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Operator methods of constructing matrix-valued Lyapunov functions

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Abstract— **We consider a two-component coupled system of differential equations with operator coefficients. In contrast to the well-known small-gain approach, we assume the presence of the exponential stability property of only one subsystem. We introduce a condition for the dominance of this subsystem, which allows us to prove new conditions for the exponential stability of a coupled system. An example of infinite networks is given. The results are compared with a small-gain approach based on the Lyapunov vector function.**

I. Introduction

Mathematical models of systems consisting of finite-dimensional and infinite-dimensional components are used to describe processes and phenomena in physics, control theory, chemical kinetics, and mathematical biology.

For example mathematical models of chemical and nuclear reactors [1], [2] as well as semiconductor lasers [3] are described by coupled systems of ordinary differential equations (ODEs) and partial differential equations (PDEs). In [4], the problem of stabilizing torsional and axial vibrations that arise in rotary drilling systems modeled by a coupled system of ODEs and PDEs is considered. We note that effective methods of stabilization of such systems were developed in [5]. Also, in recent years, with the development of various types of networks (social networks, transport and production networks), the problem of modeling networks of large size which can be idealized as infinite networks arose. The mathematical image of such networks can be an infinite system of ODEs which can be presented in an abstract form as a differential equation or a coupled system of differential equations in the Banach space [6].

The study of stability of equilibrium in such models is an important stage of their qualitative analysis. Equations linearized around the equilibrium can be presented in the form of a two-component abstract system with operator coefficients. As well as for finite-dimensional dynamic systems, the main method of stability analysis of infinite-dimensional systems is the method of Lyapunov functions, based on a scalar, vector or matrix-valued functions [7]. However, it is too conservative, since the presence of the exponential stability of all independent subsystems is not necessary for the stability of a coupled system.

The method of Lyapunov functions is convenient to use in a case of finding explicit solutions of the system is a difficult task. On the one hand, it is important to construct a Lyapunov function that would allow establishing simple enough conditions for checking stability. On the other hand, such conditions should not be too conservative.

The success of the classical and modern theory of coupled systems are based on mathematical methods that make it possible to analyze more complex structures based on the properties of independent subsystems, each of which has good dynamic properties (for example, exponential or asymptotic stability) [8]. This general paradigm is the basis of the Lyapunov vector function (VLF) methods [9], [10] and small-gain theorems [11], [12], on the basis of which simple and quite effective sufficient conditions for the stability of coupled systems are obtained.

This article proposes new methods for constructing the matrix Lyapunov function (MLF) based on which sufficient conditions for the stability of coupled two-component linear systems with operator coefficients are established. Exponential flow of only one subsystem is assumed, provided that this subsystem is dominant in the sense of the proposed spectrum separation condition.

The article consists of sixth sections. In the second section, the motivation of research and problem statement are given. In the third section, an algorithm for constructing the Lyapunov function is proposed. Based on this algorithm, theorems about the exponential stability of linear systems are proved in Section 4. In the fifth section, a countable system of differential equations are considered as an example of application of proven theorems. The last section discusses the results and prospects for further research.

Notations. Let H_i, $i=1,2$ be a real Hilbert spaces with scalar product $\langle\cdot,\cdot\rangle_{H_i}$ and norm $\|x\|_{H_i}=\sqrt{\langle x,x\rangle_{H_i}}$. The Banach space of linear bounded operators acting from H_i to H_j, $j=1,2$ with the defined norm $\|A\|=\sup_{\|x\|_{H_i}=1}\|Ax\|_{H_j}$, $A\in L(H_i,H_j)$ is denoted by $L(H_i,H_j)$. In the case of $H_i=H_j$, the notation $L(H_i,H_i)\equiv L(H_i)$ is used. The conjugate space H_i^* is identified with H_i due to Ries's theorem, i.e. $H_i^*=H_i$. If $A\in L(H_i,H_j)$, then $A^*\in L(H_j,H_i)$ denotes a linear operator conjugate to A , i.e., such that for all $x\in H_i$, $y\in H_j$ the identity $\langle y,Ax\rangle_{H_j}=\langle A^*y,x\rangle_{H_i}$ holds. If M is a subset of some topological space, then $\overline{M}$ is its closure, id_M is the identical mapping $M\to M$. If $A:\mathscr{D}(A)\to H_i$ is a linear operator, $\mathscr{D}(A)\subset H_i$ is a linear manifold, i.e., the domain of the linear operator A, and $\rho(A)\subset\mathbb{C}$ is its spectral set, i.e.,

$$\rho(A):=\{\lambda\in\mathbb{C}:R_A(\lambda):=(\lambda\,\mathrm{id}_{H_i}-A)^{-1}\in L(H_i)\}.$$

$\mathscr{R}(A):=\{Ax:x\in\mathscr{D}(A)\}$ is the set of values of the linear operator A. If $A\in L(H_i)$, $A^*=A$, then A is called a self-adjoint operator. It satisfies

$$\lambda_{\min}(A):=\inf_{\|x\|_{H_i}=1}\langle x,Ax\rangle_{H_i},\quad \lambda_{\max}(A):=\sup_{\|x\|_{H_i}=1}\langle x,Ax\rangle_{H_i}.$$

In the case when $\lambda_{\min}(A)>0$, the linear operator A will be called positive definite.

