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A note on observability of nonlinear discrete-time systems

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***Abstract*— A concept of backward observability is introduced for nonlinear discrete-time control systems. According to the new definition the state variables can be expressed as functions of inputs, outputs and their backward shifts. It is shown that the new observability definition is more general for non-reversible systems than the one used typically in the literature. In case of reversible systems it is proved that backward observability is equivalent to the standard definition of observability. The new definition allows to enlarge the class of systems for which state variables can be estimated and the observers constructed.**

I. INTRODUCTION

Observability is a fundamental concept in control theory. For an observable system one can, in principle, construct a state observer to estimate the state variables. The latter is important, since the state feedback is widely used tool in control engineering, but all state components are rarely measurable. For discrete-time systems many notions of observability are defined, see for instance [17].

In this paper we are interested in single-experiment generic observability as defined in [9], [10]. The word 'generic' means that we are not studying observability in a neighborhood of a point, but in an open and dense subset of the state space. Nor are we interested in uniform observability [5]. Single-experiment observability refers to the case when the state variables can be expressed as functions of the system output, input and a finite number of their forward shifts. From now on the word 'observability' refers to single-experiment observability. Interestingly, almost all the papers (see, for example, [9], [10], [14]) that study observability of discrete-time systems define observability using only forward shifts of the input and output variables. This is probably due to similarity to the continuous-time case. The only exception is [1] where also the backward shifts are considered in the definition of observability. In [1] it is proved that for reversible systems, when considering both backward and forward shifts of outputs and inputs in the definition of observability, then such notion of observability is equivalent to the standard observability notion, where only forward shifts are used. However, nothing is said about non-reversible case. Non-reversible systems, however, form an important class of systems. For example, the state equations in famous Brunovsky canonical form are non-reversible.

To show that backward shifts are necessary to characterize observability for non-reversible state equations, consider the following motivating example:

$$\begin{array}{rcl} x_1(t+1) & = & u(t) \\ x_2(t+1) & = & x_3(t) \\ x_3(t+1) & = & x_1(t)+x_2(t)u(t) \\ y(t) & = & x_3(t). \end{array} \qquad (1)$$

It is shown in [10] that system (1) is not observable. This means that not all state variables can be expressed as functions of the output $y(t)$, the input $u(t)$ and their forward shifts. However, they can be expressed as functions of $y(t)$, $u(t)$ and their backward shifts. From the first two equations one gets $x_1(t) = u(t-1)$, $x_2(t) = x_3(t-1) = y(t-1)$ and from the last equation $x_3(t) = y(t)$.

Motivated by above example, in this paper a concept of backward observability is defined for discrete-time systems. We say that a discrete-time system is backward observable if its state variables are equal to functions of the system output, input and a finite number of their backward shifts. It will be described how to check the property of backward observability and how the concept is related to the standard observability definition. More precisely, it will be proved that the concept of backward observability is more general than the standard observability notion, meaning that observability yields backward observability. However, for reversible systems the two concepts of observability coincide, that is, like in [1], backward observability does not add anything compared to the standard definition. It is also shown that backward observability allows to estimate the state variables of a larger class of systems than standard observability notion and that there is more freedom in selecting an output to guarantee observability of a system.

It should be noted that backward observability has similarities with the concept of constructibility defined for linear discrete-time systems, see, for example, [11]. Just as backward observability, constructibility is related to past measurements, but the latter is focused to the recovery of the final state. The concept of constructability has been generalized to the nonlinear case, see [15], [16]. In [15] constructibility analysis was done for an observable model of a robot. In [16], however, it was proved that observability yields constructibility for nonlinear discrete-time systems. Moreover, an assumption similar to backward observability, called backward distinguishability is made in [3] to design observers for discrete-time systems.

