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On the effect of the presence of an opponent in a class of LQ differential games

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Abstract— **The aim of this paper is to assess the effect of the presence of an opponent in a class of finite-horizon differential games described by scalar linear differential equations and quadratic cost functionals in which the state is penalized only at the terminal time. The contribution of the other player is quantitatively characterized by comparing the solutions of the underlying Riccati differential equations for the optimal control (in the absence of the opponent) and of the differential game. In the case of open-loop Nash equilibria, this effect can be characterized in closed form, since an analytic expression for the solutions of the coupled asymmetric differential Riccati equations can be computed. For feedback Nash equilibria a closed-form solution to the related coupled symmetric differential Riccati equations cannot be determined. Therefore an estimate of the solution is provided by relying on a functional approximation approach, allowing to characterize the effect of the presence of an opponent also in this setting.**

Index Terms— **Optimal control, Optimization, Nonlinear systems**

I. Introduction

The paramount importance of the *Differential Riccati Equations* (DRE) in multiple fields of application, spanning from mathematics, physics, engineering, and economics has led to an extensive and vibrant reasearch activity in the past decades, which is still currently active. In particular, in the field of *optimal control theory*, the application of *Dynamic Programming* (DP) (see [1], [2]) leads to an optimal solution of the finite-horizon linear quadratic regulator (LQR) problem, which is described by a state feedback characterized by the solution of a DRE (see [3], [4], [5], [6]). Such a solution can be equivalently obtained by applying *Pontryagin's Minimum Principle* (PMP) (see [7]), which yields instead an open-loop optimal solution. Nonetheless, the application of invariance arguments (see [4]) leads to a feedback synthesis of the underlying control law: it may then be immediately recognized that the invariance equation coincides with the DRE that arises via the application of DP. In the case of *linear quadratic* (LQ) differential games, (see [5], [6], [8], [9], [10]), when seeking for so-called *Nash equilibrium* strategies, a similar reasoning does not lead to the same conclusions, *i.e.*, application of the PMP together with invariance arguments yields a set of coupled DRE which is different from that provided by DP arguments. In the former case, the resulting set of coupled *asymmetric* DRE leads to the characterization of the so-called *open-loop Nash equilibrium* strategies (see [6]), while in the latter case the resulting set of coupled *symmetric* DRE allows describing the class of *feedback* (or *closed-loop*) *Nash equilibrium* strategies. Methods that provide both analytical (see [11], [12], [13], [14], [15]) as well as iterative (see, for example, [16]) solutions to asymmetric DRE have gained an interest in the past decades. Analytical methods try to directly solve the underlying equations, possibly by exploiting some structural properties exhibited by the (matrix) coefficients of the considered equations. For example in [11], a proportional relation between the matrices constituting the running cost is exploited to derive the closed-form solution to the coupled asymmetric DREs. In [13], [14] a local closed-form solution to the asymmetric DRE is provided by assuming the invertibility of the coefficient of the quadratic term, together with a similarity assumption on a suitable block matrix. On the other hand, iterative methods rely on the construction of a sequence of approximate solutions which iteratively converges to the solution of the DRE. Unfortunately in the case of symmetric DRE it is not possible to compute an analytical solution, in general, by exploiting structural properties similar to those exhibited by the asymmetric counterpart. Thus, approximation methods, like those based on series expansion (see, *e.g.*, [17], [18]) to provide estimates of such solutions, have emerged.

The main contribution of this work is to provide quantitative expressions that characterize the effect of the presence of an opponent for the class of scalar LQ differential games. In the case of open-loop Nash equilibria, a closed-form expression of such an effect can be computed, since it is possible to obtain closed-form solutions to the coupled asymmetric DRE. In this case, the presence of the opponent can be characterized in terms of a "translating" factor acting on the behavior of the optimal costate of the underlying optimal control problem of the considered player. In the case of feedback Nash equilibria, it is possible to show that the presence of an opponent always results into an improvement on the cost incurred by the considered player. An estimate of the expression, characterizing such an improvement, can be provided by relying on functional approximation arguments.

The rest of the paper is organized as follows. The class of considered problems, together with some preliminaries,

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are introduced in Section II. The main results, namely the characterization of the effect of the presence of an opponent in a class of LQ games, are provided in Sections III and IV. The former characterizes this effect, in closed-form, in the case of open-loop Nash equilibria, whereas the latter provides an estimate of this characterization in the case of feedback Nash equilibria. A discussion on the aspects concerning the construction of the estimates of the solutions to the coupled symmetric DRE, together with a numerical example are provided in Section V. Finally, some concluding remarks and a perspective on future work are given in Section VI.

