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Local Turnpike Properties in Finite Horizon Optimal Control

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***Abstract*— In optimal control, it is well known that near-optimal trajectories exhibit a turnpike property if the system is strictly dissipative at the considered equilibrium and additional technical conditions are satisfied. In this paper we extend this result to a system which is merely locally strictly dissipative. For the special case of locally positive definite stage costs we show that there exists upper and lower bounds on the optimization horizon for which a local turnpike property becomes visible. For locally strictly dissipative costs we show that the same holds under a condition on the leaving arc of the local turnpike property. Our theoretical findings are illustrated by numerical examples.**

I. Introduction

The turnpike property describes the phenomenon that a near-optimal trajectory of an optimal control problem stays close to an optimal equilibrium most of the time. Often, but not always, at the end of the time horizon the optimal trajectory moves away from the optimal equilibrium. This is the so-called leaving arc. In recent literature (for surveys see [5], [6] and for a selection of recent papers in different application areas see, e.g., [1], [3], [4], [12]), the turnpike property has been intensively studied and sufficient conditions for this property to hold have been provided for different kinds of optimal control problems. One of these conditions is strict dissipativity with respect to the stage cost [10].

In this paper we investigate the occurrence of local turnpike behavior, i.e., turnpike behavior for initial conditions near a local optimal equilibrium, under a local strict dissipativity assumption using the discrete-time cardinality turnpike definition [5]. For infinite-horizon discounted optimal control problems it was recently shown in [9] that a local strict dissipativity condition implies a local turnpike property. Adapting Theorem 4.4 from [7] to the local situation, it is relatively straightforward to prove that the same holds for finite-horizon discounted optimal control. Yet, the situation becomes significantly more complex when finite-horizon *non-discounted* optimal control problems are considered, as we do in this paper. The reason for this is the possible occurrence of leaving arcs. Hence, in this paper we first investigate the case of locally positive definite stage costs, in which no leaving arcs occur. In this situation, we can show that there exists a lower bound on the optimization horizon for which a local turnpike property occurs for all trajectories that stay near the optimal equilibrium. In addition, there exist an upper bound on the optimization horizon until which near-optimal trajectories stay near the locally optimal equilibrium. If this upper bound is larger than the aforementioned lower bound, then an interval of horizons exists for which a local turnpike property can be observed.

In the general locally strictly dissipative case, we additionally have to take the leaving arc into account. Our results show that, if the leaving arc moves into a direction where the stage cost is smaller than in the locally optimal equilibrium, then the local turnpike property vanishes while if the leaving arc moves into a region with "expensive" stage cost then the local turnpike property persists.

The paper is organized as follows: we introduce our problem class and give basic definitions of strict dissipativity and the turnpike property for optimal control problems in Section II. We move on in Section III with investigating the behavior of near-optimal trajectories. Then, we analyze the case of local strict dissipativity and use these insights to obtain a local turnpike property for the general locally strictly dissipative case in Section IV. Numerical examples and simulations illustrate our findings.

II. Setting and preliminaries

We consider discrete time nonlinear systems of the form

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0 \tag{1}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous. We denote the solution of system (1) for a control sequence $\mathbf{u} = (u(0), \dots, u(N-1)) \in (\mathbb{R}^m)^N$ and initial value $x_0 \in \mathbb{R}^n$ by $x_\mathbf{u}(\cdot, x_0)$, or short by $x(\cdot)$ if there is no ambiguity about the respective control sequence and the initial value.

We impose nonempty state and input constraint sets $\mathbb{X} \subseteq \mathbb{R}^n$ and $\mathbb{U} \subseteq \mathbb{R}^m$, respectively, and we define the combined state and control constraints $\mathbb{Z} := \mathbb{X} \times \mathbb{U}$. The set of admissible control sequences for $x_0 \in \mathbb{X}$ up to time $N \in \mathbb{N}$ is defined by

$$\mathbb{U}^N(x_0) := \{\mathbf{u} \in \mathbb{U}^N \mid x_\mathbf{u}(k, x_0) \in \mathbb{X} \; \forall \, k = 1, \dots, N-1\}.$$

We assume controlled forward invariance of $\mathbb{X}$ to ensure feasibility, i.e., $\mathbb{U}^N(x_0) \neq \emptyset$ for each initial value $x_0 \in \mathbb{X}$.

By denoting the stage cost by $\ell : \mathbb{Z} \to \mathbb{R}$ we can formulate

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the optimal control problem for horizon $N \in \mathbb{N}$

$$\begin{aligned}\min_{\mathbf{u}\in\mathbb{U}^N(x_0)} J_N(x_0,\mathbf{u}) &= \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k,x_0),u(k))\\ x(k+1) &= f(x(k),u(k)), \quad k=0,\dots,N-1\\ x(0) &= x_0, \end{aligned} \tag{2}$$

where we minimize the cost function along an optimal trajectory. The optimal value function is defined by

$$V_N(x_0) := \inf_{\mathbf{u}\in\mathbb{U}^N(x_0)} J_N(x_0,\mathbf{u}).$$

Further, we denote the set of all admissible equilibria by $\mathcal{E} = \{(x^e,u^e) \in \mathbb{Z} \mid x^e = f(x^e,u^e)\}$ and, if it exists, (x^e_g,u^e_g) denotes the strictly globally optimal equilibrium, i.e. $\ell(x^e_g,u^e_g) < \ell(x^e,u^e)$ holds for all $(x^e,u^e) \in \mathcal{E}$ with $(x^e,u^e) \neq (x^e_g,u^e_g)$.

