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Finite-dimensional adaptive observer design for reaction-diffusion system

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***Abstract*— A new finite dimensional adaptive observer is proposed for a class of linear parabolic systems. The observer is based on the modal decomposition approach and uses a classical persistent excitation condition to ensure exponential convergence of both states and parameter estimation errors to zero.**

Keywords : Reaction diffusion systems, Adaptive observer design, Time-delay systems.

I. INTRODUCTION

Adaptive state observers are used to deal with online state and parameter estimation. For finite-dimensional systems, various adaptive observers for several classes of systems have been proposed in the literature. The most important works in this area can be founded in [1], [2], [3] and references therein. These results were extended to sampled-data and delayed cases in [4], [5] ,[6] and [7]. The problem of adaptive observers design for. distributed parameter systems (DPSs) becomes a hot topic especially since the last decade. Several adaptive observers design techniques have been developed including the infinite-dimensional Luenberger observer for linear DPSs, the boundary observer, backstepping-based boundary observers for both parabolic and hyperbolic DPSs, (e.g. [8] [9] [10] and [11]) . The common feature of observers developed for DPSs is that they are governed by partially differential equations (PDEs). This fact implies that their implementation employs space discretization methods which may become computationally very hard. The aim of this contribution, is to propose more simpler adaptive observers for heat equation than those existing in the literature. More precisely, we will use the modal decomposition approach proposed in [12] and [13] to construct a new finite-dimensional adaptive observer which is described only by a finite number of ordinary differential equations (ODEs). The fact that an observer is based only on ODEs, simplifies greatly the implementation since we don't need to use the finite element method (FEM) technique which is employed for simulation of observers governed by PDEs. We also show that this result can be easily extended to delayed measurements case.

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II. SYSTEM DESCRIPTION AND ASSUMPTIONS

Consider the class of reaction –diffusion systems :

$$u_t = u_{xx} + qu \quad for\ t > 0,\ x \in (0,1) \tag{1}$$

with

$$u_x(0) = u(1) = 0 \tag{2}$$

under the measurement

$$y(t) = u(0,t) + \phi(t)\theta \tag{3}$$

where ϕ is a known uniformly bounded function and θ is a vector of unknown parameters. The term $\phi(t)\theta$ models either sensors uncertainties or faults to be detected and isolated. This uncertain term induces a difference between $u(0,t)$ and the available measure $y(t)$. The role of the adaptive observer is to provide an accurate estimation of both unmeasurable state $u(x,t)$ and the unknown vector of parameters θ. The well-known regular Sturm-Liouville eigenvalue problem $\psi''(x) + \lambda\psi(x) = 0, \quad x \in [0,1]$ with $\psi(1) = \psi'(0) = 0$, generates an increasing sequence of eigenvalues $\lambda_n = \frac{\pi^2}{4}(2n-1)^2 \quad n \geq 1$ with corresponding eigenfunctions $\psi_n(x) = \sqrt{2}cos(\sqrt{\lambda_n}x)$ for $n \geq 1$. The eigenfunctions ψ_n form an orthonormal basis of $L^2(0,1)$. Consequently the solutions of the heat equation (1) can be presented as

$$u(x,t) = \sum_{n=1}^{\infty} z_n\psi_n \tag{4}$$

where z_n are solutions of the following ODEs

$$\dot{z}_n = -\lambda_n z_n + qz_n \quad n = 1,2,\ldots. \tag{5}$$

The output y can also be expressed as follows

$$y(t) = \sqrt{2}\sum_{n=1}^{\infty} z_n(t) + \phi(t)\theta \tag{6}$$

Since λ_n is increasing sequence then we can define an integer N as the smallest integer n for which the following holds: $\lambda_n + q < 0, \forall n > N$. We assume additionally that $q \neq \lambda_N$. These assumptions imply that the modes z_n $\forall n > N$ are exponentially stable.

