Generalized Nash Equilibrium Seeking in a Class of Contractive Population Games over Networks

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Abstract—In this paper, we consider the problem of generalized Nash equilibrium (GNE) seeking in a class of contractive population games under a partial-decision information setup and subject to affine equality constraints. Namely, we consider multiple populations, each comprised of a large number of payoff-driven decision makers, and we embed a network topology ruling the exchange of information between the multiple populations. Conceptually, we consider that each population has an associated payoff provider entity, which yields the payoff signals to the agents of its corresponding population. The multiple payoff providers communicate through a (possibly non-complete) network to estimate the non-local information relevant to compute the payoff signals. As the main contribution, we formulate the dynamics of the payoff providers and we provide sufficient conditions to guarantee the asymptotic stability of the set of generalized Nash equilibria of the underlying game. To the best of our knowledge, this is the first paper to address the problem of GNE seeking in population games under partial-decision information.

I. INTRODUCTION

Consider a society with $P \in \mathbb{Z}_{>1}$ populations of decisionmaking agents and let $\mathcal{P} = \{1, 2, \dots, P\}$ be the set indexing the populations. For each population $k \in \mathcal{P}$, let $N^k \in \mathbb{Z}_{\gg 1}$ be the total number of agents that belong to population k(where we assume that N^k is large and constant over time), let $S^k = \{1, 2, ..., n^k\}$, with $n^k \in \mathbb{Z}_{\geq 2}$, be the set of decision strategies available to the agents of population k and, for each $i \in S^k$, let $\chi_i^k \in [0, 1]$ denote the proportion of agents of population k choosing strategy i, i.e., $N^k\chi^k_i$ yields the total number of agents playing strategy i in population k. For the sake of generality, we let $m^k \in \mathbb{R}_{>0}$ represent the total mass of agents of population k, and we let $x_i^k = (m^k \chi_i^k) \in [0, m^k]$ denote the portion of agents, from the total mass m^k , that are playing strategy i in population k. Clearly, if $m^k = 1$, then there is no distinction between x_i^k and χ_i^k . Under the considered framework, the strategic distribution of population \boldsymbol{k} is then captured by the vector $\mathbf{x}^{k} = \operatorname{col}(x_{i}^{k})_{i \in S^{k}} \in \Delta^{k}$, where $\Delta^{k} = \left\{ \mathbf{x}^{k} \in \mathbb{R}_{\geq 0}^{n^{k}} : \mathbf{1}_{n^{k}}^{\top} \mathbf{x}^{k} = m^{k} \right\}$ denotes the set of

possible strategic distributions for population k. Similarly, the strategic distribution of the entire society is captured by the vector $\mathbf{x} = \operatorname{col} (\mathbf{x}^k)_{k \in \mathcal{P}} \in \Delta$, where $\Delta = \Delta^1 \times \cdots \times \Delta^{\tilde{P}}$ is the set of possible strategic distributions for the entire society. Furthermore, each strategy $i \in \mathcal{S}^k$ is characterized by a fitness function $f_i^k : \mathbb{R}^n_{\geq 0} \to \mathbb{R}$. Namely, $f_i^k(\mathbf{x})$ provides the fitness value of strategy $i \in S^k$ at the strategic distribution $\mathbf{x} \in \Delta$. Throughout, we let $\mathbf{f}(\cdot) = \operatorname{col} \left(\mathbf{f}^k(\cdot) \right)_{k \in \mathcal{P}}$ be the overall fitness vector, whilst $\mathbf{f}^k(\cdot) = \operatorname{col}\left(f_i^k(\cdot)\right)_{i \in S^k}$ is the fitness vector of population k. Based on the overall fitness vector $\mathbf{f}(\cdot)$, a population game can then be defined in normal form as the tuple $G = (\mathcal{P}, \Delta, \mathbf{f}(\cdot))$, which captures the involved populations (\mathcal{P}) , the set of possible strategic distributions (Δ), and the overall fitness vector ($\mathbf{f}(\cdot)$). The overall fitness vector defines the strategic environment for the population game, and it plays the role of the so-called pseudo-gradient mapping of the game [1, Section 6].

To establish how the strategic distribution of the society evolves over time, let $t \in \mathbb{R}_{\geq 0}$ denote the continuous-time index, and let $\mathbf{x}(t)$ be the value of \mathbf{x} at time t. Moreover, let $p_i^k(t) \in \mathbb{R}$ be the payoff value perceived by the agents of population $k \in \mathcal{P}$ that are choosing strategy $i \in S^k$ at time t. Accordingly, $\mathbf{p}^k(t) = \operatorname{col}(p_i^k(t))_{i \in S^k}$ is the payoff vector of population k, and $\mathbf{p}(t) = \operatorname{col}(\mathbf{p}^k(t))_{k \in \mathcal{P}}$ is the overall payoff vector of the entire society.

Under the considered framework, the microscopic decision-making process of the agents is as follows. Each agent is equipped with a Poisson alarm clock and a revision protocol. The Poisson clocks provide independent and identically distributed strategy-revision opportunities according to an exponential distribution with rate $R \in \mathbb{R}_{>0}$, while the revision protocols are maps of the form ρ_{ij}^k : $\Delta^k \times$ $\mathbb{R}^{n^k} \to \mathbb{R}_{\geq 0}$ which characterize the probability distributions that agents use to update their strategies. Namely, if at time t an agent choosing strategy $i \in S^k$ in population $k \in \mathcal{P}$ receives a revision opportunity, then such an agent switches to strategy $j \in S^k \setminus \{i\}$ with probability $\rho_{ij}^{k}\left(\mathbf{x}^{k}\left(\tilde{t}\right), \mathbf{p}^{k}\left(\tilde{t}\right)\right) / R$, or remains at strategy i with probability $1 - (1/R) \sum_{j \in S^{k} \setminus \{i\}} \rho_{ij}^{k}\left(\mathbf{x}^{k}\left(\tilde{t}\right), \mathbf{p}^{k}\left(\tilde{t}\right)\right)$, where $\tilde{t} < t$ is an arbitrary time instant between the previous revision time of any agent of the society and time t (as in [2, Section 4.1], it is assumed that R is large enough so that these probabilities are well-defined for all times). Throughout, we assume that the revision protocols of the agents are fixed/given, i.e., we cannot design them.

