisely, for each initial condition $x_0 \in \mathcal{S}$ the tangent space can be divided in a horizontal distribution $\mathcal{H}_x$ and a vertical distribution $\mathcal{V}_x$. The property of p-dominance can be interpreted as a form of horizontal contraction [10, Section VII], in the sense that contraction is only imposed in the horizontal subspace. However, horizontal contraction is not a sufficient condition for k-contraction [19], and a bound on the expansion rate of the vertical subspace has to be imposed. This bound is obtained via (4a) paired with (4c).

This relationship between p-dominance and k-contraction explains why both properties share similar convergence results for systems evolving in a bounded set. Consider system (1) and assume $\mathcal{S}$ is compact and forward invariant. In [22] it is shown that any bounded solution converges to an equilibrium point if the system is 2-contractive. Similarly, in [13, Corollary 1], it is proven that any bounded solution converges to a fixed point if the system is 1-dominant.

B. Sufficient conditions based on matrix compounds

Sufficient conditions for k-contraction were originally given in the seminal work by Muldowney [14] and were recently rediscovered in the works [15], [23]. The remainder of this subsection focuses on briefly describing these sufficient conditions in the context of k-contraction as presented in Definition 1 and comparing them to the ones presented in Theorem 1. As previously stated, sufficient conditions provided in [14], [15] strongly depend on the use of matrix compounds. Consequently, first, we introduce the notion of multiplicative and additive compound of a matrix. More details on their computation can be found in [24].

Definition 3 (Multiplicative Compound [14])**.** *Consider a matrix $Q \in \mathbb{R}^{n\times m}$ and select an integer $k \in [1, min\{n, m\}]$. Moreover, define a minor of order k of the matrix Q as the determinant of some $k\times k$ submatrix of Q. The k^{th} multiplicative compound of Q, denoted as $Q^{(k)}$, is the $\binom{n}{k} \times \binom{m}{k}$ matrix including all the minors of order k of Q in a lexicographic order.*

Definition 4 (Additive Compound [14])**.** *Consider a matrix $Q \in \mathbb{R}^{n\times n}$ and select an integer $k \in [1, n]$. The k^{th} additive compound of Q is the $\binom{n}{k} \times \binom{n}{k}$ matrix defined as*

$$Q^{[k]} := \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(I + \varepsilon Q)^{(k)}|_{\varepsilon=0}.$$

Bearing these definitions in mind, we now reframe the sufficient condition for k-contraction presented in [14], [15] in the framework of this paper. In this case, we can consider time-varying systems of the form

$$\dot{x} = f(t, x), \qquad x \in \mathbb{R}^n \tag{9}$$

where f is sufficiently smooth with respect to its second argument and continuous with respect to the first one. Note that Definition 1 applies similarly to the case of time-varying systems (9).

Theorem 3. *Assume there exists a compact forward invariant set $\mathcal{A} \subseteq \mathbb{R}^n$, a symmetric positive definite matrix $Q \in \mathbb{R}^{\binom{n}{k}\times\binom{n}{k}}$ and a real number $\mu > 0$ such that for all $(t, x) \in \mathbb{R}_+ \times \mathcal{A}$ it holds*

$$Q\left(\frac{\partial f}{\partial x}(t, x)^{[k]}\right) + \left(\frac{\partial f}{\partial x}(t, x)^{[k]}\right)^\top Q \preceq -\mu Q. \tag{10}$$

Then, system (9) *is k-contractive on $\mathcal{S} := \mathcal{A}$.*

The proof is postponed to Section V-D.

Remark 2. *Inequality* (10) *is equivalent to the condition in [15, Theorem 9] using the logarithmic norm induced by the weighted ℓ_2 norm (e.g. [25, Equation 2.56]). However, in our statement, we allow the set $\mathcal{A}$ to be non-convex. Furthermore, for the case $k = 1$, we recover the well-known Demidovich conditions (e.g. [26]) and the proof in [11] for contraction of lengths in the context of Euclidean metrics.*

We now compare results in Theorem 3 to the ones in Theorem 1 and Proposition 1. We start by considering computational complexity. First, notice that the inertia constraint in our condition can be relaxed to obtain an unconstrained LMI, see [13, Section VI.B]. Hence, we focus on the linear system framework with distinct real eigenvalues, as Proposition 1 provides a larger set of matrix inequalities with respect to Theorem 1 and Theorem 2 ($c_k \leqslant k$). Let $M \in \mathbb{R}^{r\times r}$ be an arbitrary square matrix and $Q \in \mathbb{R}^{r\times r}$ be a symmetric matrix. Since Q is symmetric, each condition of the form $QM + M^\top Q \preceq \mu Q$ requires the computation of

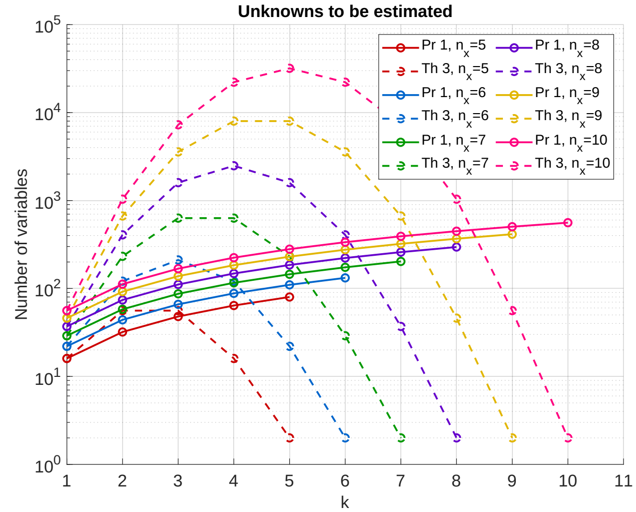

Fig. 2. Number of variables to be estimated Proposition 1 (solid) and by Theorem 3 (dashed) in function of k. Different colors refer to different n.