Finally, note that like in case of the observability property, backward shifts play a critical role in characterizing the flatness property for discrete-time systems though at first flatness was defined for discrete-time systems through simply replacing time-derivatives by forward shifts. However, re-

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cently (see [4], [7]) it was shown that allowing also backward shifts of system variables leads to a more general definition.

II. PRELIMINARIES

Recall the basic concepts of the algebraic approach based on the difference field and differential 1-forms, see [2]. For simplicity we limit ourselves to single-input single-output discrete-time nonlinear systems given by their state and output equations

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ y(t) &= h(x(t)), \end{aligned} \tag{2}$$

where $x(t) \in X \subseteq \mathbb{R}^n$ is the state, $u(t) \in U \subseteq \mathbb{R}$ is the input, $y(t) \in Y \subseteq \mathbb{R}$ is the output, $f = (f_1, \dots, f_n)^T$ and h are analytic in their arguments and X, U, Y are open subsets. Assume that there exists an analytic function $\chi(x(t), u(t))$, such that

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) \\ z(t) &:= \chi(x(t), u(t)) \end{aligned} \tag{3}$$

can be globally solved for $x(t)$ and $u(t)$, i.e., $x(t) = F(x(t+1), z(t))$, $u(t) = G(x(t+1), z(t))$ for some analytic functions F and G. This is a typical assumption in discrete-time case that allows to define uniquely a backward shift operator, see below. A stronger assumption often made in discrete-time case is system reversibility.

Definition 1: System (2) is said to be reversible if the function χ can be chosen as $\chi(x(t), u(t)) = u(t)$.
Typically, reversibility is defined by the rank condition

$$\mathrm{rank}\frac{\partial f(x(t), u(t))}{\partial x(t)} = n$$

that is assumed to hold on an open and dense subset of $X \times U$. However, the rank condition does not guarantee that globally there exist analytic functions F and G as a solution of (3).

Let $\mathcal{K}$ be the field of meromorphic functions in a finite number of variables from the set $\{x_i, u^{[k]}, z^{[-l]}; i = 1, \dots, n; k \geq 0; l > 0\}$. The variable $u^{[k]}$ corresponds to $u(t+k)$, but is seen as an independent variable of the field $\mathcal{K}$, not as a function of time t. Similar interpretation is used for x_i and $z^{[-l]}$. Define on $\mathcal{K}$ the forward shift operator $\delta : \mathcal{K} \to \mathcal{K}$ as $\delta(x) = f(x, u)$, $\delta(u^{[k]}) = u^{[k+1]}$, $\delta(z^{[-l]}) = z^{[-l+1]}$ for $l \geq 2$, $\delta(z^{[-1]}) = \chi(x, u)$ and $\delta(\psi(x, u, \dots, u^{[k]}, z^{[-1]}, z^{[-2]}, \dots, z^{[-l]})) = \psi(f(x, u), u^{[1]}, \dots, u^{[k+1]}, \chi(x, u), z^{[-1]}, \dots, z^{[-l+1]})$.
The operator δ has an inverse (backward shift) operator δ^{-1} defined by $\delta^{-1}(x) = F(x, z^{[-1]})$, $\delta^{-1}(u) = G(x, z^{[-1]})$, $\delta^{-1}(z^{[-k]}) = z^{[-k-1]}$, $\delta^{-1}(u^{[k]}) = u^{[k-1]}$ for $k \geq 1$ and $\delta^{-1}(\psi(x, u, \dots, u^{[k]}, z^{[-1]}, \dots, z^{[-l]})) = \psi(F(x, z^{[-1]}), G(x, z^{[-1]}), \dots, u^{[k-1]}, z^{[-2]}, \dots, z^{[-l-1]})$.
The pair $(\mathcal{K}, \delta)$ is an inversive difference ring corresponding to the system (2). Note that the assumption that (3) can be globally solved for $x(t)$ and $u(t)$ guarantees that forward and backward shifts of functionally independent functions remain functionally independent.