In the following we make use of comparison-functions, see [13], defined by

$$\begin{aligned}\mathcal{K} &:= \{\alpha:\mathbb{R}_+\to\mathbb{R}_+ \mid \alpha \text{ is continuous and}\\ &\qquad \text{strictly increasing with } \alpha(0)=0\}\\ \mathcal{K}_\infty &:= \{\alpha:\mathbb{R}_+\to\mathbb{R}_+ \mid \alpha\in\mathcal{K},\ \alpha \text{ is unbounded}\}\\ \mathcal{L} &:= \{\delta:\mathbb{R}_+\to\mathbb{R}_+ \mid \delta \text{ is continuous and}\\ &\qquad \text{strictly decreasing with } \lim_{t\to\infty}\delta(t)=0\}.\end{aligned}$$

Moreover, $\mathcal{B}_\varepsilon(x_0) \subseteq \mathbb{R}^n$ denotes the open ball with radius $\varepsilon > 0$ around x_0.

In this paper, we focus on the question of for which horizons we can observe a local turnpike behavior of near-optimal solutions. A key ingredient for our investigation is strict dissipativity, which goes back to [15]. The following definition provides the strict dissipativity definitions for the global and the local case.

Definition 1: (i) The system (1) is *strictly* (x,u)–*dissipative* for the stage cost ℓ at an equilibrium (x^e,u^e) if there exists a storage function $\lambda : \mathbb{X} \to \mathbb{R}$ bounded from below, and a function $\alpha \in \mathcal{K}_\infty$ such that for all $(x,u) \in \mathbb{Z}$ the inequality

$$\begin{aligned}\widetilde{\ell}(x,u) &:= \ell(x,u) - \ell(x^e,u^e) + \lambda(x) - \lambda(f(x,u))\\ &\geq \alpha(\|x-x^e, u-u^e\|)\end{aligned} \tag{3}$$

holds, where $\widetilde{\ell}$ is called the rotated stage cost.

(ii) The system (1) is *locally strictly* (x,u)–*dissipative* for the stage cost ℓ at an equilibrium (x^e,u^e) if strict dissipativity holds for all x in a neighborhood $\mathbb{X}_\mathcal{N}(x^e)$ of x^e, i.e. the inequality (3) holds for all $(x,u) \in \mathbb{X}_\mathcal{N}(x^e)\times\mathbb{U}$ with $f(x,u)\in\mathbb{X}$.

The connection between strict dissipativity and trajectory behavior is well studied, see, for instance, [7], [15], [16]. We remark that often in definitions of strict dissipativity the class $\mathcal{K}_\infty$–function α does not depend on the control u. Throughout this paper we use this stronger definition, but we will usually drop the "(x,u)" in what follows.

The following result appeared as Proposition 8.15 in [11]. It shows that strict dissipativity implies the turnpike property.

Proposition 1: Assume that system (1) is strictly dissipative for the stage cost ℓ at the global optimal equilibrium (x^e_g,u^e_g) with bounded storage function λ. Then, for each $\delta > 0$ there exists $\sigma_\delta \in \mathcal{L}$ such that for all $N,P \in \mathbb{N}$, $x_0 \in \mathbb{X}$ and $\mathbf{u} \in \mathbb{U}^N(x_0)$ with $J_N(x_0,\mathbf{u}) \leq N\ell(x^e_g,u^e_g) + \delta$ the set $\mathcal{Q}(x_0,\mathbf{u},P,N) := \{k \in \{0,\dots,N-1\} \mid \|x_{\mathbf{u}}(k,x_0) - x^e_g\| \geq \sigma_\delta(P)\}$ has at most P elements.
A similar technique as used for the proof of this proposition in [11] is used in the proof of Theorem 1 below.

We note that while the proposition makes a statement for any $N \geq 1$, it only becomes meaningful if $N > P$ and $\sigma_\delta(P)$ is reasonably small. This yields an implicit lower bound for the time horizons N for which the turnpike property can actually be observed. In the case of $\ell(x^e_g,u^e_g) = 0$, the condition on the trajectories is $J_N(x_0,\mathbf{u}) \leq \delta$. To guarantee the existence of such a $\mathbf{u}$, a reachability condition on trajectories starting in a neighborhood of the equilibrium can be used, see [8, Theorem 5.6] for details.