We introduce Hilbert space $\ell^2 := \{(x_1,\dots,x_k,\dots) : x_k \in \mathbb{R}, k \in \mathbb{N}, \sum_{k=1}^{\infty} x_k^2 < +\infty\}$ with a scalar product $\langle x,y\rangle_{\ell^2} := \sum_{k=1}^{\infty} x_k y_k$, $x, y \in \ell^2$.

II. Motivation

Consider a linear coupled system of differential equations

$$\begin{aligned} \dot{x}(t) &= Ax(t) + By(t), \quad x(0) = x_0 \in H_1, \\ \dot{y}(t) &= Cx(t) + Dy(t), \quad y(0) = y_0 \in \mathscr{D}(D), \end{aligned} \tag{1}$$

where $x \in H_1$, $y \in H_2$, $A \in L(H_1)$, $B \in L(H_2,H_1)$, $C \in L(H_1,H_2)$, D is a linear operator densely defined in H_2 (i.e., $\overline{\mathscr{D}(D)} = H_2$) and $\mathscr{D}(D)$ is its domain.

Remark 1: Systems of linear differential equations written in the abstract form (1) include a sufficiently wide class of linear infinite-dimensional systems. For example, these can be coupled systems of ordinary differential equations and differential equations with partial derivatives of various types (parabolic, hyperbolic), coupled systems of integro-differential equations and equations with partial derivatives, etc.

For an independent subsystem

$$\dot{y}(t) = Dy(t), \quad y(0) = y_0 \in \mathscr{D}(D), \tag{2}$$

we make the following assumptions.

Assumption 1: The linear operator D is densely defined in the Hilbert space H_2, $0 \in \rho(D)$ and there is a (coercive) Lyapunov function $V_2(y) = \langle y, P_{22}y\rangle_{H_2}$, where $P_{22} \in L(H_2)$ is a linear self-adjoint operator such that for some positive constants α_2, β_2 and γ_2, the inequality

$$\alpha_2\|y\|^2_{H_2} \le V_2(y) \le \beta_2\|y\|^2_{H_2}. \tag{3}$$

holds. For the complete derivative of function $V_2(y)$ along the trajectory of the linear system (2), the estimate

$$\dot{V}_2(y) = \langle Dy, P_{22}y\rangle_{H_2} + \langle y, P_{22}Dy\rangle_{H_2} \le -\gamma_2\|y\|^2_{H_2}. \tag{4}$$

holds.

Remark 2: It follows from Assumption 1 and the Loomer—Phillips Theorem ([13], p. 14, Theorem 4.3) that the linear operator D is closed and generates an exponentially stable C_0-semigroup $(e^{tD})_{t\ge 0}$ in the Hilbert space H_2, i.e., for some positive constants $M > 0$, $\omega > 0$ the inequality

$$\|e^{tD}\| \le Me^{-\omega t}, \quad t \ge 0. \tag{5}$$

holds.
Indeed, we introduce a new scalar product $(x,y)_{H_2} := \langle x, P_{22}y\rangle_{H_2}$ in the Hilbert space H_2. Taking into account (4) for all $y \in \mathscr{D}(D)$

$$\begin{aligned} (y,Dy)_{H_2} = \langle y, P_{22}Dy\rangle_{H_2} = \frac{1}{2}(\langle y, P_{22}Dy\rangle_{H_2} \\ +\langle P_{22}y, Dy\rangle_{H_2}) < 0, \end{aligned}$$

we come to the conclusion that the linear operator D is dissipative with respect to the scalar product $(.,.)_{H_2}$. The norm generated by this scalar product and the norm $\|\cdot\|_{H_2}$ are equivalent by (3). From Assumption 1, $0 \in \rho(D)$ and the openness of the set $\rho(D)$, it follows that for some $\lambda_0 > 0$ $\mathscr{R}(\lambda_0 \,\mathrm{id}_{H_2} - D) = H_2$. Thus, all the assumptions of the Loomer-Phillips Theorem are fulfilled. Using the Loomer-Phillips and Hille-Yosida theorems, one can prove that Assumption 1 guarantees well-posedness of (2).