III. Adaptive observer for Reaction-diffusion system

A. Adaptive observer structure

Following [13], we propose the following finite-dimensional adaptive observer structure where :

$$\begin{cases} \hat{u}(x,t) = \sum_{n=1}^{N} \hat{z}_n(t)\psi_n(x) \\ \dot{\hat{z}}_n(t) = -\mu_n \hat{z}_n - l_n(\hat{y}-y) + v_1 \quad n = 1,\ldots,N \\ \hat{y} = \sqrt{2}\sum_{n=1}^{N} \hat{z}_n(t) + \phi(t)\hat{\theta}. \end{cases} \tag{7}$$

Here $\mu_n = -\lambda_n - q$, l_n are observer gains, $\hat{\theta}$ is estimate of θ , v_1 is an additional signal that we will choose later on. Denote $A_N = diag(-\mu_1,\ldots,-\mu_N)$, for $n = 1,\ldots,N$, $C_N = (\sqrt{2},\ldots,\sqrt{2})$, $L_N = (l_1,\ldots,l_n)^T$ Since (A_N, C_N) is observable, we choose L_N such that $A_N - L_N C_N$ is Hurwitz. Consider the state estimation error $\tilde{Z}_N = (\tilde{z}_1,\ldots,\tilde{z}_N)^T$ where $\tilde{z}_i = \hat{z}_i - z_i$, and the estimation parameter error $\tilde{\theta} = \hat{\theta} - \theta$. Then the observation error system is expresses as follows :

$$\dot{\tilde{Z}}_N(t) = (A_N - L_N C_N)\tilde{Z}_N - L_N\phi(t)\tilde{\theta}(t) + L_N\sqrt{2}\zeta(t) + v_1 \tag{8}$$

where

$$\zeta(t) = \sum_{i=N+1}^{\infty} z_i(t). \tag{9}$$

Using [13], we can claim that

$$|\zeta(t)|^2 \leq e^{-\mu_{N+1}t}||\tilde{u}_x(x,0)||^2 \tag{10}$$

with

$$\tilde{u}_x(x,0) = \sum_{n=1}^{N} \hat{z}_n(0)\psi_n'(x) - u_x(x,0) \tag{11}$$

Consider next the decoupling transformation [2]

$$\epsilon_N(t) = \tilde{Z}_N(t) - \alpha(t)\tilde{\theta}(t) \tag{12}$$

where α is the solution of an auxiliary filter which is defined as follows :

$$\begin{cases} \dot{\alpha}(t) = (A_N - L_N C_N)\alpha(t) - L_N\phi(t) \\ v_1 = \alpha(t)\dot{\hat{\theta}} \end{cases} \tag{13}$$

From this, we deduce the ODE of ϵ_N which doesn't depend on $\tilde{\theta}$.

$$\dot{\epsilon}_N(t) = (A_N - L_N C_N)\epsilon_N(t) + \sqrt{2}\ L_N\zeta(t) \tag{14}$$

We choose L_N such that $A_N - L_N C_N$ is Hurwitz.
Since (14) is input to state stable (ISS) with respect to ζ and ζ is exponentially decaying according to [13], we conclude that (14) is exponentially stable. System (13) is ISS with respect to ϕ, whereas ϕ is uniformly bounded, implying boundedness of $|\alpha(t)|$.

1) Estimation law design: From the decoupling transformation, we can propose

$$\dot{\hat{\theta}} = -R(t)(\alpha^T(t)C_N^T + \phi^T(t))(\hat{y}(t) - y(t)) \tag{15}$$

with

$$\frac{\mathrm{d}R(t)}{dt} = R(t) - R(t)(\alpha^T(t)C_N^T + \phi^T(t))(C_N\alpha^T(t) + \phi(t))R(t) \tag{16}$$

and

$$\frac{\mathrm{d}R^{-1}(t)}{dt} = -R^{-1}(t) + (\alpha^T(t)C_N^T + \phi^T(t))(C_N\alpha^T(t) + \phi(t)) \tag{17}$$

It was already proven in [1] that if $|\alpha|$ and $|\phi|$ are bounded and if the persistent excitation condition

$$\int_t^{t+T} K^T(s)K(s)ds \geq \beta_0\mathbb{I} \tag{18}$$

with

$$K^T(t) = (\alpha^T(t)C_N^T + \phi^T(t)) \tag{19}$$

holds for some positive constant β_0 , then both $R(t)$ and $R^{-1}(t)$ are positive definite matrices and there exist two positive constants β_1 and β_2 such that the following inequalities hold :

$$\beta_1\mathbb{I}_m \leq R(t) \leq \beta_2\mathbb{I}_m \tag{20}$$

and

$$\beta_1\mathbb{I}_m \leq R^{-1}(t) \leq \beta_2\mathbb{I}_m \tag{21}$$

2) Convergence analysis: The parameter estimation error is governed by the following ODEs :

$$\dot{\tilde{\theta}} = -R(t)K^T(t)K(t)\tilde{\theta}(t) - R(t)K^T(t)(C_N\epsilon_N(t) - \sqrt{2}\zeta(t)). \tag{22}$$