$N = r(r-1)/2 + 1$ variables, namely the entries of the top triangular portion of Q and the scalar μ. Then, Theorem 3 requires $N_1 = \binom{n}{k}\left(\binom{n}{k}-1\right)/2+1$ variables while Proposition 1 requires $N_2 = kn(n-1)/2+k$ variables. To better understand how the number of variables scales with different values of k and n, see Fig. 2. Clearly, for large dimensional systems and low k, the condition in (4) is of significantly smaller computational complexity. Moreover, even in the worst case of $k = n$, Proposition 1 typically requires between 10^2 and 10^3 variables. Differently, Theorem 3 can easily reach 10^4 variables in the worst case.

Now we compare the results in terms of feedback design. We claim that the lack of matrix compounds in Theorem 1 and Proposition 1 simplifies the process of k-contractive feedback design. Consider a linear system of the form $\dot{x} = Ax + Bu$, where $u \in \mathbb{R}^q$ is the control input. Assume we want to design a state-feedback controller of the form $u = -Kx$, with K a constant matrix of adequate dimension and such that the closed-loop system is k-contractive. On one hand, Theorem 3 reduces to designing K such that condition (10) is satisfied for the closed-loop system. That is,

$$Q\left((A - BK)^{[k]}\right) + \left((A - BK)^{[k]}\right)^\top Q \preceq -\mu Q.$$

However, this is a highly nonlinear and non-convex matrix inequality, due to the strong coupling between the matrices B, K imposed by the additive matrix compound. Consequently, even for a simple linear case, a design methodology for the gain K cannot be straightforwardly derived. On the other hand, also Proposition 1 asks for conditions (4) to be verified by the closed-loop system. However, this can be transformed to a set of linear matrix inequalities by means of standard transformations [27]. For this reason, we believe that condition (4) will be crucial in the development of k-contraction design tools.

C. Comparison with condition in [18]

Consider the linear case. Previous works already investigated sufficient conditions for k-contraction that do not require the computation of the k-additive compound. As a matter of fact, [18, Theorem 17] shows that a system is k-contractive if there exist an invertible matrix $T \in \mathbb{R}^{n\times n}$ and $q \in \{1, 2, \infty\}$ such that

$$\mathrm{tr}(A) + (n-k)\mu_q(-TAT^{-1}) = \tau(A) < 0 \qquad (11)$$

where $\mu_q(\cdot)$ represents the logarithmic norm of a matrix, see e.g. [28], [29], and $\mathrm{tr}(A)$ denotes the trace of the matrix A. We highlight that Proposition 1 provides necessary and sufficient conditions while the condition (11) is only sufficient. This is made evident by the first example presented in [18, Section V], where the authors consider a diagonal matrix $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ satisfying $\mathfrak{Re}(\lambda_1) \geqslant \dots \geqslant \mathfrak{Re}(\lambda_n)$. Then, for any invertible matrix T and any $q \in \{1, 2, \infty\}$, the left-hand side in (11) reads as

$$\tau(A) = -(n-k-1)\lambda_n + \textstyle\sum_{i=1}^{n-1} \lambda_i$$

Thus, condition (11) reduces to

$$\lambda_1 + \dots + \lambda_{n-1} < (n-k-1)\lambda_n < 0. \qquad (12)$$

Recall that a spectral property of the additive compound matrix is that the eigenvalues of the matrix $A^{[k]}$ are all the possible sums of the form $\lambda_{i_1} + \lambda_{i_2} + \dots + \lambda_{i_k}$, with $i \leqslant i_1 \leqslant \dots \leqslant i_k \leqslant n$, see [15]. Thus, a necessary and sufficient condition for k-contraction is

$$\sum_{i=1}^{k} \lambda_i < 0. \qquad (13)$$

For $k = n-1$, equation (12) reduces to (13). Consequently, (12) is necessary and sufficient for $(n-1)$-contraction. However, for $n-k > 1$, it is always possible to fix a sufficiently negative eigenvalue λ_n such that (12) is not satisfied even if (13) is satisfied. Thus, condition (11) is sufficient but not necessary.

D. Infinitesimal k-contraction

Inspired by classical works on contraction theory [30], we now provide a result linking the exponential stability properties of the lifted system to the k-contraction property proposed in Definition 1. The definition of k-contraction for the lifted system was used in [19]. In this section, we recall this definition and we provide further geometrical interpretation of it, along the lines of Section II.