On the other hand, following the ideas in [3], in this paper

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Fig. 1. Considered partial-decision information setup.

we consider that the temporal evolution of the payoff vector $\mathbf{p}(t)$ is ruled by a so-called *payoff dynamics model* (PDM) as defined next.

Definition 1 (Payoff dynamics model (PDM)): A PDM is a continuous-time system of the form

$$\dot{\mathbf{q}}(t) = \mathcal{W}(\mathbf{q}(t), \mathbf{x}(t))$$
$$\mathbf{p}(t) = \mathcal{H}(\mathbf{q}(t), \mathbf{x}(t))$$

where $\mathcal{W} : \mathbb{R}^d \times \Delta \to \mathbb{R}^d$ is Lipschitz continuous, and $\mathcal{H} : \mathbb{R}^d \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n$ is continuously differentiable and Lipschitz continuous.

In words, a PDM is a continuous-time system that takes $\mathbf{x}(t)$ as input and yields $\mathbf{p}(t)$ as output. In fact, the PDM abstraction can be used to describe the aggregate learning dynamics of the society (e.g., to capture effects such as anticipation, delay, and inertia), and/or to explicitly define how the payoffs are computed in practical engineering applications (e.g., as in resource allocation problems [4]).

In this paper, we embed the multiple populations of the society within a network. More precisely, we assume that agents have direct access only to the strategic distribution of their own population, and so they must estimate any required non-local information through network-based communication. Thus, the corresponding PDM must capture the aggregate estimation and communication dynamics of the society. To model such a setup conceptually, we let each population $k \in \mathcal{P}$ have an associated high-level entity referred to as the *payoff provider* of population k. Namely, the payoff providers take as input the strategic distribution of their corresponding population, communicate with each other through a network to estimate any required non-local information, and yield the payoff vector to the agents of their population (see Fig. 1). The PDM is then the mathematical object encompassing all the payoff providers' dynamics.

Remark 1: We highlight that the payoff providers need not to be physical (tangible) entities. Namely, if the goal is to describe the underlying learning dynamics associated to the network-based estimation process, then the payoff providers are just an abstraction to model the aggregate exchange of information between populations. On the other hand, in some applications payoff providers do have a practical interpretation. For instance, in the residential demand response problem considered in [5] and [6], the role of the payoff providers is played by the electric power utilities that compute and broadcast the costs signals to the consumers.

Problem statement: The technical problem that we consider in this paper is the convergence of the society towards a generalized Nash equilibrium (GNE) as defined next [7].

Definition 2 (Generalized Nash equilibrium (GNE)): Consider a set $\Omega \subseteq \mathbb{R}^n$, and define $\mathcal{X} = \Delta \cap \Omega$. Given a population game G, characterized by a fitness vector $\mathbf{f}(\cdot)$, the set of generalized Nash equilibria of G with respect to Ω is defined as $\text{GNE}(\mathbf{f}) = \text{fix} (\arg \max_{\mathbf{x} \in \mathcal{X}} \mathbf{x}^\top \mathbf{f}(\cdot))$. Thus, we say that \mathbf{x}^* is a GNE if and only if $\mathbf{x}^* \in \text{GNE}(\mathbf{f})$.

In Definition 2, the set Ω captures coupling constraints to be considered in the decision making process of the society. As such, a GNE is a strategic distribution $\mathbf{x}^* \in \mathcal{X}$ where no agent can increase its fitness value by unilaterally changing its strategy while still satisfying the coupling constraints imposed by Ω . Throughout, we impose the Standing Assumptions 1 and 2, which characterize the class of games considered in this paper, and ensure that the set $\text{GNE}(\mathbf{f})$ is nonempty and compact [1, Lemma 6].

Standing Assumption 1: For all $k \in \mathcal{P}$, the fitness vector $\mathbf{f}^{k}(\cdot)$ is of the form $\mathbf{f}^{k}(\mathbf{x}) = \mathbf{\tilde{f}}^{k}(\mathbf{x}^{k}) - \mathbf{C}^{k\top}\mathbf{C}\mathbf{x}$, where $\mathbf{\tilde{f}}^{k}$: $\mathbb{R}_{\geq 0}^{n^{k}} \to \mathbb{R}^{n}$ is continuously differentiable and contractive in the sense that $\boldsymbol{\zeta}^{\top} \mathbf{D}_{\mathbf{x}} \mathbf{\tilde{f}}^{k}(\mathbf{x}^{k}) \boldsymbol{\zeta} \leq 0$, for all $\mathbf{x}^{k} \in \mathbb{R}_{\geq 0}^{n^{k}}$, for all $\boldsymbol{\zeta} \in \mathbb{R}^{n^{k}}$, and $\mathbf{C} = [\mathbf{C}^{1}, \cdots, \mathbf{C}^{P}]$, with $\mathbf{C}^{\ell} \in \mathbb{R}^{r \times n^{\ell}}$, for some $r \in \mathbb{Z}_{\geq 1}$ and all $\ell \in \mathcal{P}$.

Standing Assumption 2: The set Ω in Definition 2 is of the form $\Omega = \{ \mathbf{x} \in \mathbb{R}^n : \sum_{k \in \mathcal{P}} \mathbf{A}^k \mathbf{x}^k = \sum_{k \in \mathcal{P}} \mathbf{b}^k \}$, where $\mathbf{A}^k \in \mathbb{R}^{c \times n^k}, \mathbf{b}^k \in \mathbb{R}^c$, for all $k \in \mathcal{P}$, and $c \in \mathbb{Z}_{\geq 1}$. Moreover, $\operatorname{int}(\mathcal{X}) = \Delta \cap \Omega \cap \mathbb{R}^n_{>0}$ is nonempty and $\tilde{\mathbf{A}} = [\mathbf{A}^\top, \mathbf{A}_{\Delta}^\top]^\top \in \mathbb{R}^{c+P \times n}$ is full-row rank. Here, $\mathbf{A} = [\mathbf{A}^1, \cdots, \mathbf{A}^P] \in \mathbb{R}^{c \times n}$ and $\mathbf{A}_{\Delta} \in \mathbb{R}^{P \times n}$ is such that Δ can be written as $\Delta = \{ \mathbf{x} \in \mathbb{R}^n_{>0} : \mathbf{A}_{\Delta} \mathbf{x} = \operatorname{col}(m^k)_{k \in \mathcal{P}} \}$.