II. Problem statement and preliminaries

Consider a nonzero-sum scalar differential game involving two, non-cooperating, players seeking to minimize the quadratic cost functionals defined as

$$J_i(u_1, u_2) = \frac{q_{if}}{2} x(t_f)^2 + \frac{1}{2} \int_{t_0}^{t_f} r_i u_i(t)^2 dt, \tag{1}$$

subject to the linear dynamics

$$\dot{x} = ax + b_1 u_1 + b_2 u_2, \quad x(t_0) = x_0. \tag{2}$$

The variable $x : \mathbb{R} \to \mathbb{R}$ represents the state of the system, while $u_i : \mathbb{R} \to \mathbb{R}$, for $i = 1, 2$, denotes the control input of the i-th player which acts on the system, $a \in \mathbb{R}$, $b_i \in \mathbb{R}$, $i = 1, 2$, and $t_f > t_0$ is the prescribed final time. The class of scalar games has been extensively studied (see *e.g.*, [5], [6]). Moreover, the scalar quantities q_{if} and r_i are assumed to belong to $\mathbb{R}_{>0}$, for $i = 1, 2$. Such a game is referred to as a *terminal differential game* and this nomenclature can be intuitively interpreted by noting that only the terminal value of the state is penalized, in addition to the control effort over the entire time interval. While the objective of each player is to minimize its individual cost functional J_i, the outcome of the cost functional depends, due to the dynamic interconnection induced by the state of (2), also on the strategy adopted by the other player. According to the following notion of solution, a pair $(u_1^\star, u_2^\star)$ with the property that neither player has incentive to *unilaterally deviate* from the strategy $u_i^\star$, $i = 1, 2$, is sought for.

Definition 1. Consider the differential game described by (1), (2). Fix $x_0 \in \mathbb{R}$. A pair $(u_1^\star, u_2^\star)$ of strategies is a *Nash equilibrium* for the game if the inequalities

$$J_1(u_1^\star, u_2^\star) \leq J_1(u_1, u_2^\star), \tag{3a}$$

$$J_2(u_1^\star, u_2^\star) \leq J_2(u_1^\star, u_2), \tag{3b}$$

hold for any pair $(u_1, u_2^\star)$ for (3a), and for any pair $(u_1^\star, u_2)$ for (3b), with $u_i(t) = \gamma_i(t, \eta_i(t))$, where $\gamma_i \in \Gamma_i$ is an admissible strategy and the set Γ_i constitutes the *strategy space* for the i-th player, being η_i the information available to the i-th player at time $t \in [t_0, t_f]$, for $i = 1, 2$. △

It is worth noting that, depending on the available information η_i, different types of equilibria arise (see, *e.g.*, [5]). In fact, if $\eta_i(t) = x_0$ for all $t \in [t_0, t_f]$, then it is possible to define the strategy space as $\Gamma_i = \{\gamma_i(t, x_0),\ i = 1, 2,\ t \in [t_0, t_f]\}$, and the equilibrium thus obtained is the so-called *open-loop Nash equilibrium*. In the case in which the information available is given by $\eta_i(t) = x(t)$ for all $t \in [t_0, t_f]$, for $i = 1, 2$, *i.e.*, the value of the state at time t is available for all the players, then the strategy space is given by $\Gamma_i = \{\gamma_i(t, x(t)),\ i = 1, 2,\ t \in [t_0, t_f]\}$, and the equilibrium obtained in this case is called a *feedback Nash equilibrium*. It is well known (see, *e.g.*, [6]) that the game described by (1), (2) admits an open-loop Nash equilibrium for all x_0, if and only if the two-point boundary value problem

$$\dot{y} = My, \qquad Py(t_0) + Qy(t_f) = [x_0\ 0\ 0]^\top, \tag{4}$$

where

$$M = \begin{bmatrix} a & -s_1 & -s_2 \\ 0 & -a & 0 \\ 0 & 0 & -a \end{bmatrix}, \ P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ -q_{1f} & 1 & 0 \\ -q_{2f} & 0 & 1 \end{bmatrix},$$