The linearization of (9) about the trajectory $\psi^t(x_0)$ is

$$\dot{\delta_x} = \frac{\partial f}{\partial x}(t, \psi^t(x_0))\delta_x, \qquad (14)$$

where δ_x belongs to the tangent space $T_{\psi^t(x_0)}\mathbb{R}^n = \mathbb{R}^n$. Then, $\frac{\partial \psi}{\partial x}^t(x_0)\delta_{x_0}$ is a trajectory of (14) at time t initialized at δ_{x_0} at $t = 0$. From linearity, it can be deduced that $\frac{\partial \psi}{\partial x}^t(x_0)$ is the state transition matrix of (14). Then, $\frac{\partial \psi}{\partial x}^t(x_0)\delta_{x_0}$ depicts the infinitesimal displacement with respect to the solution $\psi^t(x_0)$ induced by the initial condition $x_0 + \delta_{x_0}$.

Pick any $x_0 \in \mathbb{R}^n$ and k initial conditions of the variational system in (14) $\delta^1_{x_0}, \dots, \delta^k_{x_0}$. Following [19], we define

$$X_{\mathrm{NL}}(t, x_0) := \begin{bmatrix} \frac{\partial \psi}{\partial x}^t(x_0)\delta^1_{x_0} & \cdots & \frac{\partial \psi}{\partial x}^t(x_0)\delta^k_{x_0} \end{bmatrix}.$$

Note that $X_{\mathrm{NL}}(0, x_0) = \frac{\partial \Phi_{\mathrm{loc}}}{\partial r}(r)$, where Φ_{loc} is a function whose image is an infinitesimal k-order parallelotpe with vertices at x_0 and $\delta^i_{x_0} + x_0$, namely

$$\Phi_{\mathrm{loc}}(r) = \sum_{i=1}^{k} r_i(\delta^i_{x_0} + x_0) + \left(1 - \sum_{i=1}^{k} r_i\right) x_0, \; r \in [0,1]^k.$$

We have the following result relating [19] to Definition 1.

Lemma 1 (Infinitesimal k-contraction)**.** *Consider a set $\mathcal{A} \subseteq \mathbb{R}^n$ and strictly positive constants γ and η such that the following holds for all $(t, x_0) \in \mathbb{R}_+ \times \mathcal{A}$*

$$|(X_{\mathrm{NL}}(t, x_0))^{(k)}| \leqslant \gamma e^{-\eta t} |(X_{\mathrm{NL}}(0, x_0))^{(k)}|, \qquad (15)$$

Then, system (9) *is k-contractive on $\mathcal{S} := \mathcal{A}$.*

The proof is omitted for space reasons.

V. Proofs

A. Preliminary results

We provide in this section some preliminary results that will be used in the proof of Theorem 1. First, we recall (with a mild reformulation) the following result on p-dominance [13, Theorem 1].

Theorem 4. *Suppose that system* (1) *is strictly p-dominant on a compact forward invariant set $\mathcal{A} \subset \mathbb{R}^n$ with rate $\mu > 0$ and symmetric matrix P with inertia $\mathrm{In}(P) = (p, 0, n-p)$. Then, for each $x \in \mathcal{A}$, there exists an invariant splitting $T_x\mathbb{R}^n = \mathcal{V}_x \oplus \mathcal{H}_x$, i.e. there exists a continuous mapping $\mathbf{T} : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ invertible for any $x \in \mathcal{A}$ and satisfying*

$$\mathbf{T}(x) := \begin{bmatrix} \mathbf{T_h}(x) & \mathbf{T_v}(x) \end{bmatrix}, \qquad (16a)$$

where $\mathbf{T_h} : \mathbb{R}^n \to \mathbb{R}^{n\times n-p}$ and $\mathbf{T_v} : \mathbb{R}^n \to \mathbb{R}^{n\times p}$ satisfy

$$\mathrm{Im}\ \mathbf{T_h}(x) = \mathcal{H}_x, \quad \mathrm{Im}\ \mathbf{T_v}(x) = \mathcal{V}_x. \qquad (16b)$$

Moreover, there exist a scalar $c_h > 0$ such that

$$\left|\frac{\partial \psi}{\partial x}^t(x) \begin{bmatrix} \mathbf{T_h}(x) & 0 \end{bmatrix} \delta_x\right| \leqslant c_h e^{-\mu t} \left|\begin{bmatrix} \mathbf{T_h}(x) & 0 \end{bmatrix} \delta_x\right| \qquad (16c)$$

holds for all $(t, x, \delta_x) \in \mathbb{R}_+ \times \mathcal{A} \times T_x\mathbb{R}^n$.

With this in mind, it is clear that if μ_{k-1} is strictly negative, LMI (4b) imposes a form of horizontal contraction on the system [10, Section VII]. Nonetheless, horizontal contraction is not a sufficient condition for k-contraction [19]. This motivates (4a). We clarify the effects of (4a) via the following Lemma.