It follows from Assumption 1 that the linear operator D is closed and the fact $D^{-1} \in L(H_2)$ is important for the upcoming, since $0 \in \rho(D)$. Further, we will use the notation $D^{-k} := (D^{-1})^k \in L(H_2)$, $k \in \mathbb{N}$.

We consider the problem of stability of the coupled system (1), having previously recalled the definition of exponential stability.

Definition 1: A linear system (1) is called exponentially stable if there are constants $M_0 > 0$, $\omega_0 > 0$ such that the inequality

$$\|x(t)\|_{H_1} + \|y(t)\|_{H_2} \le (\|x_0\|_{H_1} + \|y_0\|_{H_2})M_0 e^{-\omega_0 t}, \; t \ge 0. \tag{6}$$

holds.

Various variants of the Lyapunov function method can be used to study the exponential stability of the system (1).

The classical method of studying coupled systems, which arose in the 60s of the last century, is the method of VLF. This method is based on the assumption of asymptotic or even exponential stability of independent subsystems. Thus, to apply the VLF method to the system (1), we assume that the independent subsystem

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \in H_1 \tag{7}$$

is exponentially stable. In this case, it is well known [14] that for any linear self-adjoint positive definite operator $Q_1 \in L(H_1)$, $Q_1^* = Q_1$ there exists a linear self-adjoint positive definite operator $P_{11} \in L(H_1)$, $P_{11}^* = P_{11}$, which is a solution of the operator Lyapunov equation

$$A^* P_{11} + P_{11}A = -Q_1. \tag{8}$$

For an independent subsystem (7), one can construct a Lyapunov function $V_1(x) = \langle x, P_{11}x\rangle_{H_1}$, which has the following properties

$$\begin{aligned} \alpha_1\|x\|^2_{H_1} \le V_1(x) \le \beta_1\|x\|^2_{H_1}, \\ \langle x, (A^*P_{11} + P_{11}A)x\rangle_{H_1} \le -\gamma_1\|x\|^2_{H_1}, \end{aligned} \tag{9}$$

where $\alpha_1 = \lambda_{\min}(P_{11})$, $\beta_1 = \lambda_{\max}(P_{11})$ and $\gamma_1 = \lambda_{\min}(Q_1)$.

For the system (1) based on the VLF $V(x,y) = (V_1(x), V_2(y))^{\mathrm{T}}$, sufficient conditions for the exponential stability of the linear system (1) are standardly established in the form of inequality

$$\|B^*P_{11}\|\|C^*P_{22}\| < \frac{\gamma_1\gamma_2}{4}\sqrt{\frac{\alpha_1\alpha_2}{\beta_1\beta_2}}. \tag{10}$$

Remark 3: A significant advantage of the conditions (10) is their simplicity which makes it easy to check them. However, the disadvantage of these conditions is their considerable conservatism since these conditions a priori require the exponential stability of both independent subsystems.

The question arises how to investigate exponential stability in the case when an independent subsystem (7) is not exponentially stable. In this case, is it possible to obtain the same simple sufficient conditions for the stability of the coupled system (1) as the conditions (10)? To solve this problem, it is necessary to develop fundamentally new approaches, different from the classical paradigm of VLF. For the case $\dim H_1 = 1$ in [15], the Crain—Rutman theory of linear operators in semi-ordered Banach spaces is used. Another approach for various types of finite-dimensional systems are presented in [16], [17].

We recall that the matrix-valued Lyapunov function is defined as a two-index system of functions

$$U(x,y) = \begin{pmatrix} v_{11}(x) & v_{12}(x,y) \\ v_{21}(x,y) & v_{22}(y) \end{pmatrix}, \tag{11}$$

where $v_{ii} : H_i \to \mathbb{R}$, $v_{ij} : H_1 \times H_2 \to \mathbb{R}$, $v_{ij} = v_{ji}$, $i,j = 1,2$. Based on the MLF, it is possible to construct the scalar Lyapunov function $v(x,y,\eta) = \eta^{\mathrm{T}} U(x,y)\eta$, $\eta \in \mathbb{R}^2_+$, $\eta > 0$. We note that if $v_{21} = v_{12} \equiv 0$, then the MLF coincides with the VLF. In this sense, the MLF method is a certain generalization of the VLF method. The question arises about the selection of MLF elements. In the case when the independent subsystem is exponentially stable, it is advisable to choose the corresponding element of the MLF as a Lyapunov function for the corresponding independent subsystem. For example, in our case $v_{22}(y) = V_2(y) = \langle y, P_{22}y\rangle_{H_2}$. Thus, in this case, it is necessary to select the off-diagonal element v_{12} of the MLF and the diagonal element v_{11} of the MLF (11). This work is dedicated to solve these problems. To solve them, we will introduce an important assumption regarding independent subsystems, which we call the spectrum separation condition.