Based on the field $\mathcal{K}$ one defines a vector field of 1-forms as $\mathcal{E} = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}\psi | \psi \in \mathcal{K}\}$, where d is the standard differential operator. The forward and backward shift operators can be extended to $\mathcal{E}$ naturally as

$$\begin{aligned} \delta(\sum_i a_i \mathrm{d}\psi_i) &:= \sum_i \delta(a_i)\mathrm{d}\delta(\psi_i) \\ \delta^{-1}(\sum_i a_i \mathrm{d}\psi_i) &:= \sum_i \delta^{-1}(a_i)\mathrm{d}\delta^{-1}(\psi_i). \end{aligned}$$

By abuse of notations, we use the same notation for the shift operators defined on $\mathcal{K}$ and $\mathcal{E}$.

III. OBSERVABILITY DEFINITION

There are many different notions of observability for discrete-time systems, see [17]. In this section we use two notions of observability. The first, the forward observability, corresponds to the single-experiment observability, which shows whether the state variables can be expressed as functions of the system output, input and a finite number of their forward shifts. This definition is a direct generalization of the continuous-time case and is typically used in the textbooks [12], [13] and when studying discrete-time systems, see for example [8]. The second definition of observability introduced in this section is called the backward observability. The latter corresponds to the case when the state variables can be expressed as functions of the system output, input and a finite number of their backward shifts. As the past values of the input and output variables are, in general, known, unlike their future values, the definition of backward observability is much more suitable in practical applications.

A. Forward observability

We define the forward observability through the observable space of system (2), see [10]. In order to define the observable space of (2) consider the following vector spaces of 1-forms

$$\begin{aligned} \mathcal{X} &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x\} \\ \mathcal{U}_f &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u^{[k]}; k \geq 0\} \\ \mathcal{Y}_f &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{[k]}; k \geq 0\}. \end{aligned}$$

Then the vector space $\mathcal{O}_f := \mathcal{X} \cap (\mathcal{U}_f + \mathcal{Y}_f)$ is called the forward observable space of system (2). The space $\mathcal{O}_f$ is named forward observable space, because the definition relies on forward shifts of the output and input.

Definition 2: System (2) is said to be forward observable if $\dim_{\mathcal{K}} \mathcal{O}_f = n$.
It is obvious that the condition in Definition 2 is equivalent to $\mathcal{O}_f = \mathcal{X}$. The forward observable space can be computed as follows.

Lemma 1: The forward observable space is equal to $\mathcal{O}_f = \mathrm{span}_{\mathcal{K}}\{\omega_i; i = 0, \dots, n-1\}$, where

$$\omega_i = \frac{\partial y^{[i]}}{\partial x}\mathrm{d}x. \tag{4}$$

Proof: Observe that $\mathrm{d}y^{[i]} = \omega_i + \sum_{j=0}^{i-1}\left(\partial y^{[i]}/\partial u^{[j]}\right)\mathrm{d}u^{[j]}$, $i = 0, \dots, n-1$. Because $\mathrm{d}y^{[i]}$, $i \geq n$, is linearly dependent on $\mathrm{d}y, \dots, \mathrm{d}y^{[n-1]}$ and

$\mathrm{d}u, \ldots, \mathrm{d}u^{[n-1]}$, then $\mathcal{Y}_f + \mathcal{U}_f = \mathrm{span}_{\mathcal{K}}\{\omega_i, \mathrm{d}u^{[k]}; i = 0, \ldots, n-1; k \geq 0\}$. Thus, $\mathcal{O}_f = \mathcal{X} \cap (\mathcal{Y}_f + \mathcal{U}_f) = \mathrm{span}_{\mathcal{K}}\{\omega_i; i = 0, \ldots, n-1\}$. ■

Note that the 1-forms ω_i in (4) are not, in general, linearly independent and thus do not form a basis of $\mathcal{O}_f$. Instead, the 1-forms ω_i are just the generators of $\mathcal{O}_f$. To get a basis of $\mathcal{O}_f$ one has to remove the dependent elements of ω_i.