$y(t) := [x(t)\ \lambda_1(t)\ \lambda_2(t)]^\top$, s_i defined as $s_i := \frac{1}{r_i} b_i^2$ and $\lambda_i : \mathbb{R} \to \mathbb{R}$ representing the so-called *costate variables*, for $i = 1, 2$, has a solution for all x_0. The open-loop Nash equilibrium strategy is then provided by the pair $(u_1^\star, u_2^\star)$, with

$$u_i^\star(t) = -\frac{1}{r_i} b_i \lambda_i(t), \tag{5}$$

for $i = 1, 2$. The existence of a solution to the previous two-point boundary value problem is equivalent to the existence of solutions $p_1 : \mathbb{R} \to \mathbb{R}$, $p_2 : \mathbb{R} \to \mathbb{R}$, for all $t \in [t_0, t_f]$, to the *coupled asymmetric*[1] DRE

$$-\dot{p}_1 = 2ap_1 - s_1 p_1^2 - s_2 p_1 p_2, \tag{6a}$$

$$-\dot{p}_2 = 2ap_2 - s_2 p_2^2 - s_1 p_2 p_1, \tag{6b}$$

together with the terminal conditions $p_1(t_f) = q_{1f}$ and $p_2(t_f) = q_{2f}$. Moreover, the pair of strategies $(u_1^\star, u_2^\star)$ with

$$u_i^\star(t) = -\frac{1}{r_i} b_i p_i(t) e^{\int_{t_0}^{t} (a - s_1 p_1(\tau) - s_2 p_2(\tau)) d\tau} x_0, \tag{7}$$

for $i = 1, 2$, constitutes an equilibrium strategy. Since the strategy (7) can be expressed as $u_i^\star(t) = -\frac{1}{r_i} b_i p_i(t) x(t)$, where x is the trajectory of the system (2) in closed loop with $u_i^\star$, for $i = 1, 2$, this is also referred to as the *feedback synthesis* of (5).

Similarly to open-loop equilibria, also feedback Nash equilibria admit a characterization in terms of a set of *coupled symmetric* DRE, inspired by DP arguments and given by

$$-\dot{p_1} = 2ap_1 - s_1 p_1^2 - 2s_2 p_1 p_2, \tag{8a}$$

$$-\dot{p_2} = 2ap_2 - s_2 p_2^2 - 2s_1 p_2 p_1, \tag{8b}$$

[1]The property of asymmetry becomes apparent in the non-scalar case, it is employed here to distinguish the equations from those arising in the subsequent feedback Nash equilibria.

together with the final conditions $p_1(t_f) = q_{1f}$ and $p_2(t_f) = q_{2f}$, with s_1 and s_2 defined in (4). It is worth mentioning that once the solutions to the equations (8) are known, it is possible to construct the *value functions* of both players, that is

$$V_i(t,x) = \frac{1}{2}p_i(t)x^2, \tag{9}$$

for $i = 1, 2$, from which it is possible to evaluate the cost incurred by each player.

Finally, consider the *single-player* version of the problem described by the cost functional (1) and by the linear system (2), *i.e.*, the optimal control problem in which a single decision-maker seeks to minimize the cost functional

$$J(u) = \frac{q_f}{2}x(t_f)^2 + \frac{1}{2}\int_{t_0}^{t_f} ru(t)^2 dt, \tag{10}$$

subject to the linear dynamics

$$\dot{x} = ax + bu, \quad x(t_0) = x_0. \tag{11}$$

The following result, reported here without proof (see [19]) provides a closed-form solution to the underlying DRE, namely

$$-\dot{p}_{oc} = 2ap_{oc} - r^{-1}b^2p_{oc}^2, \qquad p_{oc}(t_f) = q_f. \tag{12}$$

Proposition 1. *Let*

$$\begin{aligned} p_{oc}(t) &= \left(q_f^{-1} + \int_t^{t_f} \bar{g}(\tau)d\tau\right)^{-1} e^{2a(t_f-t)} \\ &=: k_{oc}(t)^{-1}e^{2a(t_f-t)}, \end{aligned} \tag{13}$$

with $\bar{g}(t) = r^{-1}b^2e^{2a(t_f-t)} =: se^{2a(t_f-t)}$*, for all* $t \in [t_0, t_f]$*. Then* p_{oc} *in (13) is a closed-form solution of the DRE (12).* △