Lemma 2. *Consider system* (1) *and assume there exist a forward invariant compact set $\mathcal{A} \subset \mathbb{R}^n$, a positive definite matrix $P_0 \in \mathbb{R}^{n\times n}$ and a scalar μ_0 satisfying* (4a) *for all $x \in \mathcal{A}$. Then there exists a constant $c_v > 0$ such that*

$$\left|\frac{\partial \psi}{\partial x}^t(x) \begin{bmatrix} 0 & \mathbf{T_v}(x) \end{bmatrix} \delta_x\right| < c_v e^{\mu_0 t} \left|\begin{bmatrix} 0 & \mathbf{T_v}(x) \end{bmatrix} \delta_x\right| \qquad (17)$$

for all $(t, x, \delta_x) \in \mathbb{R}_+ \times \mathcal{A} \times T_x\mathbb{R}^n$, with $\mathbf{T_v}$ as in (16b)*.*

Proof. Consider the function, $W := \delta_x^\top P_0 \delta_x$. It satisfies

$$\underline{\lambda}(P_0)|\delta_x|^2 \leqslant W \leqslant \overline{\lambda}(P_0)|\delta_x|^2, \qquad (18)$$

where $\underline{\lambda}(\cdot)$ and $\overline{\lambda}(\cdot)$ represent the minimum and maximum eigenvalue of their argument, respectively. By (14), its time-derivative satisfies

$$\begin{aligned}\dot{W} &= \delta_x^\top \left(P_0 \frac{\partial f}{\partial x}(x) + \frac{\partial f}{\partial x}(x)^\top P_0 \right) \delta_x \\ &< 2\mu_0 \delta_x^\top P_0 \delta_x = 2\mu_0 W.\end{aligned}$$

Then, by Grönwall–Bellman inequality, we obtain

$$W(t) < W(0) e^{\int_0^t 2\mu_0 d\tau} = e^{2\mu_0 t} W(0), \quad \forall t \in \mathbb{R}_+.$$

Invoking (18), we obtain for all $(t, x, \delta_x) \in \mathbb{R}_+ \times \mathcal{A} \times T_x\mathbb{R}^n$

$$\left| \tfrac{\partial \psi}{\partial x}^t(x) \delta_x \right| < \sqrt{\frac{\overline{\lambda}(P_0)}{\underline{\lambda}(P_0)}} e^{\mu_0 t} |\delta_x|.$$

As $\begin{bmatrix} 0 & \mathbf{T_v}(x) \end{bmatrix} \delta_x \in T_x\mathbb{R}^n$, the result trivially follows. $\square$

Given the above results, condition (4c) can be seen as imposing a bound on the maximum expansion rate of the vertical subspace with respect to the contraction rate of the horizontal one. In particular, (4c) holds if the first is smaller than the latter. We now relate this property to infinitesimal k-contraction. As a first step, we present a technical lemma related to matrix compounds.

Lemma 3. *Consider a time-varying matrix $M(t) \in \mathbb{R}^{n \times n}$*

$$M(t) = \begin{bmatrix} H(t) & V(t) \end{bmatrix},$$

with $H(t) \in \mathbb{R}^{n \times n-p}$, $V(t) \in \mathbb{R}^{n \times p}$ and $p \in [0, n)$. Assume there exist real numbers $c_h, c_v, \alpha, \beta > 0$ such that

$$|H(t)| \leqslant c_h e^{-\alpha t}, \quad |V(t)| \leqslant c_v e^{\beta t}, \qquad \forall t \in \mathbb{R}_+. \tag{19}$$

If $\alpha > (k-1)\beta$ for some integer $k \in [p+1, n]$, there exist some real numbers $c, \varepsilon > 0$ such that

$$|M(t)^{(k)}| \leqslant c e^{-\varepsilon t}, \qquad \forall t \in \mathbb{R}_+. \tag{20}$$

Proof. Consider the elements of the compound matrix $M(t)^{(k)}$. Each one is a k^{th}-order minor of the original matrix $M(t)$, i.e., it is the determinant of a $k \times k$ submatrix of $M(t)$, see Definition 3. Since $k \geqslant p+1$, each $k \times k$ submatrix contains at least one column composed of elements of $H(t)$. That is, in the minimum case

$$M_k(t) = \begin{bmatrix} h(t) & v_1(t) & \dots & v_{k-1}(t) \end{bmatrix}, \tag{21}$$

where $M_k(t) \in \mathbb{R}^{k \times k}$ is a submatrix of $M(t)$, $h(t) \in \mathbb{R}^k$ is a vector with components of $H(t)$ and $v_i(t) \in \mathbb{R}^k$ for $i = 1, \dots, k-1$ is a vector with components of $V(t)$. In what follows, we show the elements of $M(t)^{(k)}$ are bounded. Hence, we focus on submatrices of the form (21), since their determinant represents the worst-case scenario in a stability sense. Recall that, by definition of the wedge product,

$$\det(M_k(t)) = h(t) \wedge v_1(t) \wedge \cdots \wedge v_{k-1}(t).$$

The wedge product can be represented using a basis e^i, where e^i depicts the ith canonical vector of $\mathbb{R}^n$. More specifically, by bilinearity of the wedge product, we have

$$\det(M_k(t)) = \sum_{i=1}^n h^i(t)(e^i \wedge v_1(t) \wedge \cdots \wedge v_{k-1}(t)),$$

where $h^i(t)$ is the ith element of $h(t)$. By performing similar operations on the remaining vectors we deduce

$$\det(M_k(t)) = \sum_{i_1=1}^k \cdots \sum_{i_k=1}^k h^{i_1}(t) v_2^{i_2}(t) \dots v_{k-1}^{i_k}(t) E_k, \tag{22}$$

where $E_k := (e^{i_1} \wedge e^{i_2} \wedge \cdots \wedge e^{i_k})$. By (19), we have

$$|h^i(t)| \leqslant c_h e^{-\alpha t}, \quad |v^i(t)| \leqslant c_v e^{\beta t}.$$