From Lemma 1 one gets the well-known observability rank condition

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial (y, \ldots, y^{[n-1]})^T}{\partial x} = n.$$

This also means that the input-output equation of a forward observable system has order n or, in the other words, the dimension of the forward observable space corresponds to the order of its input-output equation.

Example 1. Consider the system (1) and compute the forward observable space. Since $y = x_3$, $y^{[1]} = x_1 + x_2 u$ and $y^{[2]} = u + x_3 u^{[1]}$, then by Lemma 1 the generators of $\mathcal{O}_f$ are $\omega_1 = \mathrm{d}x_3$, $\omega_2 = \mathrm{d}x_1 + u\mathrm{d}x_2$ and $\omega_3 = u^{[1]}\mathrm{d}x_3$. Thus, since ω_1 and ω_3 are linearly dependent, one has $\mathcal{O}_f = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_3, \mathrm{d}x_1 + u\mathrm{d}x_2\}$. The dimension of $\mathcal{O}_f$ is 2, which means that the system is not forward observable.

B. Backward observability

We define the backward observability in a similar manner as forward observability, except that backward shifts of y and u are used instead of forward shifts. Define

$$\begin{aligned} \mathcal{U}_b &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}u^{[-k]}; k > 0\} \\ \mathcal{Y}_b &= \mathrm{span}_{\mathcal{K}}\{\mathrm{d}y^{[-k]}; k \geq 0\}. \end{aligned}$$

Note that compared to $\mathcal{U}_f$ the space $\mathcal{U}_b$ does not contain $\mathrm{d}u$. The reason is that for system (2) output y and its backward shifts do not depend on u and thus one does not need to include $\mathrm{d}u$ into $\mathcal{U}_b$. Of course for systems, where $y = h(x, u)$, one has to include $\mathrm{d}u$ into the space $\mathcal{U}_b$. We say that the vector space $\mathcal{O}_b := \mathcal{X} \cap (\mathcal{Y}_b + \mathcal{U}_b)$ is the backward observable space of system (2).

Definition 3: System (2) is said to be backward observable if $\dim_{\mathcal{K}} \mathcal{O}_b = n$.

The backward observable space can be computed as follows.

Lemma 2: The backward observable space is equal to $\mathcal{O}_b = \mathrm{span}_{\mathcal{K}}\{\omega_{y,i}, \omega_{u,j}; i = 0, \ldots, n-1; j = 1, \ldots, n-1\}$, where

$$\omega_{y,i} = \frac{\partial y^{[-i]}}{\partial x}\mathrm{d}x \quad \omega_{u,j} = \frac{\partial u^{[-j]}}{\partial x}\mathrm{d}x. \tag{5}$$

Proof: The proof consists of two steps. First, we show that $\mathrm{d}z^{[-1]} \in \mathcal{Y}_b + \mathcal{U}_b$. Then the second step is similar to the proof of Lemma 1.

Since the forward shifts of the input u are all functionally independent, then so must be the backward shifts of u. Thus, there exists $k \geq 1$, such that $\partial u^{[-k]}/\partial z^{[-1]} \neq 0$. Let k be the minimal integer for which the latter is satisfied. Then clearly $(u^{[-k]})^{[1]} = u^{[-k+1]} = \phi(x, \chi(x, u))$ for some function $\phi \in \mathcal{K}$. Since the selection of χ is not unique, one can choose instead of $z = \chi(x, u)$ a function $\tilde{\chi}$ as $\tilde{z} = \tilde{\chi} := \phi(x, \chi(x, u))$. The latter choice guarantees that $\mathrm{d}\tilde{z}^{[-1]} = \mathrm{d}u^{[-k]} \in \mathcal{Y}_b + \mathcal{U}_b$. By definition of spaces $\mathcal{Y}_b$ and $\mathcal{U}_b$, all the backward shifts of the elements in $\mathcal{Y}_b + \mathcal{U}_b$ belong also to $\mathcal{Y}_b + \mathcal{U}_b$. Therefore, $\mathrm{d}\tilde{z}^{[-i]} \in \mathcal{Y}_b + \mathcal{U}_b$ for $i \geq 1$.