Assumption 2: For the linear system (1), the following inequality is satisfied (*condition of separation of spectrum*)

$$q := \|A\|\|D^{-1}\| < 1. \tag{12}$$

Remark 4: Intuitively, the condition of separation of spectra can be interpreted so that the second subsystem, which is exponentially stable, is dominant and has a significant margin of stability. It is subsystem that provides stabilization of the coupled system under certain conditions on the coupling function between subsystems.

III. Construction of MLF

Without loss of the generality, we choose the vector η in the form $\eta = (1,\eta_0)^{\mathrm{T}}$, $\eta_0 > 0$. We choose the elements of the MLF $U(x,y)$ in the following form

$$v_{11}(x) = \langle x, P_{11}x\rangle_{H_1},\ v_{12}(x,y) = v_{21}(x,y) = \langle x, P_{12}y\rangle_{H_1},$$
$$v_{22}(y) = \langle y, P_{22}y\rangle_{H_2}, \tag{13}$$

where $P_{ii} \in L(H_i)$, $i = 1,2$ are linear self-adjoint operators, $P_{12} \in L(H_2, H_1)$.

We choose the linear operator P_{12} in the form

$$P_{12} = \sum_{m=0}^{\infty} (-1)^{m+1}(A^*)^m(\eta_0^{-1}P_{11}B + \eta_0 C^* P_{22})D^{-(m+1)} \in L(H_2,H_1). \tag{14}$$

The right-hand part in (14) is well-defined due to the spectrum separation condition (12) and the following estimate of the norm holds

$$\|P_{12}\| \le \frac{\|D^{-1}\|}{1-q}\|\eta_0^{-1}P_{11}B + \eta_0 C^* P_{22}\|. \tag{15}$$

The formula (14) allows to simplify the expression for $\dot v(x,y,\eta)$ if

$$\eta_0(A^*P_{12}y + P_{12}Dy) + P_{11}By + \eta_0^2 C^* P_{22} y = 0 \tag{16}$$

is taken into account for all $y \in \mathscr{D}(D)$. Taking into account (16), for the derivative of the scalar function $v(x,y,\eta) = \eta^{\mathrm{T}} U(x,y)\eta$, the following map is true

$$\begin{aligned}\dot v(x,y,\eta) &= \langle x, (A^*P_{11} + P_{11}A + \eta_0(P_{12}C + C^*P_{12}^*))x)\rangle_{H_1} \\ &+\eta_0^2(\langle Dy, P_{22}y\rangle_{H_2} + \langle y, P_{22}Dy\rangle_{H_2}) \\ &+\eta_0\langle y, (B^*P_{12} + P_{12}^*B)y)\rangle_{H_2}.\end{aligned} \tag{17}$$

The following lemma establishes an estimate for the scalar function $v(x,y,\eta)$.

Lemma 1: For all $(x,y) \in H_1 \times H_2$, the inequalities

$$\begin{aligned}\lambda_{\min}(\underline{S})(\|x\|^2_{H_1} + \|y\|^2_{H_2}) &\le v(x,y,\eta) \\ &\le \lambda_{\max}(\overline{S})(\|x\|^2_{H_1} + \|y\|^2_{H_2}),\end{aligned} \tag{18}$$

hold, where

$$\underline{S} = \begin{pmatrix} \alpha_1 & -\frac{\|D^{-1}\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\| \\ -\frac{\|D^{-1}\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\| & \eta_0^2\alpha_2 \end{pmatrix},$$
$$\overline{S} = \begin{pmatrix} \beta_1 & \frac{\|D^{-1}\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\| \\ \frac{\|D^{-1}\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\| & \eta_0^2\beta_2 \end{pmatrix} \tag{19}$$

and $\alpha_i = \lambda_{\min}(P_{ii})$, $\beta_i = \lambda_{\max}(P_{ii})$, $i = 1,2$.
The *Proof* of Lemma 1 is carried out similarly to the technique of obtaining an estimate of the Lyapunov function for a two-component system (see, for example, [16]).

We consider the problem of choosing the diagonal elements of the MLF. Since the independent subsystem (2) is exponentially stable by Assumption 1, it is appropriate to choose $v_{22}(y) = \langle y, P_{22}y\rangle_{H_2}$ (see [16]), where the linear operator P_{22} satisfies the conditions of Assumption 1. Since the independent subsystem (7) is not necessarily exponentially stable. Two ways of choosing the linear operator P_{11} are proposed below.