Moreover, the factor E_k will be either zero or an element of the canonical basis in $\mathbb{R}^n$ multiplied by plus or minus one. Thus, using the triangle inequality, one obtains

$$|\det(M_k(t))| \leqslant \kappa c_h c_v e^{(-\alpha + (k-1)\beta)t}$$

where $\kappa > 0$ is a positive constant related to the number of non-zero instances of E_k. Now, since $\alpha - (k-1)\beta > 0$ by assumption, by continuity there always exists $\varepsilon > 0$ such that $\alpha - (k-1)\beta - \varepsilon > 0$. Then,

$$|M(t)^{(k)}| = |e^{-\varepsilon t} e^{\varepsilon t} M(t)^{(k)}| \leqslant e^{-\varepsilon t} |e^{\varepsilon t} M(t)^{(k)}|.$$

By considering the worst-case (21), we have

$$e^{\varepsilon t} |\det(M_k(t))| \leqslant \bar{c} e^{(-\alpha + (k-1)\beta + \varepsilon)t},$$

for some $\bar{c} > 0$. Hence, since $\alpha - (k-1)\beta - \varepsilon > 0$, each element of $e^{\varepsilon t} M(t)^{(k)}$ is exponentially decreasing and the norm $|e^{\varepsilon t} M(t)^{(k)}|$ is uniformly bounded for all $t \in \mathbb{R}_+$, thus concluding the proof. $\square$

Leveraging on the previous Lemmas, we now provide a bound on the k multiplicative compound of the state transition matrix of the lifted system (14).

Lemma 4. *Consider system (1) and assume there exist a forward invariant compact set $\mathcal{A} \subset \mathbb{R}^n$, constants μ_0, μ_{k-1} and matrices $P_0, P_{k-1} \in \mathbb{R}^{n \times n}$ such that (4) is satisfied. Then, there exist $\varepsilon, c > 0$ such that*

$$\left| \tfrac{\partial \psi}{\partial x}^t(x)^{(k)} \right| \leqslant c e^{-\varepsilon t}, \quad \forall (t, x) \in \mathbb{R}_+ \times \mathcal{A}. \tag{23}$$

Proof. Consider (16a) in Theorem 4. Invertibility of $\mathbf{T}(x)$ yields

$$\tfrac{\partial \psi}{\partial x}^t(x) = \tfrac{\partial \psi}{\partial x}^t(x) \mathbf{T}(x) \mathbf{T}(x)^{-1} = \Psi^t(x) \mathbf{T}(x)^{-1},$$

with $\Psi^t(x) := \begin{bmatrix} \frac{\partial \psi}{\partial x}^t(x) \mathbf{T_h}(x) & \frac{\partial \psi}{\partial x}^t(x) \mathbf{T_v}(x) \end{bmatrix}$. Given any $\delta_x \in T_x\mathbb{R}^n$, consider the decomposition $\delta_x = (\delta_x^h, \delta_x^v)$, where $\delta_x^h \in \mathbb{R}^{n-p}$ and $\delta_x^v \in \mathbb{R}^p$. Then, for an arbitrary δ_x^h, inequality (16c) of Theorem 4 implies

$$\left| \tfrac{\partial \psi}{\partial x}^t(x) \mathbf{T_h}(x) \delta_x^h \right| \leqslant c_h e^{\mu_k \; 1} |\mathbf{T_h}(x) \delta_x^h|.$$

Recall the definition of matrix norm,

$$\left| \tfrac{\partial \psi}{\partial x}^t(x) \mathbf{T_h}(x) \right| := \max_{|u|=1} \left| \tfrac{\partial \psi}{\partial x}^t(x) \mathbf{T_h}(x) u \right|.$$

By selecting vector $u^\star$ such that $|u^\star| = 1$, the previous exponential relation and the triangular inequality yield

$$\begin{aligned}\left| \tfrac{\partial \psi}{\partial x}^t(x) \mathbf{T_h}(x) \right| &= \left| \tfrac{\partial \psi}{\partial x}^t(x) \mathbf{T_h}(x) u^\star \right| \\ &\leqslant c_h e^{\mu_k \; 1} |\mathbf{T_h}(x) u^\star| \leqslant c_h e^{\mu_k \; 1} |\mathbf{T_h}(x)|.\end{aligned}$$

