2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Learning control of second-order systems via nonlinearity cancellation

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***Abstract*— A technique to design controllers for nonlinear systems from data consists of letting the controllers learn the nonlinearities, cancel them out and stabilize the closed-loop dynamics. When control and nonlinearities are unmatched, the technique leads to an approximate cancellation and local stability results are obtained. In this paper, we show that, if the system has some structure that the designer can exploit, an iterative use of the data leads to a globally stabilizing controller even when control and nonlinearities are unmatched.**

I. Introduction

Controlling nonlinear systems without explicitly knowing the dynamics is a challenging problem that has motivated extensive studies on data-driven nonlinear control in recent years. On one hand, machine learning methods such as the kernel-based learning approach, e.g., [1] and [2], have been developed for learning and then control unknown nonlinear systems. However, the learned models are not always suitable for control design purposes. On the other hand, different approaches have been developed to directly synthesize controllers from data; for instance, the virtual reference feedback tuning (VRFT) approach in [3], and the online direct approach developed in [4]. Willems' fundamental lemma [5], which uses input-output data to characterize the responses of linear time-invariant systems, has been widely used in developing data-driven control approaches. Nonlinear dynamics can be approximated via data using Willems' fundamental lemma if the nonlinear system is linearized at a known equilibrium, as done in works such as [6] and [7], or written into a linear-like form. In general, to find the linear-like form, some prior knowledge on the nonlinear dynamics is needed. For example, many works assume a known dictionary of basis functions, such as the polynomial basis functions having a certain maximum degree used in [8], and the span of eigenfunctions of a Koopman operator used in [9]. Known basis functions are also used in [10] for obtaining a linear-like state-dependent representation, and in [11] for performing (approximate) nonlinearity cancellation. In practice, the dictionary of basis functions can be obtained from the physics of the plant. When the basis functions are not readily obtainable from prior knowledge, recent works [12] and [13] approximate the nonlinear dynamics as polynomial systems via Taylor's expansion for data-driven control and/or analysis. By choosing the kernel based on a priori knowledge on the system, [2] constructed data-driven predictor of the nonlinear system.

Depending on the control objectives and requirements, controllers have been developed using various approaches for the data-based representation/approximation of the nonlinear dynamics. In particular, data-based predictive control has been investigated in works such as [7] and [9], where rigorously proving the stability of the controlled system can be difficult. The sum-of-square technique is used in [8], [12], [14] with the Lyapunov method for stabilization, and in [15] with control barrier certificates for safety control. A convex-concave procedure was developed in [16] for optimal control of bilinear systems. A learning control approach via nonlinearity cancellation was proposed in [11] where the control input is designed such that the nonlinearities in the dynamics are (approximately) cancelled out and the remaining linear part is stable. When the basis functions are explicitly known and the control input matches the nonlinearities, the approach in [11] achieves exact cancellation and renders the equilibrium globally asymptotically stable. Otherwise, only approximate cancellation can be achieved, which results in a *locally* asymptotically stable equilibrium under some assumptions on the remaining nonlinearity.

Contributions. This work is inspired by the nonlinearity cancellation based learning control method in [11]. We study a class of second-order systems whose basis functions are explicitly known but the control input only directly affects one of the subsystems. The learning control objective is to render the known equilibrium *globally* asymptotically stable, which cannot be achieved by directly applying the result of [11] to the second-order system without exploiting its structure. To deal with the nonlinearity in each subsystem, we design a virtual input using data for the subsystem which the control input does not directly affect. Then, the difference between the virtual input and the actual input is described explicitly via data as the error dynamics. Finally, the control input is designed to make the equilibrium globally asymptotically stable for the overall system. In summary, by developing a virtual input and applying the idea of nonlinearity cancellation to each subsystem, the nonlinearities in both subsystems are handled. This work demonstrates that by exploiting the structure of the second-order system, the nonlinearity cancellation based learning control approach can be applied to nonlinear systems where the control input and the nonlinearities are unmatched. We note that this design can be viewed as the backstepping technique, which indicates that the proposed approach can be extended to higher order systems by applying it to each subsystem recursively.

* This publication is also part of the project Digital Twin with project number P18-03 of the research programme TTW Perspective which is (partly) financed by the Dutch Research Council (NWO).

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The rest of the paper is arranged as follows. In Section II, the learning control problem studied in this work is formulated, and the learning control approach via nonlinearity cancellation is reviewed. The proposed data-driven controller and its design procedure are presented in Section III. Section IV demonstrate an application of the proposed controller to the tunnel diode circuit. Finally, some conclusive remarks are given in Section V.

Notation. Throughout the paper, $A \succ (\succeq) 0$ denotes that matrix A is positive (semi-)definite, and $A \prec (\preceq) 0$ denotes that matrix A is negative (semi-)definite. $\|\cdot\|$ denotes the Euclidean norm.