III. The Local Turnpike Property for Locally Positive Definite Stage Costs

We start our consideration with the relation between the original and the rotated cost function

$$\begin{aligned}\widetilde{J}_N(x_0,\mathbf{u}) &:= \sum_{k=0}^{N-1}\widetilde{\ell}(x_{\mathbf{u}}(k,x_0),u(k))\\ &= J_N(x_0,\mathbf{u}) - N\ell(x^e,u^e) + \lambda(x_0) - \lambda(x_{\mathbf{u}}(N,x_0)).\end{aligned} \tag{4}$$

We like to stress that the optimal trajectories of the optimal control problem (2) with objective J_N and $\widetilde{J}_N$ do not coincide since the term $\lambda(x_{\mathbf{u}}(N,x_0))$ depends on the control sequence $\mathbf{u}$. The storage function $\lambda(x_{\mathbf{u}}(N,x_0))$ measures the cost of the last state. If we assume w.l.o.g. that $\ell(x^e,u^e) = 0$ and $\lambda(x^e) = 0$, then a negative value $\lambda(x_{\mathbf{u}}(N,x_0)) < 0$ indicates the presence of a leaving arc, which makes the analysis technical and complex. In order to simplify the analysis, we proceed step by step, beginning in this section with locally positive definite stage costs, for which no leaving arc occurs. We first consider trajectories that stay in a neighborhood of the local equilibrium and then analyze conditions under which near-optimal trajectories stay in such a neighborhood. In Subsection III-C, we develop such conditions for locally positive definite stage costs and illustrate our results numerically in Subsection III-D.

A. Local turnpikes close to equilibria

First, we consider trajectories that stay in a neighborhood of a local equilibrium (x^e,u^e). The following theorem is based on Theorem 4.3 in [9] and on Proposition 1.

Theorem 1: Consider an optimal control problem (2). Let $(x^e,u^e) \in \mathcal{E}$ and assume that system (1) is locally strictly dissipative for the stage cost ℓ at (x^e,u^e) with storage function λ. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be neighborhoods of x^e with $x^e \in \mathcal{N}_1 \subset \mathcal{N}_2 \subseteq \mathbb{X}_\mathcal{N}$ such that λ is bounded on $\mathcal{N}_1$.

Then, for all sufficiently small $\delta > 0$ there exists $\sigma_\delta \in \mathcal{L}$ such that for all solutions $x(\cdot,x_0)$ with $x_0 \in \mathcal{N}_1$, $N \in \mathbb{N}$, and $\mathbf{u} \in \mathbb{U}^N(x_0)$ satisfying $J_N(x_0,\mathbf{u}) \leq$

$N\ell(x^e,u^e)+\delta$ and $x(k,x_0)\in\mathcal{N}_2$ for all $k=0,\dots,N-1$ the set $\mathcal{Q}(x_0,\mathbf{u},P,N) := \{k \in \{0,\dots,N-1\} \mid \|(x_{\mathbf{u}}(k,x_0),u(k))-x^e\| \geq \sigma_\delta(P)\}$ has at most P elements.

Proof: We follow the proof in [11] and note that the trajectories under consideration stay in the subset $\mathcal{N}_2$.

First, we fix $\delta>0$ and claim that the assertion holds with $\sigma_\delta(P) := \sigma_\delta(P) := \alpha^{-1}((2C+\delta)/P)$ where $C>0$ is such that $\lambda(x)\leq C$ for all $x\in\mathcal{N}_1$ and $\lambda(x)\geq -C$ for all $x\in\mathbb{X}$. We prove this claim by contradiction, i.e. we assume $N,P,x_0,\mathbf{u}$ are such that $J_N(x_0,\mathbf{u})\leq N\ell(x^e,u^e)+\delta$ but $\mathcal{Q}(x_0,\mathbf{u},P,N)$ contains at least $P+1$ elements. Then, from the relation (4) we can estimate

$$\widetilde{J}_N(x_0,\mathbf{u}) \leq J_N(x_0,\mathbf{u}) - N\ell(x^e,u^e) + 2C \leq 2C+\delta.$$

Next, we use the strict dissipativity at the equilibrium (x^e,u^e) and the fact that the set $\mathcal{Q}(x_0,\mathbf{u},N,P)$ contains at least $P+1$ elements. This implies

$$\begin{aligned}\widetilde{J}_N(x_0,\mathbf{u}) &= \sum_{k=0}^{N-1}\widetilde{\ell}(x_{\mathbf{u}}(k,x_0),u(k))\\ &\geq \sum_{k=0}^{N-1}\alpha(\|x_{\mathbf{u}}(k,x_0)-x^e\|) \geq \sum_{\substack{k\in\{0,\dots,N-1\}\\ \|x_{\mathbf{u}}(k,x_0)-x^e\|\geq\sigma_\delta(P)}}\alpha(\sigma(P))\\ &\geq (P+1)\alpha(\sigma_\delta(P)) \;\geq\; (P+1)\frac{2C+\delta}{P} \;>\; 2C+\delta,\end{aligned}$$

which is a contradiction. ■

B. Qualitative trajectory behavior

In order to examine the solution behavior of an optimal control problem with a horizon large enough to reach the globally optimal equilibrium, we first need to investigate properties of trajectories that stay near a locally optimal equilibrium for a certain time and then move out of a neighborhood of this equilibrium. To this end, we use [9, Lemma 5.1], noting that its statement also holds in the non-discounted setting of this paper.