Remark 2: Standing Assumption 1 is readily satisfied in merely contractive aggregative games with affine aggregate terms (see [8, Examples 1 and 2] and [9, Section VI-A]).

Given that the revision protocols of the agents are assumed fixed/given, the only way to steer the strategic distribution of the society towards a GNE of the game G is through the payoff signals (recall the microscopic decision-making process described above). Therefore, the goal of this paper is to formulate an appropriate PDM so that the resulting payoff vector $\mathbf{p}(t)$ steers the society of agents towards the set GNE(\mathbf{f}). Note that since the fitness functions and the set Ω in general couple the decision-making process of different populations, the estimation of non-local information is indeed a requirement to yield suitable payoff signals under the partial-decision information setup of Fig. 1. In summary, the main contributions of this paper are the following:

- i) The formulation of a suitable PDM for GNE seeking in population games under the partial-decision information setup of Fig. 1.
- ii) The deduction of sufficient conditions to guarantee the

asymptotic stability of the set $GNE(\mathbf{f})$ in the limit of an infinite number of agents in each population¹.

In addition, we illustrate our results through a numerical simulation of a traffic congestion game considering large (yet finite) populations of autonomous vehicles.

Related work: The problem considered in this paper is often referred to as distributed GNE seeking (DGNES), and it has received significant attention from the classical perspective of N-player games. In such a context, the most popular approaches for DGNES are the so-called primal-dual methods. Namely, primal-dual methods exploit the connection between GNE problems, variational inequalities, and the Karush-Kuhn-Tucker (KKT) optimality conditions (refer for instance to [7], [10]). Under such approaches, players compute not only their actions (primal variables) but also some auxiliary multipliers (dual variables), which quantify in some sense the violation of the coupled constraints. By relying on consensus algorithms, players reach an agreement on the optimal values of the dual variables. Some recent approaches for DGNES in N-player games can be found in [11]–[13]. In the context of population games, on the other hand, to the best of our knowledge the problem of DGNES has not been considered yet. Related approaches in the context of distributed NE seeking (without coupling constraints) have been reported in [8], [9]. In this paper, we build upon [8], [9], but we explicitly consider the coupling constraints characterized by Ω .

Notation: We use standard font for scalars, bold font for vectors and matrices, and non-bold calligraphic font for sets. Besides, all vectors are taken as columns by default (including gradients). The set of real (integer) numbers is denoted by \mathbb{R} (\mathbb{Z}). The set of non-negative (strictly positive) real numbers is denoted by $\mathbb{R}_{>0}$ ($\mathbb{R}_{>0}$). A similar notation holds for integers. We denote the Euclidean norm by $\|\cdot\|_2$. The operators $col(\cdot)$ and $diag(\cdot)$ create a column vector and a (block) diagonal matrix of the arguments, respectively. Given a domain $\mathcal{D} \subseteq \mathbb{R}^m$ and an operator $T : \mathcal{D} \to \mathcal{D}$, fix $(T) := \{ \mathbf{z} \in \mathcal{D} : \mathbf{z} = T(\mathbf{z}) \}$ is the fixed point set of T. The gradient of a scalar-valued function f(z) is denoted $\nabla_{\mathbf{z}} f(\mathbf{z})$, and the Jacobian of a vector-valued function $\mathbf{f}(\mathbf{z})$ is denoted $D_z f(z)$. Throughout, I_d , I_d , and O_d , denote the d-dimensional identity matrix, one vector, and zero vector, respectively (we often drop the sub-index if the dimension is clear from context). A similar notation holds for the zero matrix. Finally, $\lambda_{\min}(\mathbf{M})$ yields the minimum eigenvalue of a square matrix \mathbf{M} , and \otimes denotes the Kronecker product.

II. PROPOSED PAYOFF DYNAMICS MODEL (PDM)

To formulate our proposed PDM under the partial-decision information setup of Fig. 1, we first define the graph that characterizes the communication network between the multiple payoff providers. Namely, let $\mathcal{G} = (\mathcal{P}, \mathcal{L}, \mathbf{W})$ be the directed graph (digraph) characterizing the communication network. Namely, \mathcal{P} is the set of nodes corresponding to the payoff providers/populations, $\mathcal{L} \subseteq \mathcal{P} \times \mathcal{P}$ is the set of links of possible communication, and $\mathbf{W} \in \mathbb{R}_{\geq 0}^{P \times P}$ is the weighted adjacency matrix that captures the topology of the digraph. We say that $(\ell, k) \in \mathcal{L}$ if and only if node kcan receive information from node ℓ and, for simplicity we adopt the convention that $(k, k) \notin \mathcal{L}$. Besides, $w_{k\ell} > 0$ for all $(\ell, k) \in \mathcal{L}$, and $w_{k\ell} = 0$, otherwise. Here, $w_{k\ell}$ denotes the (k, ℓ) -th element of \mathbf{W} . Furthermore, we let $\mathcal{N}_{in}^{k} = \{\ell \in \mathcal{P} : w_{k\ell} > 0\}$ denote the set of in-neighbors of node k, for all $k \in \mathcal{P}$, and we let $\mathbf{L} = \text{diag}(\mathbf{W}\mathbf{1}_{P}) - \mathbf{W}$ be the Laplacian matrix associated to \mathcal{G} .

Standing Assumption 3: The digraph G is strongly connected and weight-balanced.