II. Problem formulation and Preliminaries

In this section, we formulate the learning control problem of a class of second-order nonlinear systems, and review the data-driven control approach via nonlinearity cancellation proposed in [11].

A. Problem formulation

Consider the second-order nonlinear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + \beta_1 x_2 \\ \dot{x}_2 &= f_2(x_1, x_2) + \beta_2 u \end{aligned} \tag{1}$$

where $x_1, x_2 \in \mathbb{R}$ are the states, $u \in \mathbb{R}$ is the input, and unknown $\beta_1, \beta_2 \in \mathbb{R}$ are nonzero constants. Suppose that $f_1(0) = 0$, $f_2(0,0) = 0$, and f_1 and f_2 are continuously differentiable. We also assume that the functions f_1 and f_2 can be written as linear combinations of *known* basis functions, that is

$$f_1(x_1) = \alpha_1 \begin{bmatrix} x_1 \\ Q_1(x_1) \end{bmatrix}, \; f_2(x_1, x_2) = \alpha_2 \begin{bmatrix} x_1 \\ x_2 \\ Q_2(x_1, x_2) \end{bmatrix}$$

where $\alpha_1 \in \mathbb{R}^{1\times(1+q_1)}$ and $\alpha_2 \in \mathbb{R}^{1\times(2+q_2)}$ are unknown constant vectors, $Q_1 : \mathbb{R} \to \mathbb{R}^{q_1}$ and $Q_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{q_2}$ are continuously differentiable functions containing nonlinearities in x_1 and (x_1, x_2), respectively. Assume that $Q_1(0) = 0$. Define

$$Z_1(x_1) = \begin{bmatrix} x_1 \\ Q_1(x_1) \end{bmatrix}, \quad Z_2(x_1, x_2) = \begin{bmatrix} x_1 \\ x_2 \\ Q_2(x_1, x_2) \end{bmatrix}.$$

The system (1) can then be written into the linear-like form

$$\begin{aligned} \dot{x}_1 &= \alpha_1 Z_1(x_1) + \beta_1 x_2 \\ \dot{x}_2 &= \alpha_2 Z_2(x_1, x_2) + \beta_2 u. \end{aligned} \tag{2}$$

The objective is to use input-state data to stabilize the origin for the closed-loop system when α_i, β_i, $i = 1, 2$, are unknown. Note that if the known equilibrium is not at the origin, a coordinate transformation can be performed such that the origin is an equilibrium for the new coordinate.

Remark 1 (Second-order systems): The system (1) is an important class of dynamics for the study of nonlinear systems. Many classic nonlinear dynamical systems are in the form of (1), such as the tunnel diode circuits and the van der Pol oscillators. In this work, the structure of (1) is essential for the proposed learning control approach. By exploiting the structure, we introduce and demonstrate a learning control approach based on nonlinearity cancellation. The result can be extended to more generic nonlinear systems, such as high-order lower-triangular nonlinear systems. ■

Remark 2 (Known basis functions): To better depict the main idea of the proposed approach, this work considers a simple case where the types of functions in f_1 and f_2 are explicitly known, so that exact nonlinearity cancellation is achievable. It is not unreasonable to assume known basis functions, as in many applications such as the control of mechanical systems, basis functions can be obtained via the physics of the system. In the case where the basis functions are not explicitly known, one would resort to analyzing the impact of the neglected nonlinearities as proposed in [11], which is an interesting topic to be further investigated. ■

We denote the data sampled in an offline experiment as $\mathcal{DS} := \{u(t_k); x_1(t_k); x_2(t_k); \dot{x}_1(t_k); \dot{x}_2(t_k)\}_{k=0}^{T-1}$. Arrange the collected data and obtain the following matrices

$$\begin{aligned} X_{10} &= \begin{bmatrix} x_1(t_0) & \cdots & x_1(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{1\times T} \\ X_{20} &= \begin{bmatrix} x_2(t_0) & \cdots & x_2(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{1\times T} \\ X_{11} &= \begin{bmatrix} \dot{x}_1(t_0) & \cdots & \dot{x}_1(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{1\times T} \\ X_{21} &= \begin{bmatrix} \dot{x}_2(t_0) & \cdots & \dot{x}_2(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{1\times T} \\ Z_{10} &= \begin{bmatrix} x_1(t_0) & \cdots & x_1(t_{T-1}) \\ Q_1(x_1(t_0)) & \cdots & Q_1(x_1(t_{T-1})) \end{bmatrix} \in \mathbb{R}^{(1+q_1)\times T} \\ Z_{20} &= \begin{bmatrix} x_1(t_0) & \cdots & x_1(t_{T-1}) \\ x_2(t_0) & \cdots & x_2(t_{T-1}) \\ Q_2(x_1(t_0), x_2(t_0)) & \cdots & Q_2(x_1(t_{T-1}), x_2(t_{T-1})) \end{bmatrix} \\ &\quad \in \mathbb{R}^{(2+q_2)\times T} \\ U_0 &= \begin{bmatrix} u(t_0) & \cdots & u(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{1\times T}. \end{aligned} \tag{3}$$