Now, one has $\mathrm{d}y^{[-i]} = \omega_{y,i} + \sum_{s=1}^{i} \left(\partial y^{[-i]}/\partial z^{[-s]}\right) \mathrm{d}z^{[-s]}$, $i = 0, \ldots, n-1$. Similarly, $\mathrm{d}u^{[-j]} = \omega_{u,j} + \sum_{s=1}^{j} \left(\partial u^{[-j]}/\partial z^{[-s]}\right) \mathrm{d}z^{[-s]}$, $j = 1, \ldots, n-1$. Because $\mathrm{d}y^{[-i]}$, $\mathrm{d}u^{[-i]}$, $i \geq n$, are linearly dependent on $\mathrm{d}y, \ldots, \mathrm{d}y^{[-n+1]}$ and $\mathrm{d}u^{[-1]}, \ldots, \mathrm{d}u^{[-n+1]}$, then $\mathcal{Y}_b + \mathcal{U}_b = \mathrm{span}_{\mathcal{K}}\{\omega_{y,i}, \omega_{u,j}, \mathrm{d}z^{[-k]}; i = 0, \ldots, n-1; j = 1, \ldots, n-1; k \geq 1\}$. Thus, $\mathcal{O}_b = \mathcal{X} \cap (\mathcal{Y}_b + \mathcal{U}_b) = \mathrm{span}_{\mathcal{K}}\{\omega_{y,i}, \omega_{u,j}; i = 0, \ldots, n-1; j = 1, \ldots, n-1\}$. ■

Like in the case of forward observable space, the vectors of 1-forms $\omega_{y,i}$ and $\omega_{u,j}$ contain, in general, linearly dependent elements. Thus, they form a set of generators for $\mathcal{O}_b$, but not a basis.

Note that unlike in case of forward observability, there is no known link between the dimension of backward observable space and the order of the input-output equation of a given system. Thus, the order of the input-output equation of a backward observable system might be less than n.

Example 2. Check whether the equations (1) are backward observable. Here we take $z = x_2$, which yields

$$\begin{aligned} y &= x_3 \\ y^{[-1]} &= x_2 \\ y^{[-2]} &= z^{[-1]} \\ u^{[-1]} &= x_1 \\ u^{[-2]} &= x_3 - x_1 z^{[-1]}. \end{aligned}$$

By Lemma 2 one clearly has $\mathcal{O}_b = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$ and thus system (1) is backward observable.

We would like to stress, that the concept of backward observability is not just something related to nonlinearities. A linear system can also be backward observable and not forward observable as shown by the following example.

Example 3. Consider a linear system described by the state equations

$$\begin{aligned} x_1^{[1]} &= x_2 + x_3 \\ x_2^{[1]} &= u \\ x_3^{[1]} &= x_1 \\ y &= x_1. \end{aligned} \tag{6}$$

The forward observable space of (6) is $\mathcal{O}_f = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}(x_2 + x_3)\}$ and thus system (6) is not forward observable. However, the backward observable space can be computed as $\mathcal{O}_b = \mathrm{span}_{\mathcal{K}}\{\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3\}$ meaning that system (6) is backward observable.

IV. PROPERTIES OF THE OBSERVABILITY NOTIONS

Next we study how the two observability notions – forward and backward observability – are related. First, we show that forward observability is just a special case of backward observability.

Lemma 3: If system (2) is forward observable then it is also backward observable.