Remark 3: Overall, Standing Assumption 3 implies that rank $(\mathbf{L}) = P - 1$, $\mathbf{1}_P^\top \mathbf{L} = \mathbf{0}_P^\top$, $\lambda_{\min} (\mathbf{L} + \mathbf{L}^\top) = 0$, and that $\mathbf{L}\boldsymbol{\zeta} = \mathbf{0}_P \Leftrightarrow \boldsymbol{\zeta} \in \operatorname{span}(\mathbf{1}_P)$ [14].

According to Definition 1, the PDM yields the overall payoff vector $\mathbf{p}(t)$ as a causal map from the strategic distribution $\mathbf{x}(t)$. To cope with the partial-decision information setup of Fig. 1, in this paper we propose the PDM given by

$$\dot{\mathbf{q}}_{1}^{k}(t) = -\mathbf{q}_{1}^{k}(t) - \sum_{\ell \in \mathcal{P}} w_{k\ell} \left(\mathbf{q}_{1}^{k}(t) - \mathbf{q}_{1}^{\ell}(t) \right) - \sum_{\ell \in \mathcal{P}} w_{\ell k} \left(\mathbf{q}_{2}^{k}(t) - \mathbf{q}_{2}^{\ell}(t) \right) + \mathbf{C}^{k\top} \mathbf{x}^{k}(t)$$
(1a)

$$\dot{\mathbf{q}}_{2}^{k}(t) = \sum_{\ell \in \mathcal{P}} w_{k\ell} \left(\mathbf{q}_{1}^{k}(t) - \mathbf{q}_{1}^{\ell}(t) \right)$$
(1b)

$$\dot{\mathbf{q}}_{3}^{k}(t) = -\sum_{\ell \in \mathcal{P}} w_{k\ell} \left(\mathbf{q}_{3}^{k}(t) - \mathbf{q}_{3}^{\ell}(t) \right) \\ -\sum_{\ell \in \mathcal{P}} w_{\ell k} \left(\mathbf{q}_{4}^{k}(t) - \mathbf{q}_{4}^{\ell}(t) \right) + \mathbf{A}^{k} \mathbf{x}^{k}(t) - \mathbf{b}^{k} \quad (1c)$$

$$\dot{\mathbf{q}}_{4}^{k}(t) = \sum_{\ell \in \mathcal{P}} w_{k\ell} \left(\mathbf{q}_{3}^{k}(t) - \mathbf{q}_{3}^{\ell}(t) \right)$$
(1d)

$$\mathbf{p}^{k}(t) = \tilde{\mathbf{f}}^{k} \left(\mathbf{x}^{k}(t) \right) - P \mathbf{C}^{k \top} \mathbf{q}_{1}^{k}(t) - \mathbf{A}^{k \top} \mathbf{q}_{3}^{k}(t), \qquad (1e)$$

for all $k \in \mathcal{P}$. Our proposed PDM (1) is inspired by the so-called proportional integral consensus algorithm [15] and its interpretation is as follows. On one hand, q_1^k is used to estimate, in a distributed fashion, the non-local aggregate term $(1/P)\mathbf{Cx}$, i.e., $\mathbf{q}_1^k(t) \to (1/P)\mathbf{Cx}(t)$ as $t \to \infty$, for all $k \in \mathcal{P}$. On the other hand, \mathbf{q}_3^k represents the estimation held by the k-th payoff provider regarding the Lagrange multipliers associated to the equality constraints of Ω . In contrast, \mathbf{q}_2^k and \mathbf{q}_4^k are auxiliary state variables used to eliminate steady-state errors, i.e., \mathbf{q}_2^k and \mathbf{q}_4^k provide an integral action to ensure the consensus of q_1^k and q_3^k over all $k \in \mathcal{P}$, respectively. Finally, the payoff vector of each population k is computed based on the fitness vector, yet considering the non-local information estimated by \mathbf{q}_1^k , and on the marginal cost associated to the violation of the coupled equality constraints (captured by $\mathbf{A}^{k\top}\mathbf{q}_3^k$). In addition, we highlight the following properties of our proposed PDM (1).

Lemma 1: Consider the PDM (1) and suppose that $\dot{\mathbf{q}}_1^k(t) = \dot{\mathbf{q}}_2^k(t) = \mathbf{0}_r$, and $\dot{\mathbf{q}}_3^k(t) = \dot{\mathbf{q}}_4^k(t) = \mathbf{0}_c$, for all $k \in \mathcal{P}$. Then, it holds that

i)
$$\mathbf{q}_1^k(t) = (1/P)\mathbf{C}\mathbf{x}(t)$$
, for all $k \in \mathcal{P}$.

¹All the proofs of our theoretical results can be found at the following link: https://drive.google.com/drive/folders/ 1wS6L1-MivRaQkNV_JkV8oQdkuyH-lo6r?usp=sharing



Fig. 2. Closed-loop system formed by an EDM-PDM pair.

ii) $\mathbf{q}_3^k(t) = \hat{\mathbf{q}}_3(t)$, for some $\hat{\mathbf{q}}_3(t) \in \mathbb{R}^c$, for all $k \in \mathcal{P}$. iii) $\mathbf{x}(t) \in \Omega$.

iv) $\mathbf{p}(t) = \mathbf{f}(\mathbf{x}(t)) - \mathbf{A}^{\top} \hat{\mathbf{q}}_3(t)$, for some $\hat{\mathbf{q}}_3(t) \in \mathbb{R}^c$.

We remark that the proposed PDM (1) indeed reflects the partial-decision information setup of Fig. 1. Namely, each payoff provider is ruled by the dynamics (1) for the corresponding super-index k, and the information exchange between payoff providers regards the state vectors \mathbf{q}_1^k , \mathbf{q}_2^k , \mathbf{q}_3^k , and \mathbf{q}_4^k . We now proceed to formally analyze the stability properties of the set GNE(\mathbf{f}) under the proposed PDM.