By the system dynamics, the arranged data satisfies that

$$X_{11} = \alpha_1 Z_{10} + \beta_1 X_{20} = \begin{bmatrix} \beta_1 & \alpha_1 \end{bmatrix} \begin{bmatrix} X_{20} \\ Z_{10} \end{bmatrix}, \tag{4}$$

$$X_{21} = \alpha_2 Z_{20} + \beta_2 U_0 = \begin{bmatrix} \beta_2 & \alpha_2 \end{bmatrix} \begin{bmatrix} U_0 \\ Z_{20} \end{bmatrix}. \tag{5}$$

The problem studied in this work is to design a feedback controller $u = F(x_1, x_2)$ for the system (1) with known functions Z_1 and Z_2 using the data set $\mathcal{DS}$, such that the origin is a *globally* asymptotically stable equilibrium for the closed-loop dynamics.

Remark 3 (Identification of the dynamics): Under the assumption that $\begin{bmatrix} X_{20} \\ Z_{10} \end{bmatrix}$ and $\begin{bmatrix} U_0 \\ Z_{20} \end{bmatrix}$ have full row rank, the system parameters α_i and β_i, $i = 1, 2$, can be identified from (4) and (5), and controllers can be designed based on the identified model. These rank conditions ensure that the data is rich enough for the parameter identification. We will show a similar data richness requirement after presenting the main result of this work. ■

Remark 4 (Full state and state derivative measurement): It is assumed that the state is fully measurable, for a different approach has to be applied otherwise, which is out of the scope of this work. We note that the full measurement assumption can be restrictive in applications, and it is of importance to relax it in our future investigations. The state derivative can be well-approximated using methods such as the numerical differentiation with an approximation error, which is proportional to the sampling time. ■

B. Data-driven control via (approximate) nonlinearity cancellation

This work is inspired by the idea of data-driven control via (approximate) nonlinearity cancellation developed in [11]. For the completeness of this paper, we summarized the approach therein in what follows.

Consider the nonlinear system

$$\dot{x} = f(x) + Bu \tag{6}$$

which can be written as

$$\dot{x} = AZ(x) + Bu \tag{7}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input. The matrices $A \in \mathbb{R}^{n\times S}$ and $B \in \mathbb{R}^{n\times m}$ are unknown. Assume that $Z : \mathbb{R}^n \to \mathbb{R}^S$ is a known continuous function that contains at least all functions in the nonlinear dynamics $f(x)$. In particular, Z is supposed to take the form of

$$Z(x) = \begin{bmatrix} x \\ Q(x) \end{bmatrix} \tag{8}$$

where $Q : \mathbb{R}^n \to \mathbb{R}^{S-n}$ contains the nonlinear functions in Z.

Assume that both x and u are fully measured. The data set sampled in an experiment is denoted as $\mathcal{D} := \{x(t_k), \dot{x}(t_k), u(t_k)\}_{k=0}^{T-1}$ for some integer $T > 0$. Write the data matrices as

$$\begin{aligned} X_1 &= \begin{bmatrix} \dot{x}(t_0) & \cdots & \dot{x}(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{n\times T}, \\ U_0 &= \begin{bmatrix} u(t_0) & \cdots & u(t_{T-1}) \end{bmatrix} \in \mathbb{R}^{m\times T}, \\ Z_0 &= \begin{bmatrix} x(t_0) & \cdots & x(t_{T-1}) \\ Q(x(t_0)) & \cdots & Q(x(t_{T-1})) \end{bmatrix} \in \mathbb{R}^{S\times T}, \end{aligned}$$

which satisfy the relation

$$X_1 = AZ_0 + BU_0. \tag{9}$$

The following lemma is the continuous-time counterpart of [11, Lemma 1], which gives the data-based closed-loop representation of system (6).

Lemma 1: Consider any matrices $K \in \mathbb{R}^{m\times S}$ and $G \in \mathbb{R}^{T\times S}$ such that

$$\begin{bmatrix} K \\ I_S \end{bmatrix} = \begin{bmatrix} U_0 \\ Z_o \end{bmatrix} G \tag{10}$$

where $G = \begin{bmatrix} G_1 & G_2 \end{bmatrix}$, $G_1 \in \mathbb{R}^{T\times n}$ and $G_2 \in \mathbb{R}^{T\times(S-n)}$. Then, the controller designed as $u = KZ(x)$ leads to the closed-loop dynamics

$$\dot{x} = Mx + NQ(x) \tag{11}$$

where $M := X_1G_1$ and $N := X_1G_2$. ■

Proof: With the designed control input $u = KZ(x)$, the closed-loop dynamics is written as