First method. Let Q_1 be a self-adjoint positive definite linear operator, $r \in \mathbb{Z}_+ \cup \{\infty\}$. We consider the following operator equation for the linear operator P_{11}

$$\begin{aligned}A^*P_{11} + P_{11}A + \sum_{m=0}^{r}(-1)^{m+1}(C^*(D^{-(m+1)})^*B^*P_{11}A^m \\ +(A^*)^m P_{11}BD^{-(m+1)}C) = -Q_1.\end{aligned} \tag{20}$$

Second method. Let Q_1 be a self-adjoint positive definite linear operator, $r \in \mathbb{Z}_+ \cup \{\infty\}$. For the linear operator P_{11}, we consider the linear operator equation

$$\begin{aligned} A^*P_{11} + P_{11}A + \sum_{m=0}^{r} (-1)^{m+1}(C^*(D^{-(m+1)})^*B^*P_{11}A^m \\ +(A^*)^m P_{11}BD^{-(m+1)}C + (A^*)^m C^* P_{22} D^{-(m+1)}C \\ +C^*(D^{-(m+1)})^* P_{22}CA^m) = -Q_1. \end{aligned} \tag{21}$$

Formulas (16) and operator equation (20) (or (21)) are introduced in such a way as to provide the simplest form for analysis of the complete derivative $\dot{v}(x,y,\eta)$ of the function $v(x,y,\eta)$ for the linear system (1).

IV. Sufficient conditions for the exponential stability

Lemma 1 and formula (17) make it possible to formulate sufficient conditions for the exponential stability of the linear system (1).

Theorem 1: Assume that the conditions of Assumption 1 and the condition of separation of spectrum (12) are fulfilled for the linear system (1) and for some $r \in \mathbb{Z}_+ \cup \{\infty\}$, there exist a positive definite self-conjugate linear operator P_{11}, which satisfies the generalized Lyapunov operator equation (20), and a positive constant η_0 such that the inequalities

$$\|\eta_0^{-1}P_{11}B + \eta_0 C^* P_{22}\| < \frac{(1-q)\sqrt{\alpha_1\alpha_2}}{\|D^{-1}\|}, \tag{22}$$

$$\begin{aligned} \eta_0^2 \lambda_{\max}(G_r) + \frac{2q^{r+1}\|D^{-1}\|\|C\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\| \\ < \lambda_{\min}(Q_1), \end{aligned} \tag{23}$$

$$\begin{aligned} \lambda_{\max}(F_r(\eta_0)) + \frac{2q^{r+1}\|D^{-1}\|\|B\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\| \\ < \gamma_2\eta_0^2, \end{aligned} \tag{24}$$

are satisfied with the notation

$$\begin{aligned} F_r(\eta_0) = \sum_{m=0}^{r} (-1)^{m+1}(\eta_0^2(B^*(A^*)^m C^* P_{22} D^{-(m+1)} \\ +(D^{-(m+1)})^* P_{22} CA^m B) + B^*(A^*)^m P_{11} BD^{-(m+1)} \\ +(D^{-(m+1)})^* B^* P_{11} A^m B), \\ G_r = \sum_{m=0}^{r} (-1)^{m+1}((A^*)^m C^* P_{22} D^{-(m+1)}C \\ +C^*(D^{-(m+1)})^* P_{22} CA^m). \end{aligned}$$

Then, the linear system (1) is exponentially stable.

Remark 5: Theorem 1 reduces the study of the stability of linear system (1) to solving the operator equation (20) which greatly simplifies the problem. For example, in the case where $\dim H_1 < +\infty$, the equation (20) is a linear matrix equation that generalizes the Lyapunov equation and can be solved by standard methods of linear algebra, for example, based on the theory of direct matrix products.

Remark 6: The estimate

$$\lambda_{\max}(F_r(\eta_0)) \le \frac{2(1-q^{r+1})\|D^{-1}\|\|B\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\|.$$

can be established by direct calculations based on the spectrum separation condition (12).

This inequality allows us to replace the last inequality of the system (22) with a rougher inequality, the advantage of which is in easier verification

$$\frac{2\|D^{-1}\|\|B\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\| < \gamma_2 \eta_0^2. \tag{25}$$

Proof. First of all, we consider the case where $(x_0, y_0) \in H_1 \times \mathscr{D}(D)$. Then, $(x(t), y(t)) \in H_1 \times \mathscr{D}(D)$ for $t \ge 0$.