III. STABILITY ANALYSIS

Following the framework of [2] and [3], to analyze the stability properties of $GNE(\mathbf{f})$, we consider the limiting case where $N^k \to \infty$, for all $k \in \mathcal{P}$. Namely, when the number of agents within each population is large enough, the expected temporal evolution of the strategic distribution of the society is arbitrarily well-modeled by a so-called evolutionary dynamics model (EDM) as defined next.

Definition 3 (Evolutionary dynamics model (EDM)): An EDM is a continuous-time system of the form

$$\dot{\mathbf{x}}(t) = \boldsymbol{\mathcal{V}}\left(\mathbf{x}(t), \mathbf{p}(t)\right),\tag{2}$$

where $\mathcal{V} : \Delta \times \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous and satisfies that $\mathcal{V}(\mathbf{x}, \mathbf{p}) \in T\Delta(\mathbf{x})$, for every $\mathbf{x} \in \Delta$ and every $\mathbf{p} \in \mathbb{R}^n$. Here, $T\Delta(\mathbf{x})$ denotes the tangent cone of Δ at \mathbf{x} .

As depicted in Fig. 2, an EDM-PDM pair form a closedloop system where the EDM takes $\mathbf{p}(t)$ as input and yields $\mathbf{x}(t)$ as output, while the PDM takes $\mathbf{x}(t)$ as input and yields $\mathbf{p}(t)$ as output. Depending on the revision protocols of the agents, different EDMs might arise. Some examples include the Brown-von Neumann-Nash (BNN) dynamics [16], the replicator dynamics [17], the Smith dynamics [18], and the logit and regularized logit EDMs [19], among many others. Instead of assuming a particular form of revision protocol, in this paper we focus on the class of δ -passive EDMs, which encompasses several revision protocols at once [3].

Definition 4 (δ -passive EDM): An EDM is said to be δ passive if there exist a continuously differentiable nonnegative δ -storage function $S_{EDM} : \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, and a non-negative function $\sigma_{EDM} : \Delta \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, such that $S_{EDM}(\mathbf{x}, \mathbf{p}) = 0 \Leftrightarrow \mathcal{V}(\mathbf{x}, \mathbf{p}) = \mathbf{0}_n$ and

$$S_{EDM}(\mathbf{x},\mathbf{p},\mathbf{v}) \leq -\sigma_{EDM}(\mathbf{x},\mathbf{p},\mathbf{v}) + \mathbf{v}^{\top} \mathcal{V}(\mathbf{x},\mathbf{p}),$$

for all $\mathbf{x} \in \Delta$ and all $\mathbf{p}, \mathbf{v} \in \mathbb{R}^n$. Here, we define

$$\dot{S}_{EDM}\left(\mathbf{x},\mathbf{p},\mathbf{v}\right) = \left[\begin{array}{c} \nabla_{\mathbf{x}} S_{EDM}(\mathbf{x},\mathbf{p}) \\ \nabla_{\mathbf{p}} S_{EDM}(\mathbf{x},\mathbf{p}) \end{array}\right]^{\top} \left[\begin{array}{c} \boldsymbol{\mathcal{V}}(\mathbf{x},\mathbf{p}) \\ \mathbf{v} \end{array}\right].$$

Moreover, the function $\sigma_{EDM}(\cdot, \cdot, \cdot)$ is said to be informative if it satisfies that $\sigma_{EDM}(\mathbf{x}, \mathbf{p}, \mathbf{0}_n) = 0 \Leftrightarrow \mathcal{V}(\mathbf{x}, \mathbf{p}) = \mathbf{0}_n$.

The class of δ -passive EDMs is particularly important when considered together with a δ -antipassive PDM [3].

Definition 5 (δ -antipassive PDM): A PDM is said to be δ -antipassive if there exist a continuously differentiable nonnegative δ -storage function $S_{PDM} : \mathbb{R}^d \times \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a non-negative function $\sigma_{PDM} : \mathbb{R}^d \times \Delta \to \mathbb{R}_{\geq 0}$, such that $S_{PDM}(\mathbf{q}, \mathbf{x}) = 0 \Leftrightarrow \mathcal{W}(\mathbf{q}, \mathbf{x}) = \mathbf{0}_d$ and

$$\dot{S}_{PDM}(\mathbf{q}, \mathbf{x}, \mathbf{w}) \leq -\sigma_{PDM}(\mathbf{q}, \mathbf{x}, \mathbf{w}) - \mathbf{w}^{\top} \dot{\mathcal{H}}(\mathbf{q}, \mathbf{x}, \mathbf{w}),$$

for all $\mathbf{q} \in \mathbb{R}^d$, all $\mathbf{x} \in \Delta$ and all $\mathbf{w} \in \mathbb{R}^n$. Here, we define

$$\dot{S}_{PDM} \left(\mathbf{q}, \mathbf{x}, \mathbf{w} \right) = \begin{bmatrix} \nabla_{\mathbf{q}} S_{PDM}(\mathbf{q}, \mathbf{x}) \\ \nabla_{\mathbf{x}} S_{PDM}(\mathbf{q}, \mathbf{x}) \end{bmatrix}^{\top} \begin{bmatrix} \mathcal{W}(\mathbf{q}, \mathbf{x}) \\ \mathbf{w} \end{bmatrix}$$

$$\dot{\mathcal{H}}(\mathbf{q}, \mathbf{x}, \mathbf{w}) = D_{\mathbf{q}} \mathcal{H}(\mathbf{q}, \mathbf{x}) \mathcal{W}(\mathbf{q}, \mathbf{x}) + D_{\mathbf{x}} \mathcal{H}(\mathbf{q}, \mathbf{x}) \mathbf{w}.$$

Moreover, the function $\sigma_{PDM}(\cdot, \cdot, \cdot)$ is said to be informative if it satisfies that $\sigma_{PDM}(\mathbf{q}, \mathbf{x}, \mathbf{0}_n) = 0 \Leftrightarrow \mathcal{W}(\mathbf{q}, \mathbf{x}) = \mathbf{0}_d$.

Based on Definitions 1-5, we now provide our main supporting result to characterize the stability properties of the set GNE(f).