Since $\mathcal{A}$ is compact and $\mathbf{T}$ is continuous, $|\mathbf{T_h}(x)|$ is bounded for all $x \in \mathcal{A}$. Then, by (16c), and by (17) we obtain

$$\left|\frac{\partial \psi}{\partial x}^t(x)\mathbf{T_h}(x)\right| \leqslant c_h e^{\mu_k \;\; 1}|\mathbf{T_h}(x)| \leqslant \bar{c}_h e^{\mu_k \;\; 1}$$
$$\left|\frac{\partial \psi}{\partial x}^t(x)\mathbf{T_v}(x)\right| < c_v e^{\mu_0}|\mathbf{T_v}(x)| \leqslant \bar{c}_v e^{\mu_0}$$

for all $x \in \mathcal{A}$. Finally, by boundedness of $\mathbf{T}(x)$ and Lemma 3, we obtain

$$\left|\frac{\partial \psi}{\partial x}^t(x)^{(k)}\right| \leqslant |\Psi^t(x)^{(k)}||\mathbf{T}(x)^{-1^{(k)}}| \leqslant ce^{-\varepsilon t}$$

for all $x \in \mathcal{A}$, concluding the proof. □

B. Proof Theorem 1

Consider the k^{th} multiplicative compound of matrix $X_{\mathrm{NL}}(t, x_0)$ defined as in Section IV-D. A simple computation shows:

$$\begin{aligned} X_{\mathrm{NL}}(t, x_0)^{(k)} &= \left[\frac{\partial \psi}{\partial x}^t(x_0)\delta^1_{x_0} \quad \ldots \quad \frac{\partial \psi}{\partial x}^t(x_0)\delta^k_{x_0}\right]^{(k)} \\ &= \frac{\partial \psi}{\partial x}^t(x_0)^{(k)} X_{\mathrm{NL}}(0, x_0)^{(k)}, \end{aligned}$$

where the second inequality is derived from the Cauchy-Binet formula [31, Chapter 1]. From (4) and Lemma 4 we obtain

$$|(X_{\mathrm{NL}}(t, x_0))^{(k)}| \leqslant ce^{-\varepsilon t}|(X_{\mathrm{NL}}(0, x_0))^{(k)}|.$$

Hence, the system is infinitesimally k-contractive and the result follows by Lemma 1.

C. Proof Proposition 1

Following Theorem 3, a sufficient condition for k-contraction in linear systems is stability of $A^{[k]}$. Moreover, (13) shows that this condition is also necessary for k-contraction. Then, the remainder of the proof is based on showing that (6a)- (6b) in Proposition 1 are equivalent to $A^{[k]}$ being Hurwitz. First, we recall [32, Lemma 1, Section 3].

Lemma 5. *Assume there exists a symmetric matrix $P \in \mathbb{R}^{n\times n}$ with $\mathrm{In}(P) = (p, 0, n-p)$ and a constant μ such that*

$$A^\top P + PA \prec 2\mu P\,. \tag{24}$$

Then, matrix A has p eigenvalues with real part strictly bigger than μ and $n-p$ eigenvalues real part strictly smaller than μ.

We now present the main arguments proving sufficiency and necessity of the result in Proposition 1.

Sufficiency. Let the eigenvalues of A be ordered such that $\lambda_1 > \lambda_2 > \cdots > \lambda_n$. A spectral property of the additive compound matrix is that the eigenvalues of the matrix $A^{[k]}$ are all the possible sums of the form $\lambda_{i_1} + \lambda_{i_2} + \cdots + \lambda_{i_k}$, with $i \leqslant i_1 \leqslant \ldots \leqslant i_k \leqslant n$, see [15]. Therefore, (13) is a necessary and sufficient condition for k-contraction. Now, by Lemma 5, inequality (6a) implies $\lambda_{i+1} < \mu_i$ for all $i = 0, \ldots, k-1$. Then, by (6b) and since the eigenvalues are scalars, we have

$$\sum_{i=1}^{k} \lambda_i = \sum_{i=1}^{k} \lambda_i < \sum_{i=0}^{k-1} \mu_i < 0\,,$$

and (13) is satisfied.

Necessity. As stated in the previous step of the proof, if $A^{[k]}$ is Hurwitz, (13) is verified. Hence, by continuity, there exist a set of scalars $\varepsilon_i > 0$ such that $\sum_{i=1}^{k}(\varepsilon_i + \lambda_i) < 0$. Select $\mu_{i-1} = \varepsilon_i + \lambda_i > \lambda_i$, for $i = 1, \ldots, k-1$. We have

$$\sum_{i=0}^{k-1} \mu_i = \sum_{i=1}^{k} (\varepsilon_i + \lambda_i) < 0.$$

Now, define matrices $\hat{A}_i = A - \mu_i I$ with $i = 0, \ldots, k-1$. It is clear that, by the definition of μ_i, each matrix $\hat{A}_i$ has i negative eigenvalues and $n-i$ positive eigenvalues. Then, by [16, Theorem 2.5], there exist symmetric matrices P_i such that

$$\hat{A}^\top P_i + P_i \hat{A} = -G_i \qquad \forall i = 0, \ldots, k-1\,,$$

with $G_i \succ 0$ and $\mathrm{In}(P_i) = \mathrm{In}(\hat{A}_i) = \{i, 0, n-i\}$. Then, as $G_i \succ 0$ and by using the definition of $\hat{A}_i$, we have

$$A^\top P_i + P_i A \prec 2\mu_i P_i \qquad \forall i = 0, \ldots, k-1\,,$$

thus concluding the proof.