$$\begin{aligned} \dot{x} &= AZ(x) + BKZ(x) \\ &= \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I_S \end{bmatrix} Z(x) \\ &\overset{(10)}{=} \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ Z_o \end{bmatrix} \begin{bmatrix} G_1 & G_2 \end{bmatrix} \begin{bmatrix} x \\ Q(x) \end{bmatrix} \\ &\overset{(9)}{=} X_1G_1x + X_1G_2Q(x), \end{aligned}$$

which gives (11). □

Lemma 1 establishes a data-based representation of the closed-loop dynamics (6) composed of a linear part Mx and a nonlinear part $NQ(x)$. When the function Z is explicitly known, a semi-definite program (SDP) has been derived in [11], whose solution leads to a control gain K that cancels out the nonlinearity and renders the origin globally exponentially stable for the closed-loop system. The result for the continuous-time system (6) is presented as follows.

Theorem 1: [11, Section V.A] Consider the nonlinear system (6). With decision variables $P_1 \in \mathbb{R}^{n\times n}$, $P_1 \succ 0$, $Y_1 \in \mathbb{R}^{T\times n}$, and $G_2 \in \mathbb{R}^{T\times(s-n)}$, if the following SDP

$$Z_0Y_1 = \begin{bmatrix} P_1 \\ 0_{(S-n)\times n} \end{bmatrix}, \tag{12a}$$

$$Z_0G_2 = \begin{bmatrix} 0_{n\times(S-n)} \\ I_{S-n} \end{bmatrix}, \tag{12b}$$

$$X_1G_2 = 0_{(S-n)\times n}, \tag{12c}$$

$$X_1Y_1 + (X_1Y_1)^\top \prec 0, \tag{12d}$$

is feasible, then the controller $u = KZ(x)$ with $K = U_0 \begin{bmatrix} Y_1P_1^{-1} & G_2 \end{bmatrix}$ linearizes the closed-loop dynamics, and renders the origin globally exponentially stable. ■

The essential idea of Theorem 1 is to find a matrix G that cancels out the closed-loop nonlinearity $X_1G_2Q(x)$ and makes the time derivative of the Lyapunov function $V(x) = x^\top P_1^{-1}x$ negative for all $x \neq 0$, such that the origin is globally exponentially stable. However, there are cases where (12c) is infeasible, for which [11] proposed to minimize the nonlinearity, i.e., $\|X_1G_2\|$, and obtained local stability results. The system (1) is one of those cases, since the control input u only directly affects the x_2-subsystem, and cannot cancel out the nonlinearity in the x_1-subsystem. Hence, directly applying Theorem 1 to the system (1) leads to a *locally* stable origin, and thus cannot solve the problem formulated in the previous subsection.

III. Stabilization of Second-Order Systems via Nonlinearity Cancellation

To render the origin *globally* asymptotically stable, the nonlinearity in the x_1-subsystem also needs to be handled, despite the fact that it is not matched with the control input u. To address this problem, this work performs data-driven control via nonlinearity cancellation for each subsystem. Specifically, for the x_1-subsystem, a virtual control input is designed based on the idea of Theorem 1. Then, we establish an error subsystem that is the difference between the virtual control input and the actual input x_2 [17], [18]. Applying the nonlinearity cancellation approach again to the obtained error subsystem, the error converges to the origin globally and exponentially. Finally, an overall controller for (1) is obtained.

Specifically, for the x_1-subsystem, we present the following result that is analogous to Lemma 1.

Lemma 2: Consider any matrices $K_1 \in \mathbb{R}^{1\times(1+q_1)}$ and $G_1 \in \mathbb{R}^{T\times(1+q_1)}$ such that

$$\begin{bmatrix} K_1 \\ I_{(1+q_1)} \end{bmatrix} = \begin{bmatrix} X_{20} \\ Z_{10} \end{bmatrix} G_1. \tag{13}$$

Let G_1 be partitioned as $G_1 = \begin{bmatrix} G_{11} & G_{12} \end{bmatrix}$ where $G_{11} \in \mathbb{R}^{T\times 1}$ and $G_{12} \in \mathbb{R}^{T\times q_1}$. Then, the system

$$\dot{x}_1 = f_1(x_1) + \beta_1 x_2$$

can be written as

$$\dot{x}_1 = M_1 x_1 + N_1 Q_1(x_1) + \beta_1(x_2 - v) \tag{14}$$

where $M_1 := X_{11}G_{11}$, $N_1 := X_{11}G_{12}$, and $v := K_1 Z_1(x_1)$. ■

Proof: Let $v = K_1 Z_1(x_1)$ be the virtual input applied to the x_1-subsystem. Applying the virtual input gives the dynamics of x_1 as

$$\dot{x}_1 = \alpha_1 Z_1(x_1) + \beta_1 v + \beta_1 x_2 - \beta_1 v.$$

Consider the matrices K_1 and G_1 satisfying (13). Applying Lemma 1, we have the dynamics of the x_1-subsystem as

$$\dot{x}_1 = M_1 x_1 + N_1 Q_1(x_1) + \beta_1(x_2 - v)$$

where $M_1 = X_{11}G_{11}$ and $N_1 = X_{11}G_{12}$. □

The following result illustrates the design of K_1.