Proposition 1: Consider an arbitrary EDM-PDM pair interconnected as in Fig. 2 and define

$$\mathcal{E} = \left\{ (\mathbf{x}, \mathbf{q}) \in \Delta imes \mathbb{R}^d : egin{array}{c} \mathcal{oldsymbol{\mathcal{V}}} \left(\mathbf{x}, \mathcal{oldsymbol{\mathcal{H}}}(\mathbf{q}, \mathbf{x})
ight) = \mathbf{0}_n \ \mathcal{oldsymbol{\mathcal{W}}} \left(\mathbf{q}, \mathbf{x}
ight) = \mathbf{0}_d \end{array}
ight\},$$

i.e., \mathcal{E} is the set of equilibria of the closed-loop system. Suppose that the following conditions hold:

- C1) The EDM is δ -passive.
- C2) The PDM is δ -antipassive.
- C3) \mathcal{E} is nonempty and bounded.

Then, \mathcal{E} is Lyapunov stable under the EDM-PDM pair. Moreover, let

$$\mathcal{R} = \left\{ (\mathbf{x}, \mathbf{q}) \in \Delta \times \mathbb{R}^d : \begin{array}{c} \sigma_{EDM} \left(\mathbf{x}, \mathcal{H}(\mathbf{q}, \mathbf{x}), \mathbf{0}_n \right) = 0 \\ \sigma_{PDM} \left(\mathbf{q}, \mathbf{x}, \mathbf{0}_n \right) = 0 \end{array} \right\},$$

and suppose that, in addition to C1)-C3), it also holds that

C4) \mathcal{E} is the largest invariant set of the EDM-PDM pair within \mathcal{R} .

Then, $\boldsymbol{\mathcal{E}}$ is asymptotically stable under the EDM-PDM pair.

Proposition 2: Consider an arbitrary EDM-PDM pair interconnected as in Fig. 2, and let \mathcal{E} be defined as in Proposition 1. Suppose that the following conditions hold:

C5) For every $(\mathbf{x}^*, \mathbf{p}^*) \in \Delta \times \mathbb{R}^n$, the EDM satisfies that

$$\mathcal{V}(\mathbf{x}^*, \mathbf{p}^*) = \mathbf{0}_n \Leftrightarrow \mathbf{x}^* \in \arg \max_{\mathbf{x} \in \Delta} \mathbf{x}^\top \mathbf{p}^*.$$
 (3)

C6) For every $(\mathbf{q}^*, \mathbf{x}^*) \in \mathbb{R}^d \times \Delta$, the PDM satisfies that

$$\mathcal{W}(\mathbf{q}^*, \mathbf{x}^*) = \mathbf{0}_d \Rightarrow \begin{cases} \mathcal{H}(\mathbf{q}^*, \mathbf{x}^*) = \mathbf{f}(\mathbf{x}^*) - \mathbf{A}^\top \hat{\mathbf{q}}^* \\ \mathbf{x}^* \in \Omega, \end{cases}$$
(4)

where $\hat{\mathbf{q}}^* = \mathbf{M}\mathbf{q}^*$, for some $\mathbf{M} \in \mathbb{R}^{c \times d}$.

Then, every $(\mathbf{x}^*, \mathbf{q}^*) \in \mathcal{E}$ implies that $\mathbf{x}^* \in \text{GNE}(\mathbf{f})$.

Propositions 1 and 2 yield sufficient conditions to assert the Lyapunov and asymptotic stability of GNE(**f**) under an arbitrary EDM-PDM pair. In particular, note that Condition C4) is readily satisfied if both $\sigma_{EDM}(\cdot, \cdot, \cdot)$ and $\sigma_{PDM}(\cdot, \cdot, \cdot)$ are informative in the sense of Definitions 4 and 5, respectively. To analyze our proposed PDM under the light of Propositions 1 and 2, we first rewrite (1) in an equivalent reduced-order form.

Consider the dynamics (with state $\mathbf{z} = \operatorname{col}(\mathbf{z}_1, \mathbf{z}_2)$, input \mathbf{x} , and output \mathbf{y}) given by

$$\dot{\mathbf{z}}_1(t) = -\overline{\mathbf{Q}}(\alpha)\mathbf{z}_1(t) - \overline{\mathbf{L}}^{\top}\mathbf{z}_2(t) + \overline{\mathbf{D}}\mathbf{x}(t) - \beta\mathbf{b} \qquad (5a)$$

$$\dot{\mathbf{z}}_2(t) = \mathbf{L}\mathbf{z}_1(t) \tag{5b}$$

$$\mathbf{y}(t) = \alpha \tilde{\mathbf{f}} (\mathbf{x}(t)) - \gamma \overline{\mathbf{D}}^{\top} \mathbf{z}_1(t),$$
 (5c)

where $\overline{\mathbf{Q}}(\alpha) = \alpha \mathbf{I}_{Pe} + \overline{\mathbf{L}}^{\top}$, $\overline{\mathbf{L}} = \mathbf{L} \otimes \mathbf{I}_{e}$, $\mathbf{b} = \operatorname{col}(\mathbf{b}^{k})_{k \in \mathcal{P}}$, $\overline{\mathbf{D}} \in \mathbb{R}^{Pe \times n}$, $\mathbf{\tilde{f}}(\cdot) = \operatorname{col}(\mathbf{\tilde{f}}^{k}(\cdot))_{k \in \mathcal{P}}$, $\alpha \in \{0, 1\}$, $\beta = 1 - \alpha$, $\gamma \in \mathbb{R}$, and $e \in \mathbb{Z}_{\geq 1}$. Now, Let $\mathbf{u} = (1/\sqrt{P})\mathbf{1}_{P}$ and $\mathbf{U} = [\mathbf{\tilde{U}}, \mathbf{u}] \in \mathbb{R}^{P \times P}$ be an orthonormal matrix. Moreover, consider the change of variables $\mathbf{z}_{2} = (\mathbf{U} \otimes \mathbf{I}_{r})[\mathbf{\tilde{z}}_{2}^{\top}, \mathbf{\hat{z}}_{2}^{\top}]^{\top}$, with $\mathbf{\tilde{z}}_{2} \in \mathbb{R}^{Pe-e}$ and $\mathbf{\hat{z}}_{2} \in \mathbb{R}^{e}$. Applying such change of variables to (5) with the fact that $\mathbf{L}^{\top}\mathbf{u} = \mathbf{0}_{P}$, yields the reduced-order dynamics (with state $\mathbf{\tilde{z}} = \operatorname{col}(\mathbf{z}_{1}, \mathbf{\tilde{z}}_{2})$, input \mathbf{x} , and output \mathbf{y}) given by