Lemma 1 ([9]): Consider the optimal control problem (2) with f continuous and assume that system (1) is locally strictly (x,u)–dissipative for the stage cost ℓ at the local equilibrium $(x^e,u^e)\in\mathcal{E}$. Let $\rho>0$ be such that $\mathcal{B}_\rho(x^e)\subset\mathbb{X}_{\mathcal{N}}(x^e)$.

Then, there exists $\eta>0$ such that for each $K\geq 1$ and any trajectory $x_{\mathbf{u}}(\cdot,x_0)$ with $x_0=x(0)\in\mathcal{B}_\eta(x^e)$, $\mathbf{u}\in\mathbb{U}^N(x_0)$, and $x(K)\notin\mathcal{B}_\rho(x^e)$ there is $M\in\{0,\dots,K-1\}$ such that $x(0),\dots,x(M)\in\mathcal{B}_\eta(x^e)$ and

$$\begin{aligned}&(i)\; x(M+1)\in\mathcal{B}_\rho(x^e)\setminus\mathcal{B}_\eta(x^e)\\ \text{or }&(ii)\;\|u(M)-u^e\|\geq\eta\end{aligned}$$

holds.

C. Locally positive definite stage costs

We use the trajectory behavior from Lemma 1 to derive conditions to observe a local turnpike property. Before dealing with the challenges of a local leaving arc in Section IV, we begin our analysis by avoiding this problem by assuming that the stage cost is locally positive definite, i.e., that it is locally strictly dissipative with $\lambda\equiv 0$. The following assumption ensures, on the one hand, locally strictly dissipative positive definite stage costs and, on the other hand, a bound on the optimal value function near x^e.

Assumption 1: Let $(x^e,u^e)\in\mathcal{E}$, and assume that

(i) system (1) is locally strictly (x,u)–dissipative for the stage cost ℓ at the local equilibrium (x^e,u^e) with storage function $\lambda\equiv 0$;

(ii) $\ell(x^e,u^e)=0$;

(iii) for all $\varepsilon>0$ there exists a neighborhood of the equilibrium $\mathcal{N}(x^e)$ such that for all $x_0\in\mathcal{N}(x^e)$, and all $N\in\mathbb{N}$, the optimal value function satisfies $V_N(x_0)\leq\varepsilon$.

In the following, for an arbitrary neighborhood of the local equilibrium $\mathcal{N}$, we denote by $K(\mathcal{N})$ the minimal time instant when the stage cost may become negative after the trajectory has left the neighborhood $\mathcal{N}$, i.e., it holds that $\ell(x_{\mathbf{u}}(K(\mathcal{N}),x_0),u(K(\mathcal{N})))<0$ for some $x_0\in\mathcal{N}$, $\mathbf{u}\in\mathbb{U}^N(x_0)$ but $\ell(x_{\mathbf{u}}(k,x_0),u(k))\geq 0$ for all $k\in\{0,\dots,K(\mathcal{N})-1\}$, all $x_0\in\mathcal{N}$ and all $\mathbf{u}\in\mathbb{U}^N(x_0)$.

Note that Assumption 1 (iii) can be achieved by a reachability assumption as discussed after Proposition 1. Moreover, Assumption 1 (ii) is satisfied w.l.o.g. since it can be achieved by adding a constant to the stage cost. Hence, the stage cost and the rotated stage cost coincide in a neighborhood $\mathbb{X}_{\mathcal{N}}(x^e)$.

For optimal control problems satsfying Assumption 1, we can derive a positive lower bound for the cost of solutions leaving a neighborhood of x^e within a certain time N.

Lemma 2: Let Assumption 1 (i), (ii) hold and assume that the stage cost ℓ is bounded from below.

Then, there exists a horizon $N^\star\in\mathbb{N}$, neighborhoods $\mathcal{N}_1$, $\mathcal{N}_2$ with $x^e\in\mathcal{N}_1\subset\mathcal{N}_2\subseteq\mathbb{X}_{\mathcal{N}}(x^e)$ and $\gamma>0$ such that for all trajectories $x_{\mathbf{u}}(\cdot,x_0)$ with $x_0\in\mathcal{N}_1$, $N\leq N^\star$, and $\mathbf{u}\in\mathbb{U}^N(x_0)$ satisfying $x_{\mathbf{u}}(P,x_0)\notin\mathcal{N}_2$ for some $P\in\{1,\dots,N-1\}$ the cost function satisfies

$$J_N(x_0,\mathbf{u})\geq\gamma.$$

Here $N^\star$ depends on the problem data and is constructed in the proof.