Proposition 1: Consider the system

$$\dot{x}_1 = f_1(x_1) + \beta_1 x_2 \tag{15}$$

along with $\Omega_1 > 0$ and the following program

$$Z_{10}G_1 = I_{(1+q_1)} \tag{16a}$$
$$X_{11}G_{12} = 0_{1\times q_1} \tag{16b}$$
$$X_{11}G_{11} \le -\Omega_1. \tag{16c}$$

If the program is feasible, then $x_2 = K_1 Z_1(x_1)$ with $K_1 = X_{20}G_1$ renders the origin of (15) a globally exponentially stable equilibrium. ■

Proof: Recall that by Lemma 2, the system (15) can be written as

$$\dot{x}_1 = X_{11}G_{11}x_1 + X_{11}G_{12}Q_1(x_1) + \beta_1(x_2 - v).$$

When $x_2 = v = K_1 Z_1(x_1)$, the above equation becomes

$$\dot{x}_1 = X_{11}G_{11}x_1 + X_{11}G_{12}Q_1(x_1). \tag{17}$$

By the condition (16b), the nonlinear term $X_{11}G_{12}Q_1(x_1)$ is cancelled out, leaving the closed-loop dynamics of the x_1-subsystem as

$$\dot{x}_1 = X_{11}G_{11}x_1 \tag{18}$$

where $X_{11}G_{11}$ is Hurwitz by the condition (16c). Hence, the origin is a globally exponentially stable equilibrium of (15). □

Define the difference between the virtual input v and the actual input x_2 as $\delta := x_2 - v$. By the designed $v = K_1 Z_1(x_1)$, one can write its dynamics as

$$\begin{aligned}\dot{\delta} &= \dot{x}_2 - \dot{v} \\ &= \alpha_2 Z_2(x_1, \delta + K_1 Z_1(x_1)) + \beta_2 u - K_1 \frac{\partial Z_1}{\partial x_1}\dot{x}_1.\end{aligned}$$

Define function $\overline{Q}_2 : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{\overline{q}_2}$ as the vector containing all types of nonlinear functions in $Q_2(x_1, \delta + K_1 Z_1(x_1))$, $Z_1(x_1)$, and $\frac{\partial Z_1}{\partial x_1}\dot{x}_1$. Letting $\widetilde{Q}_2(x_1,\delta) := \begin{bmatrix} x_1^\top & \overline{Q}_2(x_1,\delta)^\top \end{bmatrix}^\top$ and

$$\overline{Z}_2(x_1,\delta) = \begin{bmatrix} \delta \\ \widetilde{Q}_2(x_1,\delta) \end{bmatrix} \tag{19}$$

gives the δ-subsystem as

$$\dot{\delta} = \overline{\alpha}_2 \overline{Z}_2(x_1,\delta) + \beta_2 u \tag{20}$$

where $\overline{\alpha}_2$ is the unknown vector satisfying

$$\overline{\alpha}_2 \overline{Z}_2(x_1,\delta) = \alpha_2 Z_2(x_1, \delta + K_1 Z_1(x_1)) - K_1 \frac{\partial Z_1}{\partial x_1}\dot{x}_1.$$

We note that the function $\overline{Q}_2$ is known because $Q_2(x_1,x_2)$ is known, and $\dot{x}_1$ can be written explicitly after designing v. Moreover, as the function $\overline{Z}_2$ is known, one can evaluate it at the data points in $\mathcal{DS}$ to obtain the data matrix $\overline{Z}_{20}$. The data of $\dot{\delta}$, denoted as Δ_1, can be obtained using X_{11} and X_{21} based on the relation $\dot{\delta} = \dot{x}_2 - K_1\frac{\partial Z_1}{\partial x_1}\dot{x}_1$.

Remark 5 (Nonlinearity in $\frac{\partial Z_1}{\partial x_1}\dot{x}_1$): Under the designed virtual input v, the nonlinearity is cancelled out, and the closed-loop dynamics of x_1 is linear and in the form of $\dot{x}_1 = M_1 x_1 + \beta_1 \delta$. Therefore, one has that

$$\dot{v} = K_1 \frac{\partial Z_1}{\partial x_1}\dot{x}_1 = M_1 K_1 \frac{\partial Z_1}{\partial x_1} x_1 + \beta_1 K_1 \frac{\partial Z_1}{\partial x_1}\delta,$$

where the nonlinearity is contained in $\frac{\partial Z_1}{\partial x_1}x_1$ and $\frac{\partial Z_1}{\partial x_1}\delta$, which is available as $Z_1(x_1)$ is known. ■

For the δ-subsystem, results similar to Lemma 2 and Proposition 1 can be established.