To prove exponential convergence of $\tilde{\theta}$, let us consider the following Lyapunov function for (23):

$$V = \tilde{\theta}^T R^{-1}(t)\tilde{\theta}. \tag{23}$$

Then after simple computations, we deduce that the time-derivative of V satisfies the following inequality

$$\dot{V}(t) = -V(t) + |C_N\epsilon_N(t) - \sqrt{2}\zeta(t)|^2 \tag{24}$$

Since both ϵ_N and ζ converge exponentially to zero , then the comparison Lemma, allows us to conclude that $\tilde{\theta}$ will also converge exponentially to zero. On the other hand, form (15), we can deduce that $|Z_N(t)|^2 \leq 2|\epsilon_N(t)|^2 + 2|\alpha(t)|^2|\tilde{\theta}(t)|^2$. Since α is bounded and $|\epsilon_N|$ converges exponentially to zero, then Z_N is also exponentially convergent. Using the Parseval's equality, we have

$$||\hat{u}(\cdot,t) - u(\cdot,t)||^2 = |Z_N(t)|^2 + \sum_{n\geq N+1} z_n^2(t). \tag{25}$$

Since $\sum_{n\geq N+1} z_n^2(t) \leq e^{-\mu_{N+1}t}||u(\cdot,0)||^2$, then we can also conclude that $||\hat{u}(.,t) - u(.,t)||$ converges exponentially to zero. We are now in position to state the following result.

Proposition 3.1: Consider system (1) and adaptive observer described by (7), (13) and (15). Then under condition (18) with K defined in (19) , the estimation errors $||\tilde{u}(.,t)||$ and $|\tilde{\theta}(t)|$ converge exponentially to zero.

IV. Extension to the case of delayed output

In this section, we extend the above results to the case of delayed output with known and bounded fast varying delay $\tau(t) \geq 0$ (without any constraints on the delay derivative). In this case

$$y(t) = u(0, t-\tau(t)) + \phi(t-\tau(t)))\theta \tag{26}$$

whereas (7) becomes

$$\begin{cases} \hat{u}(x,t) = \sum_{n=1}^{N} \hat{z}_n(t)\psi_n(x) \\ \dot{\hat{z}}_n(t) = -\mu_n \hat{z}_n - l_n(\hat{y} - y) + v_1 \quad n = 1, \dots, N \\ \hat{y} = \sqrt{2}\sum_{n=1}^{N} \hat{z}_n(t-\tau(t)) + \phi(t-\tau(t))\hat{\theta}(t) \end{cases} \tag{27}$$

Using the same decoupling transformations, we obtain :

$$\begin{cases} \dot{\alpha}(t) = A_N\alpha(t) - L_N C_N \alpha(t-\tau(t)) - L_N\phi(t-\tau(t)) \\ v_1 = \alpha(t)\dot{\hat{\theta}} + L_n C_N(\hat{\theta}(t-\tau(t)) - \hat{\theta}(t)) \end{cases} \tag{28}$$

From this, we deduce the delayed differential equation for ϵ_N which doesn't depend on $\tilde{\theta}$.

$$\dot{\epsilon}_N(t) = A_N\epsilon_N(t) - L_N C_N \epsilon_N(t-\tau(t)) + \sqrt{2}\, L_N\zeta(t-\tau(t)) \tag{29}$$

where $\zeta(t)$ is defined in (9). Inspired by [6] we propose the following adaptive law:

$$\begin{aligned} \dot{\hat{\theta}} &= -R(t)K^T(t-\tau(t))(\hat{y}(t) - y(t)) \\ &+ R(t)K^T(t-\tau(t))C_N\alpha(t-\tau(t))(\hat{\theta}(t-\tau(t)) - \hat{\theta}(t)) \end{aligned} \tag{30}$$

and$K(t)$ given by (19). The parameter estimation error is governed by the following ODE:

$$\begin{aligned} \dot{\tilde{\theta}} &= -R(t)K^T(t-\tau(t))K(t-\tau(t))\tilde{\theta}(t) \\ &- R(t)K^T(t-\tau(t))(C_N\epsilon_N(t-\tau(t)) - \sqrt{2}\zeta(t-\tau(t))) \end{aligned} \tag{31}$$

Note that, the equation (29) is identical to (22) of [13]. Then LMI (25) of Theorem 1 in [13]) gives a bound of the delay ensuring that system (29) is exponentially stable. The term $\zeta(t-\tau(t))$ remains exponentially converging as in [13], which implies the exponential convergence of $\tilde{\theta}(t)$ under persistent excitation condition (18).