Proof: The cost function can be split into three parts depending on the trajectory behavior. For this, we define $\mathcal{N}_1 := \mathcal{B}_\eta(x^e)$ and $\mathcal{N}_2 := \mathcal{B}_\rho(x^e)$ with $\eta,\rho>0$ from Lemma 1. Then, the trajectory under consideration satisfies the assumptions of Lemma 1 and, thus, there exists $M\in\{0,\dots,P\}$ such that either $x(M+1)\in\mathcal{B}_\rho(x^e)\setminus\mathcal{B}_\eta(x^e)$ or $\|u(M)-u^e\|\geq\eta$ holds. Due to the strict (x,u)–dissipativity we obtain in both cases

$$\begin{aligned}\widetilde{\ell}(x_{\mathbf{u}}(M,x_0),u(M)) &\geq \alpha(\|x_{\mathbf{u}}(M,x_0)-x^e\|)\geq\alpha(\eta),\\ \widetilde{\ell}(x_{\mathbf{u}}(M,x_0),u(M)) &\geq \alpha(\|u(M)-u^e\|)\geq\alpha(\eta),\end{aligned}$$

respectively, and we abbreviate $\alpha(\eta)=:\theta$. Further, Lemma 1 implies $\ell(x_{\mathbf{u}}(k,x_0),u(k))\geq 0$, for all $k=0,\dots,M-1$, because $x_{\mathbf{u}}(k,x_0)\in\mathcal{B}_\eta(x^e)$, where $\ell((x_{\mathbf{u}}(k,x_0),u(k)))\geq 0$

due to Assumption 1 (i). We set $K := K(\mathcal{B}_\eta(x^e))$ as defined after Assumption 1. Hence, the transition costs

$$\begin{aligned}
&\sum_{k=M}^{K-1} \ell(x_{\mathbf{u}}(k, x_0), u(k)) \\
&= \ell(x_{\mathbf{u}}(M, x_0), u(M)) + \sum_{k=M+1}^{K-1} \ell(x_{\mathbf{u}}(k, x_0), u(k)) \\
&\geq \ell(x_{\mathbf{u}}(M, x_0), u(M)) \geq \alpha(\eta) = \theta
\end{aligned}$$

are bounded from below by the cost generated at the time step at which the trajectory leaves $\mathcal{B}_\eta(x^e)$.

Next, we observe that boundedness from below implies the existence of $\ell_{\min} < 0$ with $\ell(x_{\mathbf{u}}(k, x_0), u(k)) \geq \ell_{\min}$ for all $k = K, \ldots, N-1$. Together this yields

$$\begin{aligned}
J_N(x_0, \mathbf{u}) &= \sum_{k=0}^{N-1} \ell(x_{\mathbf{u}}(k, x_0), u(k)) \\
&= \sum_{k=0}^{M-1} \ell(x_{\mathbf{u}}(k, x_0), u(k)) + \sum_{k=M}^{K-1} \ell(x_{\mathbf{u}}(k, x_0), u(k)) \\
&\quad + \sum_{k=K}^{N-1} \ell(x_{\mathbf{u}}(k, x_0), u(k)) \\
&\geq \theta + (N-K)\ell_{\min}
\end{aligned}$$

We now show that the assertion holds for any $\gamma \in (0, \theta)$. For this purpose, we show the existence of $N^\star$ with

$$\theta + (N-K)\ell_{\min} \geq \gamma$$

for all $N \in \mathbb{N}$ with $N \leq N^\star$. This is equivalent to

$$N\ell_{\min} \geq \gamma - \theta + K\ell_{\min} \Leftrightarrow N \leq \frac{\gamma - \theta}{\ell_{\min}} + K$$

since $\ell_{\min} < 0$. Hence, $N^\star = \left\lfloor \frac{\gamma-\theta}{\ell_{\min}} \right\rfloor + K(\mathcal{B}_\eta(x^e))$ satisfies the assertion. ■

Remark 1: (i) Approximating the transition costs with the lower bound θ may seem like a rough estimate. However, it provides the advantage that we do not need to know the exact behavior of the trajectory when leaving the neighborhood of the local equilibrium. Hence, θ provides a bound for the transition costs that is independent of the number of steps and the reason that causes these costs. Nevertheless, we need to know the time instant K after which the trajectory may be in a region with negative stage cost, to give an upper bound on the horizon N.

(ii) The value $\gamma > 0$ can be seen as an upper bound for the cost of all trajectories that stay in a neighborhood of the local equilibrium for all time steps $k = 0, \ldots, N-1$.

We are now able to formulate a local turnpike property.

Theorem 2: Consider the optimal control problem (2) with continuous f and stage cost ℓ bounded from below. Let Assumption 1 hold.

Then, there exists a neighborhood of the equilibrium $\mathcal{N}(x^e)$ and a horizon $N^\star \in \mathbb{N}$, specified in the proof, such that for all sufficiently small $\delta > 0$ there exists $\sigma_\delta \in \mathcal{L}$ depending on δ and on α and λ from Definition 1, but not on $N^\star$, such that for all solutions $x_{\mathbf{u}}(\cdot, x_0)$ with $x_0 \in \mathcal{N}(x^e)$, $N \leq N^\star$, and $\mathbf{u} \in \mathbb{U}^N(x_0)$ satisfying $J_N(x_0, \mathbf{u}) \leq V_N(x_0) + \delta$ the set $\mathcal{Q}(x_0, \mathbf{u}, P, N) := \{k \in \{0, \ldots, N-1\} \mid \|(x_{\mathbf{u}}(k, x_0), u(k)) - x^e\| \geq \sigma_\delta(P)\}$ has at most P elements.