The result provided in Proposition 1 characterizes, in closed form, the expression of the optimal control law and of the value function associated to the optimal control problem described by (10) and (11). In fact it follows immediately that

$$u^\star_{oc}(t) = -\frac{1}{r}bp_{oc}(t)x(t), \qquad V_{oc}(t,x) = \frac{1}{2}p_{oc}(t)x^2, \tag{14}$$

with p_{oc} given by (13). To characterize the effect of the presence of an opponent in (1), (2) an expression similar to the one provided by (13) is sought for. More precisely the objective can be reformulated as the task of computing, if it exists, a function Ψ_i with the property that

$$\begin{aligned} p_i(t) &= \left(q_{if}^{-1} + \int_t^{t_f} \bar{g}_i(\tau)d\tau + \Psi_i(t)\right)^{-1} e^{2a(t_f-t)} \\ &=: k_i(t)^{-1}e^{2a(t_f-t)}, \end{aligned} \tag{15}$$

with $\Psi_i : [t_0, t_f] \to \mathbb{R}$, $\bar{g}_i$ defined as in (13), for $i = 1, 2$, solves (6) or (8). The structure exhibited by the right-hand side of (15) emphasizes the contribution of the opponent, which is encoded into Ψ_i (compare (15) with (13)), on the behavior of the cost incurred by the considered player. Knowledge of Ψ_i, once such an expression is known, would permit a quantitative analysis of the effect of the presence of an opponent by considering, in the case of feedback Nash equilibria, the ratio

$$\frac{V_{i_{oc}}(t,x)}{V_i(t,x)} = \frac{p_{i_{oc}}(t)}{p_i(t)} = \frac{k_i(t)}{k_{i_{oc}}(t)}. \tag{16}$$

This allows then to deduce the measure in which the opponent concurs to improve, or to worsen, the cost incurred by the considered player.

Remark 1. Once k_i on the second line of (15) is known, Ψ_i is given by

$$\Psi_i(t) = k_i(t) - q_{if}^{-1} - \int_t^{t_f} \bar{g}_i(\tau)d\tau, \tag{17}$$

for $i = 1, 2$, and for any $t \in [t_0, t_f]$. ▲

Even in the open loop case, namely for equations (6), the knowledge of Ψ_i allows determining the effect of the presence of an opponent on the costate variable of the considered player.

III. Closed-form solutions to the OL-NE coupled scalar DRE

The main results are introduced by first considering the characterization in closed form of open-loop Nash equilibrium strategies while the analysis of feeback Nash equilibria is postponed to section IV. Such a closed-form solution, in fact, allows for a direct comparison with (13) arising in the corresponding optimal control setting.

Proposition 2. *Suppose that there exist unique solutions* p_1 *and* p_2 *to the DRE (6a) and (6b), together with the prescribed terminal conditions. Let* p_i *be given by (15), for* $i = 1, 2$*. Then* p_i *satisfies the coupled DRE (6) if and only if* k_i*, for* $i = 1, 2$*, satisfies the system*

$$\dot{k}_1 = -\bar{g}_1(t) - \bar{g}_2(t)\frac{k_1(t)}{k_2(t)}, \qquad k_1(t_f) = q_{1f}^{-1}, \tag{18a}$$

$$\dot{k}_2 = -\bar{g}_2(t) - \bar{g}_1(t)\frac{k_2(t)}{k_1(t)}, \qquad k_2(t_f) = q_{2f}^{-1}. \tag{18b}$$

△

In order to improve clarity of the forthcoming statements, the following definition recalls the notion of *first integral* of an *ordinary differential equation* (see [20] and [21]).

Definition 2. Let f be a vector field defined in a domain $[t_0, t_f] \times U$, $U \subset \mathbb{R}^n$ and let $\xi : [t_0, t_f] \times U \to \mathbb{R}$ be such that $\xi \in C^1([t_0, t_f] \times U)$. Then ξ is called a *first integral* of the equation $\dot{\sigma} = f(t, \sigma)$ if its *Lie derivative* along the vector field f vanishes for all $(t, \sigma) \in [t_0, t_f] \times U$, namely

$$L_f\xi = \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial \sigma}f(t, \sigma(t)) \equiv 0.$$

△

The equations (18) constitute a system of differential equations with rational vector field. The following statement shows that (18) admits a simple first integral, which is subsequently instrumental for computing a closed-form expression of the solution to (18) (hence to the asymmetric DRE (6) via (15)).