A. Extension to sampled-data case

We suppose that the output

$$y(t_k) = u(0, t_k) + \phi(t_k)\theta \tag{32}$$

is available only at sampling instants t_k which constitute an increasing sequence defined as follows : $0 = t_0 < t_1 < \dots, < t_k < \dots, \lim_{t_k \to \infty} = \infty$ with $t_{k+1} - t_k \leq h$, is the maximum allowable sampling period. It is well known [14] that the sampled-data case can be reformulated as time-delay one where the delay $\tau(t) = t - t_k$ for all $t \in [t_k, t_{k+1})$. From (26),(27) and (29) we easily propose the following sampled-data finite dimensional adaptive observer :
For $t \in [t_k, t_{k+1})$

$$\begin{cases} \hat{u}(x,t) = \sum_{n=1}^{N} \hat{z}_n(t)\psi_n(x) \\ \dot{\hat{z}}_n(t) = -\mu_n\hat{z}_n - l_n(\hat{y} - y(t_k)) + v_1 \quad n = 1, \dots, N \\ \hat{y} = \sqrt{2}\sum_{n=1}^{N} \hat{z}_n(t_k) + \phi(t_k))\hat{\theta}(t) \\ \dot{\alpha}(t) = A_N\alpha(t) - L_N C_N\alpha(t_k) - L_N\phi(t_k) \\ v_1 = \alpha(t)\dot{\hat{\theta}} + L_n C_N(\hat{\theta}(t_k) - \hat{\theta}(t)) \\ \dot{\hat{\theta}} = -R(t)K^T(t-\tau(t))(\hat{y}(t) - y(t)) \\ +R(t)K^T(t-\tau(t))C_N\alpha(t_k)(\hat{\theta}(t_k) - \hat{\theta}(t))) \end{cases} \tag{33}$$

V. Example

In this section we illustrate our observer on the system governed by (1) with $q = 3$ and the output $y = u(0,t) + (2 - \cos(10t))\theta$. We choose $N = 2$ and $L = (23.2, \quad 1.1)^T$ derived from (LMI (25) of Theorem 1 in [13]) with initial condition $u_0(x) = 1$ and constant delay $\tau = 0.1$. Simulations of the estimations error and $\hat{\theta}$ are given in Figure 1 for the constant delay $\tau = 1$ and in Figure 2 for the sampled-data case with $t_{k+1} - t_k \leq 0.7$.

VI. Conclusion

In this paper, we presented a new adaptive observer for reaction diffusion equation. Our algorithm ensures good performances and is based only on a finite number of ODEs. Further results concerning other classes of PDEs are under investigation

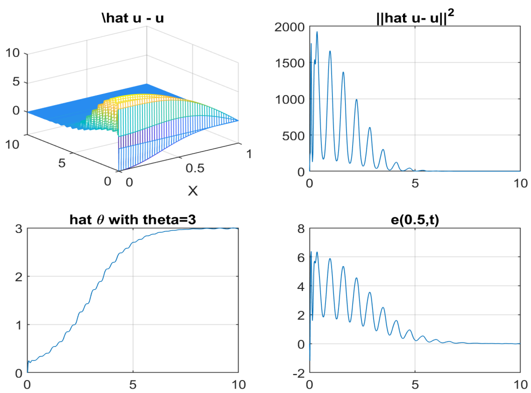

Fig. 1. Simulations with delay $\tau = 0.1$

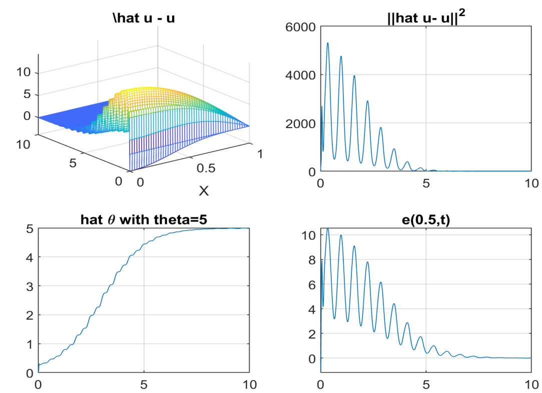

Fig. 2. Simulations of sampled-data case with $h = 0.7$

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