Proof: If system (2) is forward observable, then the forward observable space has dimension n. This means that the expressions of $y, \ldots, y^{[n-1]}, u, \ldots, u^{[n-2]}$ are functionally independent. Compute the $(n-1)$-step backward shifts of the latter expressions to get

$$\begin{array}{rcl} y^{[-i]} &=& h_i(x, z^{[-1]}, \ldots, z^{[-i]}), \ \ i = n-1, \ldots, 0, \\ u^{[-j]} &=& g_j(x, z^{[-1]}, \ldots, z^{[-j]}), \ \ j = n-1, \ldots, 1. \end{array} \tag{7}$$

Since shifting functions back does not make them functionally dependent, then the functions on the right-hand side of (7) are functionally independent and one can solve the equations (7) for x and $z^{[-i]}$, $i = 1, \ldots, n-1$. Therefore, the state variable x can be written as function of $y^{[-i]}$, $i = 0, \ldots, n-1$ and $u^{[-j]}$, $j = 1, \ldots, n-1$. This means that the backward observability space $\mathcal{O}_b$ contains $\mathrm{d}x_i$, $i = 1, \ldots, n$, and thus its dimension must be equal to n. ■

Note that the opposite is not, in general, true, i.e., if system (2) is backward observable, then it is not always forward observable. This can also be seen from Examples 1 and 2. However, when system (2) is reversible, then the concepts of forward and backward observabilities coincide.

Lemma 4: If system (2) reversible, then backward observability is equivalent to forward observability.

Proof: Since by Lemma 3 forward observability always yields backward observability, it remains to show that for reversible systems backward observability yields forward observability. In the case of reversible systems one can always take $z = u$. Since backward shifts of z are independent variables in the field $\mathcal{K}$, then $\partial u^{[-k]}/\partial x \equiv 0$ for all $k \geq 1$ and, by Lemma 2, one has $\mathcal{O}_b = \mathrm{span}_{\mathcal{K}}\{\omega_{y,i}; i = 0, \ldots, n-1\}$. Because system (2) is assumed to be backward observable, then

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(y, \ldots, y^{[-n+1]})^T}{\partial x} = n$$

or in other words, $y^{[-k]}$, $k = 0, \ldots, n-1$, $u^{[-s]}$, $s = 1, \ldots, n-1$, must be functionally independent. Therefore, also $y^{[k]}$, $k = 0, \ldots, n-1$, $u^{[s]}$, $s = 0, \ldots, n-2$, must be functionally independent, because the latter expressions are obtained by forward shifting everything $n-1$ times and it is known that forward shifts of functionally independent functions remain functionally independent. Thus,

$$\mathrm{rank}_{\mathcal{K}} \frac{\partial(y, \ldots, y^{[n-1]})^T}{\partial x} = n$$

must be true, which means that system (2) is forward observable. ■

V. APPLICATIONS

In this section we show the usefulness of backward observability in observer design as well as for selecting sensor locations for observation of system states.

A. Observer design

Here we demonstrate via example that backward observability enlarges the class of systems for which an observer can be constructed. For nonlinear systems, which can be transformed either into the classical or extended observer form, observer design is a simple task [6]. In both cases forward observability is assumed to derive the conditions for the existence of such transformation and to find the necessary (parametrized) state transformation itself. In this subsection we show that a weaker assumption of backward observability could be used instead.

Consider a non-linear system in the extended observer form

$$\begin{array}{rcl} x_1^{[1]} &=& x_2 + \varphi_1(y, \ldots, y^{[-N]}, u, \ldots, u^{[-N]}) \\ &\vdots& \\ x_{n-N}^{[1]} &=& x_{n-N+1} + \varphi_{n-N}(y, \ldots, y^{[-N]}, u, \ldots, u^{[-N]}) \\ && -\psi(y^{[-N]}, u^{[-N]}) \\ x_{n-N+1}^{[1]} &=& x_{n-N+2} \\ &\vdots& \\ x_{n-1}^{[1]} &=& x_n \\ x_n^{[1]} &=& \psi(y, u) \\ y &=& x_1, \end{array} \tag{8}$$

where $0 \leq N \leq n-1$, $\psi = y$ for reversible systems and $\psi = u$ for non-reversible systems. The use of a function ψ in the latter form is to guarantee that the equations (8) satisfy the assumption made in Section II to define uniquely the backward shift operator.