$$\dot{\mathbf{z}}_{1}(t) = -\overline{\mathbf{Q}}(\alpha)\mathbf{z}_{1}(t) - \overline{\mathbf{L}}_{\tilde{\mathbf{U}}}^{\top} \widetilde{\mathbf{z}}_{2}(t) + \overline{\mathbf{D}}\mathbf{x}(t) - \beta \mathbf{b} \qquad (6a)$$

$$\tilde{\mathbf{z}}_2(t) = \overline{\mathbf{L}}_{\tilde{\mathbf{U}}} \mathbf{z}_1(t) \tag{6b}$$

$$\mathbf{y}(t) = \alpha \tilde{\mathbf{f}} (\mathbf{x}(t)) - \gamma \overline{\mathbf{D}}^{\top} \mathbf{z}_1(t),$$
(6c)

where $\overline{\mathbf{L}}_{\widetilde{\mathbf{U}}} = (\widetilde{\mathbf{U}}^{\top} \mathbf{L}) \otimes \mathbf{I}_{e}$. Here, $\widehat{\mathbf{z}}_{2}$ has been eliminated as it corresponds to an uncontrollable state vector that does not affect the remaining state variables or the output. As argued in [9, Remark 5], under matching initial conditions, the dynamics (5) and (6) have the same input-output trajectories. Thus, to analyze (5), it suffices to consider (6).

For the forthcoming analyses we define two instances of (5) referred to as systems Σ_a and Σ_b , respectively. Namely, Σ_a has the form (5) with e = r, $\alpha = 1$ $\beta = 0$, $\gamma = P$, $\mathbf{D} = \text{diag} (\mathbf{C}^k)_{k\in\mathcal{P}}$, and \mathbf{z}_1 and \mathbf{z}_2 play the roles of $\mathbf{q}_1 = \text{col} (\mathbf{q}_1^k)_{k\in\mathcal{P}}$ and $\mathbf{q}_2 = \text{col} (\mathbf{q}_2^k)_{k\in\mathcal{P}}$, respectively. On the other hand, Σ_b has the form (5) with e = c, $\alpha = 0$, $\beta = 1$, $\gamma = 1$, $\mathbf{D} = \text{diag} (\mathbf{A}^k)_{k\in\mathcal{P}}$, and \mathbf{z}_1 and \mathbf{z}_2 play the roles of $\mathbf{q}_3 = \text{col} (\mathbf{q}_3^k)_{k\in\mathcal{P}}$ and $\mathbf{q}_4 = \text{col} (\mathbf{q}_4^k)_{k\in\mathcal{P}}$, respectively. Under such formulations, it follows that our proposed PDM (1) is equivalent to the parallel interconnection of systems Σ_a and Σ_b . More precisely, Σ_a captures the dynamics (1a) and (1b), while Σ_b captures the dynamics (1c) and (1d), for all $k \in \mathcal{P}$. The output (1e) is given by the sum of the outputs of Σ_a and Σ_b . Consequently, to analyze (1) we can consider the equivalent-reduced order forms (6) of Σ_a and Σ_b .

Based on the former discussion and without loss of generality, to analyze our proposed PDM (1) we consider

its equivalent reduced-order PDM given by

$$\dot{\mathbf{q}}_{1}(t) = -\left(\mathbf{I}_{Pr} + \overline{\mathbf{L}}_{a}\right)\mathbf{q}_{1}(t) - \overline{\mathbf{L}}_{\tilde{\mathbf{U}}a}^{\top}\tilde{\mathbf{q}}_{2}(t) + \overline{\mathbf{C}}\mathbf{x}(t) \quad (7a)$$

$$\mathbf{q}_2(t) = \mathbf{L}_{\tilde{\mathbf{U}}a} \mathbf{q}_1(t) \tag{7b}$$

$$\dot{\mathbf{q}}_3(t) = -\mathbf{L}_b \mathbf{q}_3(t) - \mathbf{L}_{\tilde{\mathbf{U}}b} \tilde{\mathbf{q}}_4(t) + \mathbf{A} \mathbf{x}(t) - \mathbf{b}$$
(7c)

$$\tilde{\mathbf{q}}_4(t) = \mathbf{L}_{\tilde{\mathbf{U}}b} \mathbf{q}_3(t) \tag{7d}$$

$$\mathbf{p}(t) = \alpha \mathbf{\tilde{f}} (\mathbf{x}(t)) - P \mathbf{\overline{C}}^{\top} \mathbf{q}_1(t) - \mathbf{\overline{A}}^{\top} \mathbf{q}_3(t),$$
(7e)

where $\overline{\mathbf{L}}_a$, $\overline{\mathbf{L}}_{\tilde{\mathbf{U}}a}$, and $\overline{\mathbf{L}}_b$, $\overline{\mathbf{L}}_{\tilde{\mathbf{U}}b}$, are defined as $\overline{\mathbf{L}}$, $\overline{\mathbf{L}}_{\tilde{\mathbf{U}}}$, but with the corresponding dimension $e \in \{r, c\}$, respectively. In addition, $\overline{\mathbf{C}} = \operatorname{diag} (\mathbf{C}^k)_{k \in \mathcal{P}}$, and $\overline{\mathbf{A}} = \operatorname{diag} (\mathbf{A}^k)_{k \in \mathcal{P}}$. Moreover, we define $\tilde{\mathbf{q}} = \operatorname{col}(\mathbf{q}_1, \tilde{\mathbf{q}}_2, \mathbf{q}_3, \tilde{\mathbf{q}}_4) \in \mathbb{R}^d$, with d = 2Pr + 2Pc - r - c.