Taking into account (20), we obtain

$$\begin{aligned} \dot{v}(x,y,\eta) \le \langle x, (-Q_1 \\ + \sum_{m=r+1}^{\infty} (-1)^{m+1}((A^*)^m P_{11} BD^{-(m+1)}C \\ +C^*(D^{-(m+1)})^* B^* P_{11} A^m) + \eta_0^2 G_r \\ +\eta_0^2 \sum_{m=r+1}^{\infty} (-1)^{m+1}((A^*)^m C^* P_{22} D^{-(m+1)}C \\ +C^*(D^{-(m+1)})^* P_{22} CA^m))x\rangle_{H_1} \\ -\gamma_2\eta_0^2\|y\|_{H_2}^2 + \langle y, F_r(\eta_0)y\rangle_{H_2} \\ +\langle y, \sum_{m=r+1}^{\infty} (-1)^{m+1}(B^*(A^*)^m(P_{11}B + \eta_0^2 C^* P_{22})D^{-(m+1)} \\ +(D^{-(m+1)})^*(B^*P_{11} + \eta_0^2 P_{22}C)A^m B)y\rangle_{H_2} \\ = \langle x, (-Q_1 \\ + \sum_{m=r+1}^{\infty} (-1)^{m+1}((A^*)^m(P_{11}B + \eta_0^2 C^* P_{22})D^{-(m+1)}C \\ +C^*(D^{-(m+1)})^*(B^*P_{11} + \eta_0^2 P_{22}C)A^m) \\ +\eta_0^2 G_r)x\rangle_{H_1} \\ -\gamma_2\eta_0^2\|y\|_{H_2}^2 + \langle y, F_r(\eta_0)y\rangle_{H_2} \\ +\langle y, \sum_{m=r+1}^{\infty} (-1)^{m+1}(B^*(A^*)^m(P_{11}B + \eta_0^2 C^* P_{22})D^{-(m+1)} \\ +(D^{-(m+1)})^*(B^*P_{11} + \eta_0^2 P_{22}C)A^m B)y\rangle_{H_2} \end{aligned}$$

Using the spectrum separation condition (12) and the Cauchy—Buniakovsky inequality, we obtain the estimate

$$\begin{aligned} \dot{v}(x,y,\eta) \le \Big(-\lambda_{\min}(Q_1) + \eta_0^2\lambda_{\max}(G_r) \\ + \frac{2q^{r+1}\|C\|\|D^{-1}\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\|\Big)\|x\|_{H_1}^2 \\ + \Big(-\gamma_2\eta_0^2 + \lambda_{\max}(F_r(\eta_0)) \\ + \frac{2q^{r+1}\|B\|\|D^{-1}\|}{1-q}\|P_{11}B + \eta_0^2 C^* P_{22}\|\Big)\|y\|_{H_2}^2. \end{aligned} \tag{26}$$

It follows from (26) and the condition of Theorem 1 that for some constant $\mu > 0$, we have

$$\dot{v}(x,y,\eta) \le -\mu(\|x\|_{H_1}^2 + \|y\|_{H_2}^2). \tag{27}$$

Therefore,

$$\begin{aligned} \|x(t)\|_{H_1}^2 + \|y(t)\|_{H_2}^2 \le \frac{1}{\lambda_{\min}(\underline{S})} v(x(t), y(t), \eta) \\ \le \frac{\lambda_{\max}(\overline{S})}{\lambda_{\min}(\underline{S})} e^{-\frac{t\mu}{\lambda_{\max}(\overline{S})}}(\|x_0\|_{H_1}^2 + \|y_0\|_{H_2}^2), \quad t \ge 0, \end{aligned} \tag{28}$$

For arbitrary initial conditions, the validity of the inequality (28) follows from the fact that $\overline{H_1 \times \mathscr{D}(D)} = H_1 \times H_2$. The theorem is proved.

We note that the practical application of Theorem 1 involves excluding the free parameter η_0 from the system of inequalities (22). In the case when it is difficult to exclude the parameter η_0, explicit conditions of exponential stability may not be available. Then, it is worth to use the second method of choosing the linear operator P_{11}.

Theorem 2: Assume that for a linear system (1), the conditions of Assumption 1, the condition of separation of spectrum (12) are satisfied, and for some $r \in \mathbb{Z}_+ \cup \{\infty\}$ there exists a positive definite self-adjoint linear operator P_{11} that satisfies the generalized Lyapunov operator equation (21) such that the inequalities

$$\|P_{11}B + C^*P_{22}\| < \frac{(1-q)\sqrt{\alpha_1\alpha_2}}{\|D^{-1}\|},$$
$$\frac{2q^{r+1}\|D^{-1}\|\|C\|}{1-q}\|P_{11}B + C^*P_{22}\| < \lambda_{\min}(Q_1), \tag{29}$$
$$\lambda_{\max}(F_r(1)) + \frac{2q^{r+1}\|D^{-1}\|\|B\|}{1-q}\|P_{11}B + C^*P_{22}\| < \gamma_2.$$

hold. Then the linear system (1) is exponentially stable.

Proof of this theorem is analogue to the proof of the previous one. We do not provide details.