Proof: Let $\delta > 0$, $\varepsilon > 0$ from Assumption 1, and $\gamma > 0$ from Lemma 2 such that $\varepsilon + \delta < \gamma$. Then, by Assumption 1 (iii) there exists a neighborhood $\widetilde{\mathcal{N}}$ of x^e such that for all $x_0 \in \widetilde{\mathcal{N}}$, $N \in \mathbb{N}$ it holds that $V_N(x_0) \leq \varepsilon$. We set $\mathcal{N}(x^e) := \widetilde{\mathcal{N}} \cap \mathcal{N}_1$ with $\mathcal{N}_1$ from Lemma 2, and conclude for $N^\star$ from Lemma 2 and for trajectories with $x_0 \in \mathcal{N}(x^e)$, $N \leq N^\star$, $\mathbf{u} \in \mathbb{U}^N(x_0)$ additional satisfying $x_{\mathbf{u}}(P, x_0) \notin \mathcal{N}_2$ for some $P \in \{1, \ldots, N-1\}$, that $J_N(x_0, \mathbf{u}) \geq \gamma$ which is a contradiction to $V_N(x_0) \leq \varepsilon < \gamma - \delta$. Hence, these trajectories cannot leave $\mathcal{N}_2$ from Lemma 2, i.e., $x_{\mathbf{u}}(k, x_0) \in \mathcal{N}_2$ for all $k \in \{0, \ldots, N\}$. For these trajectories Theorem 1 is applicable since strict (x, u)–dissipativity and $J_N(x_0, \mathbf{u}) \leq \varepsilon + \delta < \gamma$ holds. Thus, we can deduce the required turnpike property. ■

As discussed after Proposition 1, the turnpike property will only become visible for sufficiently large N, determined by σ_δ. In addition, here the upper bound $N^\star$ on N from Lemma 3 occurs. Hence, it depends on the interplay of these two bounds whether the local turnpike property will visibly occur. The example in the next section shows that this may indeed happen. We note that this is similar to the situation for discounted optimal control problems, where analogous lower and upper bounds on the discount rate exist, see [9].

D. A numerical example

In this section we illustrate our theoretical findings from above by a numerical example. We used the `nMPyC`–package, see [14], for solving optimal control problems numerically. The following example shows that the local turnpike property may indeed occur before N is so large that the solutions leave the neighborhood of x^e and turn towards the global equilibrium.

Example 1: Consider the dynamics $x^+ = x + u$ with $\mathbb{Z} = \mathbb{R} \times \mathbb{R}$ and stage cost

$$\ell(x, u) = 0.1445x^4 - 1.397x^3 + 3.2525x^2 + 5u^2$$

The corresponding optimal control problem has a local equilibrium at $(x^e, u^e) = (0, 0)$ and the global one at $(x^e_g, u^e_g) = (5, 0)$. Moreover, the stage cost exhibits local strict (x, u)–dissipativity at the equilibrium $(x^e, u^e) = (0, 0)$ with $\lambda \equiv 0$ since in a neighborhood of (x^e, u^e) the stage cost has a convex quadratic behavior.

Further, we remark that the globally optimal equilibrium $(5, 0)$ is reachable for each horizon $N \in \mathbb{N}$ and for each initial value $x_0 \in \mathbb{R}$. When setting $x_0 = 0.2$, Assumption 1 is fulfilled.

In Figure 1, the optimal trajectories of the optimal control problem are visualized for different horizons N. There, we observe that the local turnpike property appears for the horizons $N = 2, \ldots, 6$, while it is favorable to go to the global equilibrium for $N \geq 7$.

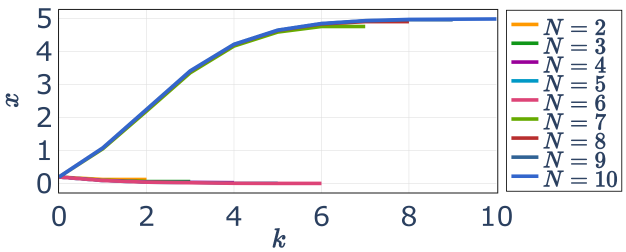

Fig. 1. Example 1 for different horizons N and $x_0 = 0.2$

IV. The Local Turnpike Property for Locally Strictly Dissipative Stage Costs

We now proceed with locally strictly dissipative stage costs, i.e., with general continuous storage functions $\lambda \not\equiv 0$. We adapt the result from Theorem 2 to this class of optimal control problems. It turns out that the result is generally not transferable since the local leaving arc and especially the costs of the leaving arc are of fundamental importance. Therefore, it is no longer sufficient to consider only the horizon N and the initial value. Rather, we also need to consider the local leaving arc behavior. To illustrate this issue, we consider first a globally strictly dissipative problem, which we modify to be locally dissipative in Example 3.

Example 2: Consider the dynamics $x^+ = x/2 + u$ with $\mathbb{Z} = \mathbb{R} \times \mathbb{R}$ and stage cost $\ell(x, u) = (x-1)^2 + u^2 - 0.2$ with the optimal equilibrium $(x_g^e, u_g^e) = (0.8, 0.4)$ and $\ell(x_g^e, u_g^e) = 0$. The corresponding optimal control problem is strictly dissipative with $\lambda(x) = 0.8x - 0.64$ (cf. [2, Proposition 4.3]), and, thus, exhibits the turnpike property.