Proposition 3. *Consider the system (18) and define* $\xi_i : \mathbb{R}^2 \to \mathbb{R}$ *as*

$$\xi_i(k_1, k_2) := \frac{k_i}{k_j}, \tag{19}$$

for $i = 1, 2$, $j = 1, 2$ *with* $j \neq i$. *Then* ξ_i *is a first integral of the system (18), for* $i = 1, 2$. *Moreover,* $\xi_i(k_1(t), k_2(t)) = q_{jf} q_{if}^{-1}$ *for all* $t \in [t_0, t_f]$, *for* $i = 1, 2$, $j = 1, 2$, *with* $j \neq i$. △

The following statement discusses how the knowledge of the first integral ξ_i, for either $i = 1$ or $i = 2$, is instrumental for providing a closed-form solution to (18)

Proposition 4. *Consider the system (18) and let*

$$k_i(t) = q_{if}^{-1} + \int_t^{t_f} \bar{g}_i(\tau) d\tau + \frac{q_{jf}}{q_{if}} \int_t^{t_f} \bar{g}_j(\tau) d\tau, \tag{20}$$

with $\bar{g}_i(t) = s_i e^{2a(t_f - t)}$, *for* $i = 1, 2$, $j = 1, 2$, $j \neq i$ *and for all* $t \in [t_0, t_f]$. *Then* k_i *is a closed-form solution of the system (18), for* $i = 1, 2$. △

The previous results allow characterizing the presence of an opponent for the considered class of differential games as follows. In particular, note that, in the case of the open-loop game, the costate is given, for all $t \in [t_0, t_f]$, by $\lambda_i(t) = p_i(t)x(t) = k_i(t)^{-1} e^{2a(t_f - t)} x(t)$, with k_i given by (20), whereas for, the case of the optimal control, it is given by $\lambda_{i_{oc}}(t) = p_{i_{oc}}(t)x(t) = k_{i_{oc}}(t)^{-1} e^{2a(t_f - t)} x(t)$, with $k_{i_{oc}}$ given by (13), for $i = 1, 2$. This allows concluding that the presence of an opponent has the effect of translating the optimal costate, and this translation depends, for $i = 1, 2$, on the factor Ψ_i in (20), namely

$$\Psi_i(t) = \frac{q_{jf}}{q_{if}} \int_t^{t_f} \bar{g}_j(\tau) d\tau = \frac{s_i}{2a} \frac{q_{jf}}{q_{if}} (e^{2a(t_f - t)} - 1).$$

IV. Estimation of the solutions to the FB-NE coupled scalar DRE

To put the equations (6) and (8), arising in open-loop Nash and feedback Nash equilibria, respectively, into the correct perspective, it is worth observing that in the scalar case the key difference is the presence of the factor 2 on the right-hand side. As shown below, such a factor has significant consequences on the analysis. By pursuing a different strategy with respect to that yielding a closed-form solution to the coupled DRE (6), the following results provide an estimate of the solution to the coupled symmetric DRE (8). The construction is achieved by relying on a functional approximation approach. Thus, once the estimate has been computed, it is possible to quantitatively characterize (via (16)), the effect of the presence of an opponent on the cost incurred by the considered player. The following result is the analogous of Proposition 2 in the previous section.

Proposition 5. *Suppose that there exist unique solutions* p_1 *and* p_2 *to the coupled symmetric DRE (8a), (8b) together with the prescribed terminal conditions. Let* p_i *be given by (15), for* $i = 1, 2$. *Then* p_i *satisfies the coupled DRE (8) if and only if* k_i, *for* $i = 1, 2$, *satisfies the system*

$$\dot{k}_1 = -\bar{g}_1(t) - 2\bar{g}_2(t) \frac{k_1(t)}{k_2(t)}, \qquad k_1(t_f) = q_{1f}^{-1}, \tag{21a}$$

$$\dot{k}_2 = -\bar{g}_2(t) - 2\bar{g}_1(t) \frac{k_2(t)}{k_1(t)}, \qquad k_2(t_f) = q_{2f}^{-1}. \tag{21b}$$

△

Since the sign of Ψ_i given by (17), with k_i solutions to (21), over the time interval $[t_0, t_f]$, plays a crucial role in the subsequent analysis, the following statement establishes that Ψ_1 and Ψ_2 are always positive over the considered time interval.

Proposition 6. *The functions* Ψ_i *given by (17) are such that* $\Psi_i(t) \geq 0$, *for* $i = 1, 2$, *and for all* $t \in [t_0, t_f]$, *provided that* $k_i(t) k_j(t)^{-1} \in C^1([t_0, t_f])$, *for* $i = 1, 2$, $j = 1, 2$, *and* $j \neq i$. △

The result provided by Proposition 6 allows characterizing, as stated below, the effect of the presence of an opponent on the cost incurred by the considered player.