V. Examples and Comparison of Results

A. Exponential ℓ^2-stability of a countable system

Let $b := (b_1, \dots, b_k, \dots) \in \ell^2$, $c := (c_1, \dots, c_k, \dots) \in \ell^2$, $(\lambda_k)_{k=1}^{\infty} \subset \mathbb{R}$,

$$0 < \lambda_1 < \dots < \lambda_k < \dots, \quad \lambda_k \to +\infty, \quad k \to \infty.$$

We denote $y := (y_1, \dots, y_k, \dots) \in \ell^2$, and a linear operator $D := \operatorname{diag}(-\lambda_1, \dots, -\lambda_k, \dots)$ with the domain

$$\mathscr{D}(D) = \{y \in \ell^2 \, : \, \sum_{k=1}^{\infty} \lambda_k^2 y_k^2 < +\infty\}$$

which acts according to the rule

$$Dy = -(\lambda_1 y_1, \dots, \lambda_k y_k, \dots).$$

We present the countable linear system in an abstract form as a coupled system on the product of spaces $\mathbb{R} \times \ell^2$:

$$\begin{aligned} \dot{x} &= ax + \langle b, y\rangle_{\ell^2}, \quad x(0) = x_0 \in \mathbb{R}, \\ \dot{y} &= cx + Dy, \quad y(0) = y_0 \in \mathscr{D}(D). \end{aligned} \tag{30}$$

We verify the conditions of Assumption 1. The linear operator D is densely defined since its domain $\mathscr{D}(D)$ contains a set of finite sequences that is dense in ℓ^2. Let $P_{22} = \mathrm{id}_{\ell^2}$. Then, for all $y \in \mathscr{D}(D)$,

$$\langle Dy, P_{22}y\rangle_{\ell^2} + \langle y, P_{22}Dy\rangle_{\ell^2} = -2\sum_{k=1}^{\infty} \lambda_k y_k^2 \le -2\lambda_1 \|y\|_{\ell^2}^2. \tag{31}$$

Thus, the conditions of Assumption 1 are fulfilled and $\gamma_2 = 2\lambda_1$.

Based on the VLF method, from (10), we obtain the conditions of exponential stability

$$a < 0, \quad \|b\|_{\ell^2}\|c\|_{\ell^2} < |a|\lambda_1. \tag{32}$$

Theorem 1 is applied to investigate the exponential stability of the linear system (30). We define

$$\Delta_1 := (2\langle b,c\rangle_{\ell^2} - (\lambda_1 - |a|)^2)^2 - 4\|b\|_{\ell^2}^2\|c\|_{\ell^2}^2,$$
$$\Delta_2 = \langle b,c\rangle_{\ell^2}^2 - \|b\|_{\ell^2}^2\|c\|_{\ell^2}^2 + \lambda_1^2(\lambda_1 - |a|)^2.$$

Corollary 1: Let the linear system (30) be such that the following inequalities hold

$$|a| < \lambda_1, \quad a + \sum_{k=1}^{\infty} \frac{c_k b_k}{\lambda_k - a} < 0, \tag{33}$$

$$\langle b,c\rangle_{\ell^2} + \|b\|_{\ell^2}\|c\|_{\ell^2} < \frac{1}{2}(\lambda_1 - |a|)^2, \tag{34}$$

$$\|b\|_{\ell^2}^2\|c\|_{\ell^2}^2 - \langle b,c\rangle_{\ell^2}^2 < \lambda_1^2(\lambda_1 - |a|)^2. \tag{35}$$

We denote

$$\sigma^{\pm} := \frac{\|b\|_{\ell^2}^2}{-\langle b,c\rangle_{\ell^2} \pm \sqrt{\Delta_2}},$$
$$\psi^{\pm} := \frac{-2\langle b,c\rangle_{\ell^2} + (\lambda_1 - |a|)^2 \pm \sqrt{\Delta_1}}{2\|c\|_{\ell^2}^2}$$

Then, when $\|b\|_{\ell^2}\|c\|_{\ell^2} \le \lambda_1(\lambda_1 - |a|)$, we have the inequality

$$\max\Big\{\sigma^+, \psi^-\Big\} < \min\Big\{-\frac{a + \sum_{k=1}^{\infty} \frac{c_k b_k}{\lambda_k - a}}{\sum_{k=1}^{\infty} \frac{c_k^2}{\lambda_k - a}}, \psi^+\Big\} \tag{36}$$

and in the case when $\|b\|_{\ell^2}\|c\|_{\ell^2} > \lambda_1(\lambda_1 - |a|)$, $\langle b,c\rangle_{\ell^2} < 0$, we have the inequality

$$\max\Big\{\sigma^+, \psi^-\Big\} < \min\Big\{-\frac{a + \sum_{k=1}^{\infty} \frac{c_k b_k}{\lambda_k - a}}{\sum_{k=1}^{\infty} \frac{c_k^2}{\lambda_k - a}}, \psi^+, \sigma^-\Big\}, \tag{37}$$

which guarantee the exponential ℓ^2-stability of the linear system (30).