Next, we characterize some key properties of the reducedorder PDM (7) and the resulting EDM-PDM pair.

Lemma 2: The reduced-order PDM (7) satisfies Conditions C2) and C6) of Propositions 1 and 2, respectively.

Lemma 3: Consider an EDM-PDM pair interconnected as in Fig. 2. Suppose that the EDM satisfies Condition C5) of Proposition 2, and that the PDM is given by (7). Then, Condition C3) of Proposition 1 holds.

Lemma 4: Consider an EDM-PDM pair interconnected as in Fig. 2. Suppose that the EDM is δ -passive with informative $\sigma_{EDM}(\cdot, \cdot, \cdot)$, and that it satisfies Condition C5) of Proposition 2. Also, suppose that the PDM is given by (7). Then, Condition C4) of Proposition 1 holds.

Theorem 1: Consider an EDM-PDM pair interconnected as in Fig. 2, where the EDM is δ -passive with informative $\sigma_{EDM}(\cdot, \cdot, \cdot)$ and satisfies Condition C5) of Proposition 2, and the PDM is given by (1). Then, the set GNE(**f**) is asymptotically stable under the EDM-PDM pair.

IV. AN ILLUSTRATIVE EXAMPLE

As illustration, we consider an application of our developed theory in the context of congestion games [20]. Namely, consider P = 4 populations of autonomous vehicles, and suppose that each population $k \in \mathcal{P}$ seeks to travel from an origin O^k to a destination D^k using the available routes connecting O^k to D^k . It is assumed that the routes are shared with the other populations, and so the goal for the agents is to minimize the congestion of the routes. Without loss of generality, we assume that there is a total of r = 3routes, and for simplicity we allow all populations to use all routes, i.e., $n^k = r$, for all $k \in \mathcal{P}$. As such, we let $\mathbf{C}^k \in \mathbb{R}^{r \times n^k}$ define a bipartite graph between population k and the routes in S^k , which for our case is $\mathbf{C}^k = \mathbf{I}_r$, for all $k \in \mathcal{P}$. Consequently, the sum $\sum_{\ell \in \mathcal{P}} \mathbf{C}^{\ell} \mathbf{x}^{\ell} = \mathbf{C} \mathbf{x}$ yields the overall society's allocation over all routes (here, $\mathbf{C} \in \mathbb{R}^{r \times n}$ is constructed as in Standing Assumption 1).

Based on the considered framework, we let the fitness vector of each population $k \in \mathcal{P}$ be of the form $\mathbf{f}^k(\mathbf{x}) = -\Theta^k \mathbf{x}^k - P \mathbf{C}^{k\top} \mathbf{C} \mathbf{x}$, where $\Theta^k \in \mathbb{R}_{\geq 0}^{n^k \times n^k}$ is a diagonal matrix encoding the preferences of population k over their available routes. Moreover, since each population k may have a different origin O^k , and the multiple origins might be spatially distributed over some geographical region, we



Fig. 3. Temporal evolution of $d_{\mathbf{x}^*}(t) = \|\mathbf{x}(t) - \mathbf{x}^*\|_2 / \|\mathbf{x}(0) - \mathbf{x}^*\|_2$, where $\mathbf{x}^* \in \text{GNE}(\mathbf{f})$ is the achieved GNE. The top plot depicts the limiting case with infinite populations, i.e., the EDM, while the bottom plot regards the considered scenario with finite populations.

embed a non-complete communication network topology among the populations. Without loss of generality, we let the digraph \mathcal{G} be a directed cycle with unitary weights.

Regarding the constraint set Ω , we assume that certain routes must be used at specific congestion levels, compelling autonomous vehicles to follow these routes and limit their use of others. Such a setup can enhance traffic network efficiency, especially when routes are shared with unpredictable non-autonomous vehicles. For simplicity, we set c = 2 and randomly sample the constraint set Ω , ensuring compliance with Standing Assumption 2.

For all $k \in \mathcal{P}$, we let N^k be randomly drawn so that each population has between 1000 and 2000 agents, and we let $m^k = 1$. In addition, we randomly sample the diagonal elements of Θ^k from the interval [0, 1]. Regarding the decision-making process of the agents, we consider the revision protocol $\rho_{ij}^k(\mathbf{p}^k) = \min(\max(p_j^k - p_i^k, 0), \nu^k),$ for all $i, j \in \mathcal{S}^k$. Here, $\nu^k \in \mathbb{R}_{>0}$ satisfies $\nu^k \leq R/(n^k - 1).$ Namely, ν^k is a saturation parameter to ensure that the switching probabilities of the microscopic decision-making process are well defined for any R (see Section I). Without loss of generality, we let R = 1 and $\nu^k = 1/(n^k - 1)$. The resulting evolutionary dynamics are the so-called Smith dynamics [18] with an added saturation. Following [9, Section II-C], it is straightforward to show that the corresponding EDM is δ -passive with informative $\sigma_{EDM}(\cdot, \cdot, \cdot)$ and that it satisfies Condition C5) of Proposition 2. Hence, the result of Theorem 1 holds. Figure 3 depicts a numerical simulation of our considered setup, which validates our theoretical results.

V. CONCLUDING REMARKS

In this paper, we have considered the problem of generalized Nash equilibrium seeking in a class of merely contractive population games, subject to affine equality constraints and under a partial-decision information setup. More precisely, we have explicitly considered a network topology that rules the information exchange between the multiple populations of decision-making agents, and we have formulated a suitable payoff dynamics model that guarantees the asymptotic stability (in the limiting case with infinite populations) of the set of generalized Nash equilibria of the underlying population game. The provided results are applicable to a broad class of decision-making protocols (i.e., those yielding δ -passive evolutionary dynamics models), and have been validated numerically on a large-scale congestion game. Future research should explore extensions to more general games and coupled convex inequality constraints.

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