An observer can be easily constructed for (8) as follows (see also [6]):

$$\begin{array}{rcl} \hat{x}_1^{[1]} &=& \hat{x}_2 + \varphi_1(y, \ldots, y^{[-N]}, u, \ldots, u^{[-N]}) \\ && +k_1(y - \hat{x}_1) \\ &\vdots& \\ \hat{x}_{n-N}^{[1]} &=& \hat{x}_{n-N+1} + \varphi_{n-N}(y, \ldots, y^{[-N]}, u, \ldots, u^{[-N]}) \\ && -\psi(y^{[-N]}, u^{[-N]}) + k_{n-N}(y - \hat{x}_1) \\ \hat{x}_{n-N+1}^{[1]} &=& \hat{x}_{n-N+2} + k_{n-N+1}(y - \hat{x}_1) \\ &\vdots& \\ \hat{x}_{n-1}^{[1]} &=& \hat{x}_n + k_{n-1}(y - \hat{x}_1) \\ \hat{x}_n^{[1]} &=& \psi(y, u) + k_n(y - \hat{x}_1), \end{array} \tag{9}$$

where the matrix $K = (k_1, \ldots, k_n)^T$ is chosen such that all eigenvalues of $A - KC$ are in the open unit disc, where $C = (1, 0, \cdots, 0)$ and

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Based on our motivating non-reversible example (1), we will show that the assumption of forward observability is actually unnecessary to take the system into the extended observer form.

Example 4. Consider the system (1). By simple inspection one can see that equations (1) can be taken into an extended observer form by a state transformation $\xi_1 = x_3$, $\xi_2 = x_1$ and $\xi_3 = x_2$:

$$\begin{aligned} \xi_1^{[1]} &= \xi_2 + y^{[-1]}u \\ \xi_2^{[1]} &= \xi_3 + u - y^{[-1]} \\ \xi_3^{[1]} &= y \\ y &= \xi_1, \end{aligned} \quad (10)$$

where $y^{[-1]} = x_2 = \xi_3$ Therefore, one is able to construct, as above, an observer for system (10) to estimate its state variables. These estimations can then be used to estimate the state variables of (1), although the system (1) is not forward observable.

Example 4 shows that backward observability enlarges the class of discrete-time systems (2) for which an observer can be constructed.

B. Sensor location

Besides being able to estimate the states for a larger class of systems, backward observability also provides more freedom in selecting which function of state coordinates to measure (if possible) to guarantee observability.

Here we address the following problem. Given the state equations

$$x(t+1) = f(x(t), u(t)), \quad (11)$$

where $x(t) \in X \subseteq \mathbb{R}^n$ is the state, $u(t) \in U \subseteq \mathbb{R}$ is the input, find an output function $y(t) = h(x(t))$, such that system (11) is backward observable with respect to the chosen output. A general solution to the latter problem is difficult to find, because the function f may have complicated structure. This is why we consider here systems of the form (11) which are static state feedback linearizable. Such systems can be taken into the form

$$\begin{aligned} \xi_1^{[1]} &= \xi_2 \\ &\vdots \\ \xi_{n-1}^{[1]} &= \xi_n \\ \xi_n^{[1]} &= g(\xi, u) \end{aligned} \quad (12)$$

by a state transformation $\xi = (\xi_1, \ldots, \xi_n)^T = \Phi(x)$. The specific structure of equations (12) allows to compute the backward shifts of system variables easily. This allows us to prove the following result.