Proposition 7. *Consider the ratio given by (16), in which* $k_{i_{oc}}$ *and* k_i *are provided by (13) and (15), respectively. Then*

$$V_i(t, x) \leq V_{i_{oc}}(t, x), \tag{22}$$

for $i = 1, 2$, *and for all* $(t, x) \in [t_0, t_f] \times \mathbb{R}$. △

From (22) it is then possible to conclude that the presence of an opponent in a scalar differential game is beneficial to both players in improving their incurred cost with respect to the one incurred in the case in which the opponent is absent. As anticipated at the beginning of this section, since it is not possible to provide an explicit first integral for the system (21) as carried out in the previous section, the following result provides instead an estimate of $k_i(t) k_j(t)^{-1}$, for all $t \in [t_0, t_f]$. The latter in turn can be used to provide an estimate of the solutions k_1 and k_2 of the system (21). From these estimates it is finally possible to quantitatively characterize the effect of an opponent on the cost incurred by the considered player (via (16)).

Proposition 8. *Suppose that there exist unique solutions* p_1 *and* p_2 *to the coupled symmetric DRE (8a), (8b) together with the prescribed terminal conditions. Consider the solutions* k_1 *and* k_2 *to the system (21) and suppose that* $k_i(t) k_j(t)^{-1} \in C^1([t_0, t_f])$, *for* $i = 1, 2$, $j = 1, 2$, *and* $j \neq i$. *Define* [2]

$$\xi_i(t) := \sum_{l=0}^{N} \omega_{i,l} t^l, \tag{23}$$

where $\omega_{i,l} \in \mathbb{R}$ *are scalar coefficients. Then, for any* $\epsilon > 0$ *and* $\epsilon_1 > 0$ *there exist an integer* N *and* $\omega_{i,l}^{\star}$, $l = 0, \ldots, N$,

[2] Although the simplest polynomial basis has been employed in the construction of (23), this may be immediately generalized to alternative sets.

such that, for $i = 1, 2,\ j = 1, 2,$ with $j \neq i$,

$$\left| \frac{k_i(t)}{k_j(t)} - \xi_i^\star(t) \right| < \epsilon,$$

for all $t \in [t_0, t_f]$. Moreover, let

$$\bar{k}_i(t) = q_{if}^{-1} + \int_t^{t_f} \bar{g}_i(\tau) d\tau + \bar{\Psi}_i(t), \tag{24}$$

with

$$\bar{\Psi}_i(t) = \int_t^{t_f} 2\bar{g}_j(\tau)\xi_i^\star(\tau)d\tau = \frac{s_i}{a}(e^{2a(t_f - t)} - 1)\omega_{i,0}^\star + \frac{s_i}{a}\left(\sum_{l=1}^{N}\sum_{\nu=0}^{l}\left(\frac{e^{2a(t_f-t)}}{(2a)^{l-\nu}} \cdot \frac{t^\nu}{\nu!} - \frac{1}{(2a)^{l-\nu}} \cdot \frac{t_f^\nu}{\nu!}\right)\omega_{i,l}^\star\right). \tag{25}$$

Then, the solutions k_i of the system (14) satisfy

$$|k_i(t) - \bar{k}_i(t)| < \epsilon_1,$$

for all $t \in [t_0, t_f]$, for $i = 1, 2$. △

V. Implementation aspects and simulation results

A few aspects related to the practical construction of the estimates of the solutions k_1 and k_2 to the system (21) are discussed in detail in this section. Note that Proposition 8, ensures the existence of N scalar coefficients $\omega_{i,l}$, $l = 0, \dots, N$ such that, for $i = 1, 2$ the estimates $\bar{k}_i$ are sufficiently close to the actual solutions k_i of the system (21), without providing a procedure to determine such values. This is precisely the objective of this section. Since the solutions k_1 and k_2 are not known in advance, the backward-construction of the estimates $\bar{k}_1$ and $\bar{k}_2$ is carried out, locally in time and state, as reported in Algorithm 1. It is worth noting that $\bar{k}_i$, in line 9 of Algorithm 1, is obtained by evaluating (24) and (25), for $t \in T^\mu$, namely

$$\bar{k}_i(t) = \bar{k}_i(t_f^\mu) + \int_t^{t_f^\mu} \bar{g}_i(\tau) d\tau + \bar{\Psi}_i(t), \tag{26}$$

and

$$\bar{\Psi}_i(t) = \int_t^{t_f^\mu} 2\bar{g}_j(\tau)\xi_i^{\star\mu}(\tau)d\tau = \frac{1}{a}(\bar{g}_i(t) - \bar{g}_i(t_f^\mu))\omega_{i,0}^{\star\mu} + \frac{1}{a}\left(\sum_{l=1}^{N}\sum_{\nu=0}^{l}\left(\frac{\bar{g}_i(t)}{(2a)^{l-\nu}} \cdot \frac{t^\nu}{\nu!} - \frac{\bar{g}_i(t_f^\mu)}{(2a)^{l-\nu}} \cdot \frac{(t_f^\mu)^\nu}{\nu!}\right)\omega_{i,l}^{\star\mu}\right). \tag{27}$$