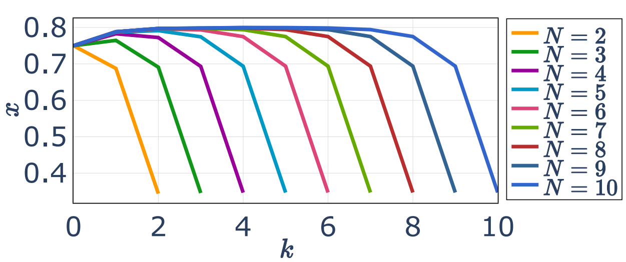

Fig. 2. Example 2 for different horizons N and $x_0 = 0.75$

Figure 2 visualizes the optimal trajectories for different horizons N and initial value $x_0 = 0.75$. We can observe a global turnpike property and that all optimal trajectories have a leaving arc tending towards 0 because the costs of keeping with control effort the trajectory at the equilibrium are higher than the costs of the leaving arc, which is also reflected in the negative cost functions, i.e., $J_N(x_0, \mathbf{u}) \approx -0.14$ for all horizons $N = 2, \ldots, 10$. Looking at relation (4), one sees that $\lambda(x(N, x_0))$ must be negative for this to happen. Indeed, the last state of the optimal trajectories for all horizons N is approximate $x(N, x_0) \approx 0.347$ and, thus, $\lambda(0.347) = -0.3624 < 0$.

Example 2 shows that if we start in a neighborhood of the optimal equilibrium, the optimal trajectories leave this neighborhood and have a negative cost. This means that Lemma 2 does no longer hold. We will thus now aim at a replacement for this lemma when $\lambda \not\equiv 0$.

To this end, we will use the following assumption, which relaxes Assumption 1.

Assumption 2: Let $(x^e, u^e) \in \mathcal{E}$ and assume that

(i) the system (1) is locally strictly (x, u)–dissipative for the stage cost ℓ at the local equilibrium (x^e, u^e) with continuous storage function λ bounded from below, and it holds that $\lambda(x^e) = 0$.

(ii) the stage cost ℓ is bounded from below, and satisfies $\ell(x^e, u^e) = 0$.

The next lemma shows that the statement of Lemma 2 still holds if λ is nonnegative along the considered trajectory.

Lemma 3: Consider an optimal control problem (2) with f continuous. Let Assumption 2 and Assumption 1 (iii) hold. Then, there exist a horizon $N^\star \in \mathbb{N}$, neighborhoods $\mathcal{N}_1$, $\mathcal{N}_2$ with $x^e \in \mathcal{N}_1 \subset \mathcal{N}_2 \subseteq \mathbb{X}_\mathcal{N}(x^e)$ and $\gamma > 0$ such that for all trajectories $x(\cdot, x_0)$ with $x_0 \in \mathcal{N}_1$, $N \leq N^\star$, and $\mathbf{u} \in \mathbb{U}^N(x_0)$ satisfying $x(P, x_0) \notin \mathcal{N}_2$ for some $P \in \{1, \ldots, N-1\}$ and $\lambda(x_\mathbf{u}(k, x_0)) \geq 0$ for all $k = 0, \ldots, N-1$ the cost function

$$J_N(x_0, \mathbf{u}) \geq \gamma$$

is bounded from below.

Proof: To show the existence of $N^\star$, we proceed as in the proof of Lemma 2: we define $\mathcal{N}_1 := \mathcal{B}_\eta(x^e)$ and $\mathcal{N}_2 := \mathcal{B}_\rho(x^e)$ with $\eta, \rho > 0$ from Lemma 1 and conclude that there exists a time instant $M \in \{0, \ldots, P\}$ such that either $x(M+1) \in \mathcal{B}_\rho(x^e) \setminus \mathcal{B}_\eta(x^e)$ or $\|u(M) - u^e\| \geq \eta$ holds. Until this time instant M the strict (x, u)–dissipativity holds such that we can consider the modified costs, i.e.,

$$\sum_{k=0}^{M} \ell(x_\mathbf{u}(k, x_0), u(k))$$

$$= -\lambda(x_0) + \sum_{k=0}^{M} \widetilde{\ell}(x_\mathbf{u}(k, x_0), u(k)) + \lambda(x_\mathbf{u}(M+1, x_0)).$$

Since the storage function λ is continuous, $x_0 \in \mathcal{N}_1$, and $\lambda(x^e) = 0$ holds, due to Assumption 2 (i), we can conclude that $\lambda(x_0) \leq \hat{\varepsilon}$ with $\hat{\varepsilon} > 0$ arbitrary small if we choose $\mathcal{N}_1$ sufficiently small. Further, we know $\lambda(x_\mathbf{u}(M+1, x_0)) \geq 0$, which leads to the same setting as in Lemma 2 if we replace γ by $\gamma - \hat{\varepsilon} > 0$. This yields the existence of the required upper bound $N^\star$ with the same construction as in the proof of Lemma 2. ■

Combining the techniques from Section III with the lemma above and the trajectory behavior, we can show that for horizons small enough, either the near-optimal trajectories stay in a neighborhood of the local equilibrium, or they must have an incentive to leave this neighborhood in the form of a negative storage function.