D. Proof of Theorem 3

Consider $\Phi \in \mathcal{I}_k$, where $\mathcal{I}_k$ is defined in (2). To simplify notation, let us denote for all (r, t) in $[0, 1]^k \times \mathbb{R}_+$

$$\Gamma(r, t) = \psi^t \circ \Phi(r)\ ,\ \Gamma_r(r, t) = \frac{\partial \Gamma}{\partial r}(r, t),$$
$$v(r, t) = \left(\Gamma_r(r, t)^{(k)}\right)^\top P^{(k)}\, \Gamma_r(r, t)^{(k)}.$$

For all (r, t) in $[0, 1]^k \times \mathbb{R}_+$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(r, t) = f(t, \Gamma(r, t)).$$

Then, by the chain rule, it follows that the point $\Gamma_r(r, t)$ evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_r(r, t) = \frac{\partial^2 \Gamma}{\partial r \partial t}(r, t) = \frac{\partial f}{\partial x}(t, \Gamma(r, t))\Gamma_r(r, t)$$

for all (r, t) in $[0, 1]^k \times \mathbb{R}_+$. Since these dynamics are linear, following similar steps to the ones presented in [15, Section 2.5], we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma_r(r, t)^{(k)} = \frac{\partial f}{\partial x}(t, \Gamma(r, t))^{[k]}\Gamma_r(r, t)^{(k)}. \tag{25}$$

Next, fix a symmetric positive definite matrix P such that $Q = P^{(k)}$. Then, since $\Gamma_r(r, t) \in \mathbb{R}^{n\times k}$, from the Cauchy-Binet formula [31, Chapter 1] the following equality holds

$$\begin{aligned} &\det\left(\Gamma_r(r, t)^\top P\, \Gamma_r(r, t)\right) \\ &\qquad = \left(\Gamma_r(r, t)^{(k)}\right)^\top P^{(k)}\, \Gamma_r(r, t)^{(k)} = v(r, t). \end{aligned} \tag{26}$$

Then, using the previous notation, the volume $\ell^k(\psi^t \circ \Phi)$ of $\psi^t \circ \Phi$ computed according to (3) takes the form

$$\ell^k(\psi^t \circ \Phi) = \int_{[0,1]^k} \sqrt{v(r, t)}\mathrm{d}r\,.$$

In turn, for all (r,t) in $[0,1]^k \times \mathbb{R}_+$, it evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell^k(\psi^t \circ \Phi) = \int_{[0,1]^k} \frac{\mathrm{d}}{\mathrm{d}t}\sqrt{v(r,t)}\,\mathrm{d}r = \int_{[0,1]^k} \frac{\mathrm{He}\left\{\left(\Gamma_r(r,t)^{(k)}\right)^\top Q \frac{\mathrm{d}}{\mathrm{d}t}\Gamma_r(r,t)^{(k)}\right\}}{2\sqrt{v(r,t)}}\mathrm{d}r$$

with the compact notation $\mathrm{He}\{A\} := A + A^\top$. Hence, for all (r,t) in $[0,1]^k \times \mathbb{R}_+$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell^k(\Phi) = \int_{[0,1]^k} \frac{1}{2\sqrt{v(r,t)}}\left(\Gamma_r(r,t)^{(k)}\right)^\top \times \mathrm{He}\left\{Q\frac{\partial f}{\partial x}(t,\Gamma(r,t))^{[k]}\right\}\Gamma_r(r,t)^{(k)}\,\mathrm{d}r\,.$$

Invoking inequality (10), the previous relation implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\ell^k(\psi^t \circ \Phi) \leqslant \int_{[0,1]^k} -\frac{\mu\, v(r,t)}{2\sqrt{v(r,t)}}\,\mathrm{d}r \leqslant -\frac{\mu}{2}\int_{[0,1]^k}\sqrt{v(r,t)}\,\mathrm{d}r \leqslant -\frac{\mu}{2}\ell^k(\psi^t \circ \Phi)$$

for all (r,t) in $[0,1]^k \times \mathbb{R}_+$. The result follows by Grönwall's lemma.

VI. Conclusions

We presented new alternative conditions for k-contraction that do not rely on matrix compounds. The proposed conditions reduce the k-contraction analysis to solving a set of LMIs. Moreover, these conditions provide a direct link between the p-dominance theory and k-contraction one.

Future works will focus on extending the proposed conditions to the context of time-varying systems and Riemannian metrics, similar to the context of 1-contraction, see, e.g. [11], [30]. Furthermore, we believe that the proposed conditions can be used to develop new tools for k-contractive feedback design, so that to extend existing conditions for standard 1-contraction, see, e.g. [4], [6], [7] and references therein.

References

[1] R. G. Sanfelice and L. Praly, "Convergence of Nonlinear Observers on $\mathbb{R}^n$ With a Riemannian Metric (Part I)," *IEEE Transactions on Automatic Control*, vol. 57, no. 7, pp. 1709–1722, 2012.

[2] G. Russo and M. di Bernardo, "Solving the rendezvous problem for multi-agent systems using contraction theory," in *48h IEEE Conference on Decision and Control*, 2009, pp. 5821–5826.

[3] Z. Aminzare and E. D. Sontag, "Synchronization of diffusively-connected nonlinear systems: Results based on contractions with respect to general norms," *IEEE Transactions on Network Science and Engineering*, vol. 1, no. 2, pp. 91–106, 2014.

[4] I. R. Manchester and J.-J. E. Slotine, "Control contraction metrics: Convex and intrinsic criteria for nonlinear feedback design," *IEEE Transactions on Automatic Control*, vol. 62, no. 6, pp. 3046–3053, 2017.