Lemma 3: Consider any matrices $K_2 \in \mathbb{R}^{1\times(2+\bar{q}_2)}$ and $G_2 \in \mathbb{R}^{T\times(2+\bar{q}_1)}$ such that

$$\begin{bmatrix} K_2 \\ I_{(2+q_2)} \end{bmatrix} = \begin{bmatrix} U_0 \\ \overline{Z}_{20} \end{bmatrix} G_2. \tag{21}$$

Let G_2 be partitioned as $G_2 = \begin{bmatrix} G_{21} & G_{22} \end{bmatrix}$ where $G_{21} \in \mathbb{R}^{T\times 1}$ and $G_{22} \in \mathbb{R}^{T\times(1+\bar{q}_2)}$. Then, the system

$$\dot{\delta} = \overline{\alpha}_2 \overline{Z}_2(x_1,\delta) + \beta_2 u$$

with $u = K_2 \overline{Z}_2(x_1,\delta)$ can be written as

$$\dot{\delta} = M_2\delta + N_2 \widetilde{Q}_2(x_1,\delta) \tag{22}$$

where $M_2 := \Delta_1 G_{21}$ and $N_2 := \Delta_1 G_{22}$. ■

Proposition 2: Consider the system

$$\dot{\delta} = \overline{\alpha}_2 \overline{Z}_2(x_1,\delta) + \beta_2 u \tag{23}$$

along with $\Omega_2 > 0$ and the following program

$$\overline{Z}_{20} G_2 = I_{2+\bar{q}_2} \tag{24a}$$
$$\Delta_1 G_{22} = 0_{1\times(1+\bar{q}_2)} \tag{24b}$$
$$\Delta_1 G_{21} \leq -\Omega_2 \tag{24c}$$

If the program is feasible, then $u = K_2 \overline{Z}_2(x_1, \delta)$ with $K_2 = U_0 G_2$ renders the origin of (23) a globally exponentially stable equilibrium. ■

The proof of Lemma 3 and Proposition 2 is similar to that of Lemma 2 and Proposition 1, and thus is omitted.

We outline the data-driven control design procedure in the pseudo-algorithm Algorithm 1,

Algorithm 1 Data-driven control of the second-order system (1) via nonlinearity cancellation

Step 1. Collect the data set $\mathcal{DS}$ and obtain the data matrices X_{20}, X_{11}, X_{21}, Z_{10}, and U_0
Step 2. Design data-driven virtual input $v = K_1 Z_1(x_1)$ for $\dot{x}_1 = \alpha_1 Z_1(x_1) + \beta_1 x_2$ (Lemma 2 and Proposition 1)
Step 3. Find the function $\overline{Z}_2$ that represents the dynamics of the error $\delta = x_2 - v$ as $\dot{\delta} = \overline{\alpha}\overline{Z}_2(x_1, \delta) + \beta_2 u$
Step 4. Evaluate $\overline{Z}_2$ and $\dot{\delta}$ at the data points in $\mathcal{DS}$ to obtain $\overline{Z}_{20}$ and Δ_1
Step 5. Design data-driven controller $u = K_2 \overline{Z}_2(x_1, \delta)$ (Lemma 3 and Proposition 2)

The main result of the data-driven control design is summarized as follows.

Theorem 2: Consider the nonlinear system (1) with the data set $\mathcal{DS}$. For any fixed $\Omega_1 > 0$, $\Omega_2 > 0$, and the decision variables $G_1 = \begin{bmatrix} G_{11} & G_{12} \end{bmatrix}$ with $G_{11} \in \mathbb{R}^{T\times 1}$, $G_{12} \in \mathbb{R}^{T\times q_1}$, and $G_2 = \begin{bmatrix} G_{21} & G_{22} \end{bmatrix}$ with $G_{21} \in \mathbb{R}^{T\times 1}$, $G_{22} \in \mathbb{R}^{T\times(1+\bar{q}_2)}$, if the programs (16) and (24) are feasible, then the controller

$$\begin{aligned} u &= U_0 G_2 \overline{Z}_2(x_1, \delta) \\ \delta &= x_2 - X_{20} G_1 Z_1(x_1) \end{aligned} \tag{25}$$

where $\overline{Z}_2(x_1, \delta)$ is defined in (19), renders the origin globally asymptotically stable for the closed-loop system (1). ■