Corollary 1: Consider an optimal control problem (2) with f continuous. Let Assumption 2 and Assumption 1 (iii) hold.

Then, there exist a horizon $N^\star \in \mathbb{N}$ from Lemma 3, neighborhoods $\mathcal{N}_1$, $\mathcal{N}_2$ with $x^e \in \mathcal{N}_1 \subset \mathcal{N}_2 \subseteq \mathbb{X}_\mathcal{N}(x^e)$ and

$\gamma > 0$ such that for all trajectories $x(\cdot, x_0)$ with $x_0 \in \mathcal{N}_1$, $N \leq N^\star$, and $\mathbf{u} \in \mathbb{U}^N(x_0)$ satisfying $J^N(x_0, \mathbf{u}) < \gamma$ at least one of the following two properties holds:

(i) $x(k, x_0) \in \mathcal{N}_2$ for all $k = 0, \dots, N-1$;

(ii) there exists $\widetilde{N} < N$ such that $\lambda(x_{\mathbf{u}}(\widetilde{N}, x_0)) < 0$.

Proof: We use the notation and setting from Lemma 3 and assume that (ii) does not hold. Then $\lambda(x_{\mathbf{u}}(\widetilde{N}, x_0)) \geq 0$ holds for all $\widetilde{N} < N$ and thus Lemma 3 implies (i). ■

If case (i) holds in Corollary 1, then by applying Theorem 1 we can conclude that the trajectories under consideration exhibits a local turnpike property. A special case of this situation is the locally positive definite case we discussed in Section III-C.

In contrast, in case (ii) it depends on the interplay between λ and ℓ whether a local turnpike property can occur. If $\lambda \geq 0$ holds along the solution that leads from the local equilibrium to the region where $\ell < 0$, then we can again follow the proof of Theorem 1 and can conclude that local turnpike property occurs. If, however, ℓ happens to be negative along this solution, then the leaving arc may directly link to the approaching arc of the globally optimal equilibrium.

To illustrate these two situations, we modify, Example 2 to the following example. Because of the one-dimensional state space, we can construct two stage costs that illustrate the two cases in which directions the local leaving arc can tend.

Example 3: Consider the dynamics and the stage cost of Example 2 and combine the stage cost ℓ with

$$l_1(x, u) = (x+2)^2 + u^2 - 1.2,$$
$$l_2(x, u) = (x-2)^2 + u^2 - 1.2,$$

respectively, from which we construct the new stage costs

$$\ell_1(x, u) = \min(\ell(x, u), l_1(x, u)),$$
$$\ell_2(x, u) = \min(\ell(x, u), l_2(x, u)).$$

For both problems, we set $\mathbb{Z} = \mathbb{R} \times \mathbb{R}$ and the initial state $x_0 = 0.75$. Constructing the optimal control problems in this way, we end up with global equilibria at $(-1.6, -0.8)$ or $(1.6, 0.8)$, respectively, and the local equilibrium at $(0.8, 0.4)$. In case of ℓ_1, the global equilibrium lies in the direction of the local leaving arc, which extends into negative direction, see Example 2. As such, there is no incentive for the optimal trajectory to stay in the local equilibrium. In contrast, in case of ℓ_2, the global equilibrium is exactly in the other direction, i.e., $\lambda > 0$ holds along all paths from the local to the global optimal equilibrium. Hence, for small horizons it is more favorable to stay in the local equilibrium.

In Figure 3, we observe that the optimal trajectories corresponding to the optimal control problem with stage cost ℓ_1 exhibit a global turnpike property at $x_g^e = -1.6$ as soon as it is reachable, i.e., for $N \geq 3$. In comparison, the trajectories corresponding to the problem with stage cost ℓ_2 exhibit a local turnpike property at approximately 0.8 for N small enough, i.e., $N \leq 4$. Only for $N \geq 5$ it is rewarding to pay the transfer costs to the global optimal equilibrium.

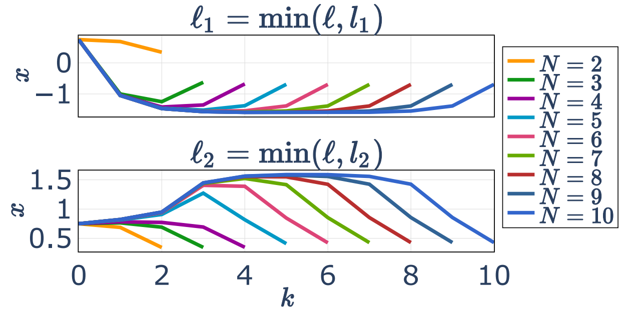

Fig. 3. Illustration of Example 3 for different horizons N, $x_0 = 0.75$

V. Conclusion

We have shown, in a discrete-time setting, that local strict dissipativity implies a local turnpike property if the initial value is close enough to the local equilibrium, the horizon N is short enough, and the local leaving arc does not extend into a region with lower stage cost than in the local equilibrium. For optimal trajectories without leaving arc, we essentially recover the results for discounted optimal control problems from [9], in which the discount factor plays a similar role as the horizon length in this paper.

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