Once $\bar{k}_i$ is known, $\bar{k}_j$ can be directly computed as $\bar{k}_j = \bar{k}_i(\xi_i^{\star\mu})^{-1}$, for all $t \in T^\mu$. Finally note that the procedure is guaranteed to converge by having assumed that $k_i(t)k_j(t)^{-1} \in C^1([t_0, t_f])$, for $i = 1, 2$, $j = 1, 2$, and $j \neq i$.

The simulations are carried out by selecting, for the first player, $q_{1f} = 1$, $r_1 = 1$, $b_1 = 1$, whereas, for the second player one has that $q_{2f} = 2$, $r_2 = 1$, and $b_2 = 0.5$. Finally, the parameters a and t_f have been selected as $a = 2$ and $t_f = 0.3s$, respectively, whereas the approximating polynomial has been chosen to be ξ_2, with $N = 5$. For the iterative construction of the estimates, the length of the time window has been selected as $\delta = 10^{-2}$, whereas the compact sets K^μ have been selected as square boxes with side length $\Delta k = 5 \cdot 10^{-2}$. The time histories of the estimation errors of the solutions k_1 and k_2 of the system (21) are reported in Figure 1, where the portion of trajectory related to an iteration is enclosed between two black dots. From Figure 1 it is possible to infer that both estimation errors are in the order of magnitude of 10^{-2}. Figure 2 depicts the time histories of the ratios $\frac{V_{ioc}}{V_i}$, for $i = 1, 2$, from which it is possible to appreciate, according to Proposition 7, that both are greater than 1, then implying that both players gain advantage, in terms of incurred cost, by the presence of the respective opponent. In particular, Figure 2 shows that the second player is the one that gains the most by the presence of his opponent, namely the player one, which has a more attenuated improvement of his incurred cost.

Algorithm 1

Require: Final time $t_f > 0$, time window length $\delta > 0$, terminal condition $(k_1(t_f), k_2(t_f))$, degree of the approximating polynomial N

1: $\mu \leftarrow 0$, $t_f^\mu \leftarrow t_f$, $t_0^\mu \leftarrow t_f^\mu - \delta$
2: $(\bar{k}_1(t_f^\mu), \bar{k}_2(t_f^\mu)) \leftarrow (k_1(t_f), k_2(t_f))$
3: **while** $t_0^\mu \geq 0$ and $t_f^\mu > t_0$ **do**
4: $\quad T^\mu \leftarrow [t_f^\mu - \delta, t_f^\mu]$
5: $\quad$ define the compact set $K^\mu \subset \mathbb{R}^2$, centered on $(\bar{k}_1(t_f^\mu), \bar{k}_2(t_f^\mu))$
6: $\quad$ get $\omega_{i,l}^{\star\mu}$, $l = 1, \dots, N$, by solving the constrained optimization problem given by

$$\min_{\omega_{i,l}^\mu} \max_{(t,k_1,k_2)\in T^\mu \times K^\mu} \left(\frac{\bar{g}_i(t)k_j - \bar{g}_j(t)k_i}{k_j^2} - \sum_{l=1}^{N} l\omega_{i,l}^\mu t^{l-1}\right)^2$$

7: $\quad \omega_{i,0}^{\star\mu} \leftarrow \frac{\bar{k}_i(t_f^\mu)}{\bar{k}_j(t_f^\mu)} - \sum_{l=1}^N \omega_{i,l}^{\star\mu}(t_f^\mu)^l$, for $i = 1, 2$, $j = 1, 2$ with $j \neq i$
8: $\quad$ **for** all $t \in T^\mu$ **do**
9: $\quad\quad$ evaluate $\bar{k}_i(t)$ from (24), for $i = 1, 2$
10: $\quad\quad$ **if** $\bar{k}_i(t) \in \partial K^\mu$ **then**
11: $\quad\quad\quad \mu \leftarrow \mu + 1$, $t_f^\mu \leftarrow t$
12: $\quad\quad\quad t_0^\mu \leftarrow t_f^\mu - \delta$ and go to line 16
13: $\quad\quad$ **end if**
14: $\quad$ **end for**
15: $\quad \mu \leftarrow \mu + 1$, $t_f^\mu \leftarrow t$, $t_0^\mu \leftarrow t_f^\mu - \delta$
16: $\quad (\bar{k}_1(t_f^\mu), \bar{k}_2(t_f^\mu)) \leftarrow (\bar{k}_1(t), \bar{k}_2(t))$
17: $\quad$ **if** $t_0^\mu < 0$ **then**
18: $\quad\quad t_0^\mu \leftarrow t_0$
19: $\quad$ **end if**
20: **end while**