Proof: Under the given assumptions, the closed-loop system (1), (25) in the (x_1, δ) coordinates is given by

$$\begin{aligned} \dot{x}_1 &= X_{11} G_{11} x_1 + \beta_1 \delta \\ \dot{\delta} &= \Delta_1 G_{21} \delta \end{aligned} \tag{26}$$

where $X_{11} G_{11} \leq -\Omega_1 < 0$ and $\Delta_1 G_{21} \leq -\Omega_2 < 0$. Then, the origin is a globally exponentially stable equilibrium for (26). Recall that $x_2 = \delta + X_{20} G_1 Z_1(x_1)$. Therefore, the origin is a globally asymptotically stable equilibrium of the closed-loop system (1) and (25). □

Remark 6 (Data richness): To have (16a) and (24a) feasible, a necessary requirement is that Z_{10} and $\overline{Z}_{20}$ have full row rank, which indicates that the data is rich enough for designing a controller. Remark 3 mentions that to identify the system parameters $\begin{bmatrix} \beta_1 & \alpha_1 \end{bmatrix}$ and $\begin{bmatrix} \beta_2 & \alpha_2 \end{bmatrix}$ via data, one needs the full row rank condition for $\begin{bmatrix} X_{20} \\ Z_{10} \end{bmatrix}$ and $\begin{bmatrix} U_0 \\ Z_{20} \end{bmatrix}$, respectively. The rank condition of Z_{10} is weaker than that of $\begin{bmatrix} X_{20} \\ Z_{10} \end{bmatrix}$ for the x_1-subsystem, but $\begin{bmatrix} X_{20} \\ Z_{10} \end{bmatrix}$ having full row rank brings certain advantages, as discussed in [11]. On the other hand, if the system is first identified using data, $\begin{bmatrix} \beta_2 & \alpha_2 \end{bmatrix}$ is calculated for the x_2-subsystem, while our proposed approach deals with $\begin{bmatrix} \beta_2 & \overline{\alpha}_2 \end{bmatrix}$ of the error dynamics δ. Hence, for general cases, it is difficult to compare the strictness of the rank conditions of $\overline{Z}_{20}$ and $\begin{bmatrix} U_0 \\ Z_{20} \end{bmatrix}$. ■

Remark 7 (Lyapunov function): A Lyapunov function can also be synthesized from the proposed design procedure. Define

$$V(x, \delta) = \frac{1}{2}\beta_1^{-2} x_1^2 + \frac{1}{2}\delta^2.$$

It holds that the time derivative of $V(x, \delta)$ along the trajectory of the closed-loop dynamics satisfies

$$\dot{V}(x, \delta) \leq -\beta_1^{-2}\Omega_1 x_1^2 - \Omega_2 \delta^2.$$

■

IV. An Example

Consider the tunnel diode circuit having the dynamics

$$\begin{aligned} \dot{x}_1 &= 0.5(-h(x_1) + x_2) \\ \dot{x}_2 &= 0.2(-x_1 - 1.5x_2 + u) \end{aligned} \tag{27}$$

where the tunnel diode current function is

$$\begin{aligned} h(x_1) = 17.76x_1 - 103.79x_1^2 + 229.62x_1^3 \\ - 226.31x_1^4 + 83.726x_1^5. \end{aligned}$$

When a nominal value of input $u = 1.2$ is applied, the system has 3 equilibria, among which $(0.063, 0.76)$ and $(0.884, 0.21)$ are stable equilibria, and $(0.286, 0.61)$ is a saddle point. The objective in this example is to design a controller such that $(0.286, 0.61)$ is globally asymptotically stable for the closed-loop system.

As the equilibrium is not at the origin, we define new variables $\bar{x}_1 = x_1 - 0.286$, $\bar{x}_2 = x_2 - 0.61$, and $\bar{u} = u - 1.2$ such that the origin is a saddle point of the dynamics

$$\begin{aligned} \dot{\bar{x}}_1 &= 0.5\left[-h(\bar{x}_1 + 0.286) + (\bar{x}_2 + 0.61)\right] \\ \dot{\bar{x}}_2 &= 0.2\left[-(\bar{x}_1 + 0.286) - 1.5(\bar{x}_2 + 0.61) + 1.2 + \bar{u}\right]. \end{aligned} \tag{28}$$

In what follows, we design the input $\bar{u}$ using $\begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}^\top$ such that the origin is globally asymptotically stable for (28).

By setting $Z_1(\bar{x}_1) = \begin{bmatrix} \bar{x}_1 & \bar{x}_1^2 & \bar{x}_1^3 & \bar{x}_1^4 & \bar{x}_1^5 \end{bmatrix}^\top$ and $Z_2(\bar{x}_1, \bar{x}_2) = \begin{bmatrix} \bar{x}_1 & \bar{x}_2 \end{bmatrix}^\top$, the dynamics (28) is in the linear-like form (2).