[5] M. Giaccagli, D. Astolfi, V. Andrieu, and L. Marconi, "Sufficient conditions for global integral action via incremental forwarding for input-affine nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6537–6551, 2022.

[6] M. Giaccagli, V. Andrieu, S. Tarbouriech, and D. Astolfi, "LMI conditions for contraction, integral action and output feedback stabilization for a class of nonlinear systems," *Automatica*, 2023.

[7] ——, "Infinite gain margin, contraction and optimality: An LMI-based design," *European Journal of Control*, vol. 68, p. 100685, 2022.

[8] S. Zoboli, S. Janny, and M. Giaccagli, "Deep learning-based output tracking via regulation and contraction theory," in *22nd IFAC World Congress*, 2023.

[9] D. Angeli, J. E. Ferrell Jr, and E. D. Sontag, "Detection of multistability, bifurcations, and hysteresis in a large class of biological positive-feedback systems," *Proceedings of the National Academy of Sciences*, vol. 101, no. 7, pp. 1822–1827, 2004.

[10] F. Forni and R. Sepulchre, "A differential lyapunov framework for contraction analysis," *IEEE Transactions on Automatic Control*, vol. 59, no. 3, pp. 614–628, 2014.

[11] V. Andrieu, B. Jayawardhana, and L. Praly, "Characterizations of global transversal exponential stability," *IEEE Transactions on Automatic Control*, vol. 66, no. 8, pp. 3682–3694, 2020.

[12] F. Forni and R. Sepulchre, "A dissipativity theorem for p-dominant systems," in *IEEE 56th Conference on Decision and Control*, 2017, pp. 3467–3472.

[13] ——, "Differential dissipativity theory for dominance analysis," *IEEE Transactions on Automatic Control*, vol. 64, no. 6, pp. 2340–2351, 2019.

[14] J. Muldowney, "Compound matrices and ordinary differential equations," *Rocky Mountain Journal of Mathematics*, vol. 20, no. 4, pp. 857–872, 1990.

[15] C. Wu, I. Kanevskiy, and M. Margaliot, "k-contraction: Theory and applications," *Automatica*, vol. 136, p. 110048, 2022.

[16] T. Stykel, "Stability and inertia theorems for generalized lyapunov equations," *Linear Algebra and its Applications*, vol. 355, no. 1-3, pp. 297–314, 2002.

[17] R. Ofir, M. Margaliot, Y. Levron, and J.-J. Slotine, "A sufficient condition for k-contraction of the series connection of two systems," *IEEE Transactions on Automatic Control*, vol. 67, no. 9, pp. 4994–5001, 2022.

[18] O. Dalin, R. Ofir, E. B. Shalom, A. Ovseevich, F. Bullo, and M. Margaliot, "Verifying k-contraction without computing k-compounds," *arXiv preprint:2209.01046*, 2022.

[19] C. Wu and D. V. Dimarogonas, "From partial and horizontal contraction to k-contraction," *arXiv preprint:2208.14379*, 2022.

[20] F. Forni and R. Sepulchre, "Differentially positive systems," *IEEE Transactions on Automatic Control*, vol. 61, no. 2, pp. 346–359, 2016.

[21] D. Angeli and E. Sontag, "Monotone control systems," *IEEE Transactions on Automatic Control*, vol. 48, no. 10, pp. 1684–1698, 2003.

[22] M. Y. Li and J. S. Muldowney, "On R.A. Smith's Autonomous Convergence Theorem," *Rocky Mountain Journal of Mathematics*, vol. 25, no. 1, pp. 365 – 378, 1995.

[23] D. Angeli, M. A. Al-Radhawi, and E. D. Sontag, "A robust lyapunov criterion for nonoscillatory behaviors in biological interaction networks," *IEEE Transactions on Automatic Control*, vol. 67, no. 7, pp. 3305–3320, 2022.

[24] M. Fiedler, *Special matrices and their applications in numerical mathematics*. Courier Corporation, 2008.

[25] F. Bullo, "Contraction theory for dynamical systems," *Kindle Direct Publishing*, vol. 1, 2022.

[26] A. Davydov, S. Jafarpour, and F. Bullo, "Non-euclidean contraction theory for robust nonlinear stability," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6667–6681, 2022.

[27] C. Scherer, P. Gahinet, and M. Chilali, "Multiobjective output-feedback control via LMI optimization," *IEEE Transactions on Automatic Control*, vol. 42, no. 7, pp. 896–911, 1997.

[28] Z. Aminzare and E. D. Sontag, "Contraction methods for nonlinear systems: A brief introduction and some open problems," in *53rd IEEE Conference on Decision and Control*, 2014, pp. 3835–3847.

[29] T. Strom, "On logarithmic norms," *SIAM Journal on Numerical Analysis*, vol. 12, no. 5, pp. 741–753, 1975.

[30] W. Lohmiller and J.-J. E. Slotine, "On contraction analysis for non-linear systems," *Automatica*, vol. 34, no. 6, pp. 683–696, 1998.

[31] S. M. Fallat and C. R. Johnson, *Totally nonnegative matrices*. Princeton university press, 2022.

[32] R. A. Smith, "The poincaré–bendixson theorem for certain differential equations of higher order," *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, vol. 83, no. 1-2, pp. 63–79, 1979.