We consider a numerical example choosing $a = 0.25$, $\lambda_k = k^2$, $b := \tau\widehat{b}$, $c := \tau\widehat{c}$, coefficient of amplification of couples between subsystems $\tau > 0$,

$$\widehat{b} := \big(1, \dots, \frac{1}{k}, \dots\big) \in \ell^2, \quad \widehat{c} := -\big(1, \dots, \frac{1}{k^{3/2}}, \dots\big) \in \ell^2.$$

For these parameters, $\|\widehat{b}\|_{\ell^2} = 1.2825$, $\|\widehat{c}\|_{\ell^2} = 1.0964$, $\langle\widehat{b},\widehat{c}\rangle_{\ell^2} = -1.3415$, $\sum_{k=1}^{\infty} \frac{\widehat{c}_k\widehat{b}_k}{k^2 - a} = -1.3912$, $\sum_{k=1}^{\infty} \frac{\widehat{c}_k^2}{k^2 - a} = 1.3725$. Corollary 1 (Theorem 1) guarantees exponential ℓ^2-stability of the linear system (30) for the values of the gain coefficient $\tau \in (0.5728, 0.81)$. Since $a > 0$, the first subsystem is unstable, so the VLF method is not applicable in this case.

For the parameter value $a = -0.25$ and the same λ_k, b, c, we find $\sum_{k=1}^{\infty} \frac{\widehat{c}_k\widehat{b}_k}{k^2 - a} = -0.8519$, $\sum_{k=1}^{\infty} \frac{\widehat{c}_k^2}{k^2 - a} = 0.835$. Corollary 1 (Theorem 1) guarantees the exponential ℓ^2-stability of the linear system (30) for the values of the amplification factor $\tau \in (0, 0.9572)$. Since $a < 0$, the first subsystem is exponentially stable, so it is possible to use the VLF method and the conditions of the exponential stability (10). These conditions lead to gain values in the interval $(0, 0.4216)$.

We denote

$$\theta := -\frac{1 + 2\sum\limits_{k=1}^{\infty} \frac{c_k^2}{\lambda_k - a}}{2\left(a + \sum\limits_{k=1}^{\infty} \frac{c_k b_k}{\lambda_k - a}\right)}.$$

Application of Theorem 2 leads to the following sufficient conditions for exponential stability (30).

Corollary 2: Let the linear system (30) be such that the following inequalities hold

$$\begin{aligned} &|a| < \lambda_1, \quad a + \sum_{k=1}^{\infty} \frac{c_k b_k}{\lambda_k - a} < 0, \\ &\|\theta^{1/2} b + \theta^{-1/2} c\|_{\ell^2} < \lambda_1 - |a|, \\ &\|b\|_{\ell^2} \|\theta b + c\|_{\ell^2} < \lambda_1(\lambda_1 - |a|). \end{aligned} \tag{38}$$

Then, the linear system (30) is exponentially ℓ^2-stable.

VI. Conclusion

The main result of the report is the explicit formula (16) for determining the off-diagonal elements of the MLF and the operator equations (20) (or (21)) that define the diagonal element corresponding to a possibly unstable subsystem. These results develop the results of [16] for a wider class of infinite-dimensional dynamical systems.

The example given in section 5 show that the proposed method of constructing Lyapunov functions is applicable in the case when one of the subsystems is unstable, and therefore the known methods of studying the stability of coupled systems based on VLF or small-gain theorems are not applicable. The example also show that for the case when both subsystems are exponentially stable and both approaches are applicable, our approach leads to less conservative stability conditions. A significant advantage of the proposed approach is its simplicity, because in fact the problem of the stability of a coupled system is reduced to solving an operator equation which contains only bounded operators and is reduced to a matrix equation in the case $\dim H_1 < \infty$. Therefore, our proposed approach significantly expands the capabilities of the Lyapunov function method for a wide class of infinite-dimensional systems. A certain limitation of our approach is the separation of spectrum condition (12). In further research, it is planned to weaken this condition or abandon it. A possible alternative to the condition (12) is the assumption of complete continuity of the linear operators A and D^{-1} which is natural in the case when the second subsystem is parabolic. In this case, it becomes possible to use the Ries—Schauder theory of completely continuous operators in Hilbert space. Another direction of generalization of the obtained results is the consideration of linear systems in Banach spaces which allows to study L^p-stability.

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