An experiment is conducted over the time interval $[0, 5]$ with the initial condition $\bar{x}(0) = \begin{bmatrix} 1 & -1 \end{bmatrix}^\top$ and the input $u = 0.1\ \sin(t)$. The sampling period is 0.1 and the length of the data is $T = 50$. We assume that we have perfect measurement of the state, input, and the state derivative.

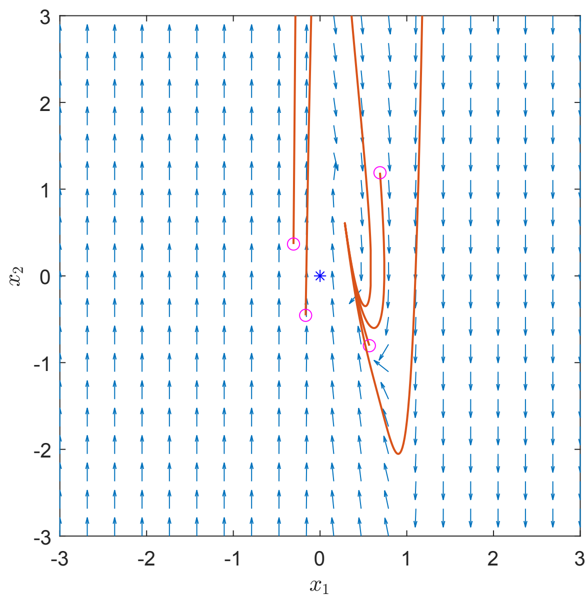

Fig. 1. Phase portrait of the closed-loop system under the designed data-driven controller.

Setting $\Omega_1 = 1$, we solve the program (16) using MOSEK in Matlab and obtain the virtual input

$$v = 83.72\bar{x}_1^5 - 106.72\bar{x}_1^4 + 39.334\bar{x}_1^3 + 1.705\bar{x}_1^2 - 7.639\bar{x}_1.$$

The resulting closed-loop system of $\bar{x}_1$ is $\dot{\bar{x}}_1 = -2.0\bar{x}_1 + 0.5\delta$, which shows that the nonlinearity is cancelled out.

Using the designed v, we obtain the nonlinearity in the δ-subsystem as

$$\overline{Q}_2(\bar{x}_1, \delta) = \begin{bmatrix}\bar{x}_1^2 & \bar{x}_1^3 & \bar{x}_1^4 & \bar{x}_1^5 & \bar{x}_1\delta & \bar{x}_1^2\delta & \bar{x}_1^3\delta & \bar{x}_1^4\delta\end{bmatrix}^\top.$$

Obtain the matrices $\overline{Z}_{20}$ and Δ_1, and set $\Omega_2 = 1$. Then, we solve the program (15), which leads to the control input

$$\begin{aligned}u = {} & 65.933\bar{x}_1 - 27.598\delta + 8.5249\bar{x}_1\delta + 295.0\bar{x}_1^2\delta \\ & - 1067.2\bar{x}_1^3\delta + 1046.5\bar{x}_1^4\delta - 31.542\bar{x}_1^2 - 1121.0\bar{x}_1^3 \\ & + 4108.7\bar{x}_1^4 - 4060.4\bar{x}_1^5.\end{aligned}$$

Under this control input, the dynamics of the δ-subsystem is

$$\begin{aligned}\dot{\delta} = {} & -2.0\delta + 4.745 \times 10^{-9}\bar{x}_1 - 2.5877 \times 10^{-9}\bar{x}_1\delta \\ & - 4.0952 \times 10^{-8}\bar{x}_1^2\delta - 5.5285 \times 10^{-7}\bar{x}_1^3\delta \\ & + 3.3291 \times 10^{-8}\bar{x}_1^4\delta + 1.3422 \times 10^{-7}\bar{x}_1^2 \\ & - 1.122 \times 10^{-7}\bar{x}_1^3 + 1.4539 \times 10^{-6}\bar{x}_1^4 \\ & - 4.6163 \times 10^{-7}\bar{x}_1^5 + 8.8818 \times 10^{-18},\end{aligned}$$

which shows that the nonlinearities are cancelled out. The phase portrait of the original closed-loop system (27) is illustrated in Fig. 1, which shows the globally asymptotic stability of the origin.

V. Conclusions and Future Works

For a class of second-order nonlinear systems, this work shows that by exploiting its structure and applying the learning control approach via nonlinearity cancellation to each subsystem, nonlinearities in both subsystems can be handled. For the subsystem not directly affected by the control input, a virtual input is designed for cancelling the nonlinearity. By dealing with the subsystems respectively, the proposed approach leads to a *globally* asymptotically stable equilibrium, in contrast to the locally asymptotically stable equilibrium when the control approach is applied to the system as a whole. More practical issues, such as noisy data and neglected nonlinearity, are important topics to be addressed in our future studies. The price to pay for solving those issues is a more complex control design.

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