VI. Conclusions and further work

The effect of the presence of an opponent in a class of finite-horizon scalar differential games has been characterized and studied in detail for open-loop and feedback Nash equilibria. In the former case it has been shown that this effect can be characterized in closed-form by relying on the notion of first integral of an ODE. In this case, the

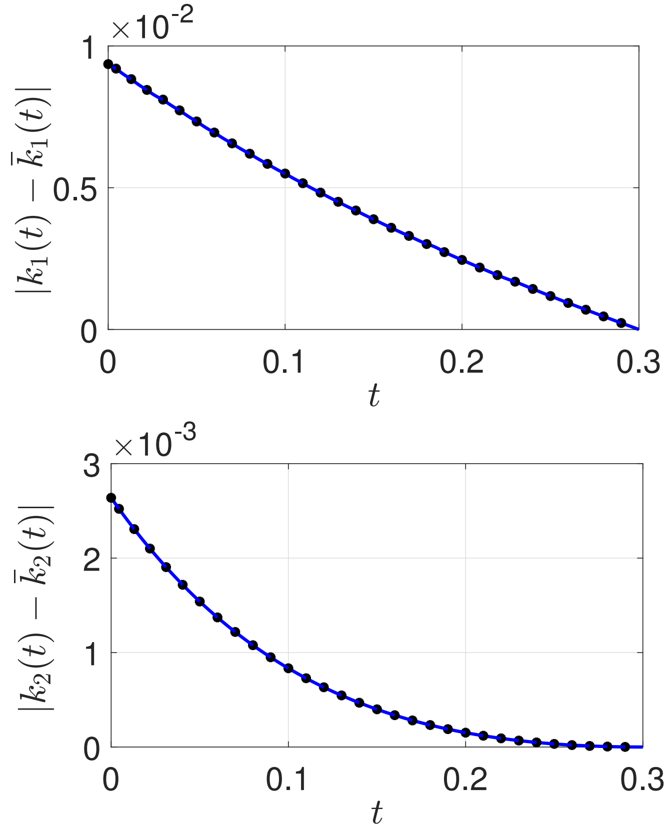

Fig. 1: Time histories of the estimation errors of the solutions k_1 and k_2 of the system (21), for the considered case.

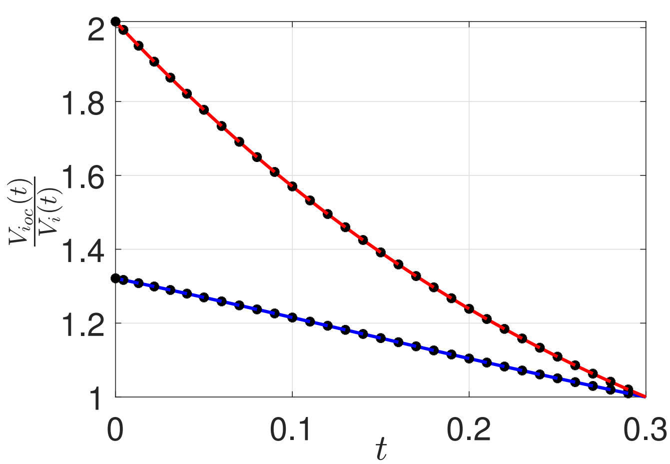

Fig. 2: Time histories of the ratios $\frac{V_{1_{oc}}}{V_1}$ (blue) and $\frac{V_{2_{oc}}}{V_2}$ (red), for the considered case. According to Proposition 7, the interaction between the two players helps both to improve their incurred cost with respect to the case in which the respective opponent is absent.

presence of an opponent is translated into a modification of the optimal costate of the underlying optimal control problem of the considered player. In the latter one, it has been proved formally that the presence of an opponent is always beneficial for the players to improve their incurred cost with respect to the case in which no opponent is present. Since a closed-form characterization similar to the one derived for the open-loop case cannot be determined, a different strategy has been pursued, by relying on a functional approximation, leading to the construction of an estimate of the solutions to a particular system of ODEs, allowing to quantitatively estimate the effect of the presence of an opponent for this class of differential games. Further work will involve the extension of the results to the case of non-scalar and more general differential LQ games.

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