Lemma 5: Any function $h(x) \in \mathcal{K}$ will guarantee that a static state feedback linearizable system (11) is backward observable with respect to the output $y = h(x)$.

Proof: We show that any function $H(\xi)$ will guarantee that system (12) is backward observable with respect to output $y = H(\xi)$. Then the output $y = H(\Phi(x))$ will guarantee backward observability of system (11).

For system of the form (12) one can always choose $z = \xi_1$. Then one gets that

$$\begin{aligned} \xi_1^{[-1]} &= z^{[-1]} \\ \xi_2^{[-1]} &= \xi_1 \\ &\vdots \\ \xi_{n-1}^{[-1]} &= \xi_{n-2} \\ \xi_n^{[-1]} &= \xi_{n-1} \\ u^{[-1]} &= \alpha_1(\xi, z^{[-1]}), \end{aligned} \quad (13)$$

where α_1 is obtained by shifting $\xi_n^{[1]} = g(\xi_1, \ldots, \xi_n, u)$ back once, which gives $\xi_n = g(z^{[-1]}, \xi_1, \ldots, \xi_{n-1}, u^{[-1]})$, and solving the latter for $u^{[-1]}$. Clearly one has $\partial\alpha_1/\partial\xi_n \neq 0$. Using the rules (13) compute the backward shifts of u as

$$\begin{aligned} u^{[-1]} &= \alpha_1(z^{[-1]}, \xi_1, \ldots, \xi_n) \\ u^{[-2]} &= \alpha_2(z^{[-2]}, z^{[-1]}, \xi_1, \ldots, \xi_{n-1}) \\ &\vdots \\ u^{[-n+1]} &= \alpha_{n-1}(z^{[-n+1]}, \ldots, z^{[-1]}, \xi_1, \xi_2) \\ u^{[-n]} &= \alpha_n(z^{[-n]}, \ldots, z^{[-1]}, \xi_1). \end{aligned} \quad (14)$$

Since $\partial\alpha_1/\partial\xi_n \neq 0$ and $\xi_n^{[-1]} = \xi_{n-1}$, then one must have $\partial\alpha_2/\partial\xi_{n-1} \neq 0$. Continuing in a similar way one has $\partial\alpha_k/\partial\xi_{n-k+1} \neq 0$ for $k = 1, \ldots, n$. Thus, one can solve (14) for x, which gives for $i = 1, \ldots, n$

$$x_i = \beta_i(z^{[-n]}, \ldots, z^{[-1]}, u^{[-n]}, \ldots, u^{[-1]}). \quad (15)$$

As in the proof of Lemma 2 one can show that $\mathrm{d}z^{[-j]} \in \mathcal{Y}_b + \mathcal{U}_b$, which means that (15) yields $\mathrm{d}x_i \in \mathcal{Y}_b + \mathcal{U}_b$. Thus, system (12) is backward observable with respect to the output function $y = H(x)$. ■

VI. CONCLUSIONS

A notion of backward observability was defined for discrete-time control systems and shown to be more general than the notion of forward observability usually used in the literature. Backward observability allows to design observers for a larger class of systems. Also, a larger choice of output functions will guarantee observability, meaning that there is more freedom in selecting sensor locations.

The future work will include extension of known results to the case when backward observability is assumed instead of the observability defined by the forward shifts of the output. In particular, when solving the problem of transforming discrete-time state equations into an extended observer form, usually forward observability is assumed. However, as shown in this paper a system in the extended observer form can be backward observable, but not forward observable. Thus, the solution to the problem, when weaker assumption of backward observability is considered, is missing. Also, it has been shown in [10] that the forward observable space is not always integrable, meaning that discrete-time equations cannot be always decomposed into observable and unobservable subsystems like in the continuous-time case. Our hypothesis is that backward observability may solve this issue, i.e., the backward observable space is always integrable and allows decomposition into observable and unobservable subsystems.

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