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On stability of second-order nonlinear time-delay systems without damping

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***Abstract*—For a second-order system with time delays and power nonlinearity of the degree higher than one, which does not contain a velocity-proportional damping term, the conditions of local asymptotic stability of the zero solution are proposed. The result is based on application of the Lyapunov-Razumikhin approach, and it is illustrated by simulations. Our local stability conditions for nonlinear systems are less restrictive than stability conditions of the corresponding linear models.**

I. Introduction

Design and analysis of time-delay systems are very complex issues that are demanded nowadays in different fields of science and technology [1], [2], [3], [4]. A reason of such a popularity is that any kind of communications in networked systems is usually accompanied by various lags and samplings, which are frequently modeled by time delays [5], [6].

Usually, appearance of delays degrades the quality of transients in dynamical systems, and may even lead to instability [7]. However, in some cases, which deal, for example, with compensation of periodic disturbances or approximation of derivatives, introduction of delays can improve robustness and convergence rates in the closed-loop systems [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18], [19] at the price of an accurate and sophisticated stability analysis.

An interesting and challenging case, where emergence of a small delay provides asymptotic stability to a neutrally stable planar system, was studied in [1]. In the works [20], [21], a second-order linear time-varying system is considered including a delayed position term with a negative gain, and it is proven that it can be stabilized by a position feedback with positive gain and with a sufficiently small delay (without velocity damping). This kind of dynamics is omnipresent in modeling the mechanical systems controlled through a communication network. The contribution of this paper consists in extension of those results to a nonlinear case. Using the Lyapunov-Razumikhin approach [2], [4] we will demonstrate that in the nonlinear setting the asymptotic stability of the origin can be guaranteed for any value of the delay (under some mild restrictions) with the attraction domain dependent on the magnitude of delays and other parameters.

The outline of this work is as follows. Preliminaries are given in Section II. The considered analysis problem is described in Section III. The main stability results are formulated in Section IV. An illustrative example is shown in Section V.

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II. Preliminaries

The real numbers are denoted by $\mathbb{R}$, $\mathbb{R}_+ = \{s \in \mathbb{R} : s \geq 0\}$, and $|s|$ is an absolute value for $s \in \mathbb{R}$. Euclidean norm for a vector $x \in \mathbb{R}^n$ is defined as $\|x\|$. We denote by $C([a, b], \mathbb{R}^n)$, $-\infty < a < b < +\infty$ the Banach space of continuous functions $\phi : [a, b] \to \mathbb{R}^n$ with the uniform norm $\|\phi\|_C = \sup_{a\leq\varsigma\leq b} \|\phi(\varsigma)\|$.

A continuous function $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0) = 0$; it belongs to class $\mathcal{K}_\infty$ if it is also radially unbounded.

A. Useful inequalities

The *Young's inequality* claims that for any $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ and $\gamma > 0$, $\delta > 0$ [22]:

$$\mathfrak{a}^\gamma \mathfrak{b}^\delta \leq \frac{1}{p}\mathfrak{a}^{\gamma p} + \frac{p-1}{p}\mathfrak{b}^{\frac{\delta p}{p-1}}$$

for any $p > 1$.

Using the properties of homogeneous functions the following results can be obtained:

Lemma 1. [23] *Let $\mathfrak{a}, \mathfrak{b} \in \mathbb{R}_+$ and $\ell > 0$, $\alpha > 0$, $\beta > 0$, $\gamma > 0$, $\delta > 0$ be given, then*

$$\mathfrak{a}^\alpha + \mathfrak{b}^\beta - \ell\mathfrak{a}^\gamma\mathfrak{b}^\delta \geq 0$$

provided that

1) $\max\{\mathfrak{a}^\alpha, \mathfrak{b}^\beta\} \geq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$ *and* $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} < 1$,
2) $\max\{\mathfrak{a}^\alpha, \mathfrak{b}^\beta\} \leq \ell^{\frac{1}{1-\frac{\gamma}{\alpha}-\frac{\delta}{\beta}}}$ *and* $\frac{\gamma}{\alpha} + \frac{\delta}{\beta} > 1$.

Lemma 2. [24] *For $x, y \in \mathbb{R}^n$ denote*

$$\mathcal{W}(x, y) = \|x\|^\alpha + \|y\|^\beta + c_1\|x\|^\eta\|y\|^\zeta - c_2\|x\|^\gamma\|y\|^\delta,$$

where $c_1, c_2, \alpha, \beta, \gamma, \delta, \eta, \zeta$ are positive constants. Let

$$\frac{\eta}{\alpha} + \frac{\zeta}{\beta} < 1,$$

then the function $\mathcal{W}(x, y)$ is positive definite for any values of c_1 and c_2 if and only if

$$\gamma + \delta\frac{\alpha - \eta}{\zeta} > \alpha, \quad \gamma\frac{\beta - \zeta}{\eta} + \delta > \beta.$$

B. Lyapunov-Razumikhin stability conditions

Consider an autonomous functional differential equation of retarded type [2]:

$$dx(t)/dt = f(x_t),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ and $x_t \in C([-\tau, 0], \mathbb{R}^n)$ is the state function, $x_t(s) = x(t+s)$, $-\tau \leq s \leq 0$ and $\tau > 0$ is a finite delay; $f : C([-\tau, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is a continuous functional, $f(0) = 0$, and is such that solutions in forward time for the system (1) exist and are unique [2]. The representation (1) includes pointwise or distributed time-delay systems.

For a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ the upper directional Dini derivative is defined as follows:

$$D^+V(x)v = \limsup_{h\to 0^+} \frac{V(x + hv) - V(x)}{h}$$

for any $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

The Lyapunov-Razumikhin method can be formulated as follows:

Theorem 1. *[2] Let there exist a locally Lipschitz continuous Lyapunov-Razumikhin function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that*
(i) for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and all $x \in \mathbb{R}^n$:

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|);$$

(ii) for some $\alpha, \gamma \in \mathcal{K}$, with $\gamma(s) > s$ for all $s > 0$, and all $\varphi \in C([-\tau, 0], \mathbb{R}^n)$:

$$\max_{\theta\in[-\tau,0]} V(\varphi(\theta)) \leq \gamma \circ V(\varphi(0)) \Rightarrow$$
$$D^+V(\varphi(0))\, f(\varphi) \leq -\alpha(|\varphi(0)|).$$

Then the system (1) *is globally asymptotically stable at the origin.*

Restricting the domain of admissible values for the state, the conditions of local asymptotic stability can be formulated.

III. Statement of the Problem

In [20], [21], the problem of asymptotic stability for the scalar equation

$$\ddot{x}(t) + ax(t - h) - bx(t - g) = 0 \tag{2}$$

was studied, where $x \in \mathbb{R}$, a and b are constant positive coefficients, h and g are constant positive delays. The system (2) has no damping proportional to the velocity $\dot{x} \in \mathbb{R}$, and in the delay-free case (when $h = g = 0$) under the restriction $a > b$ it has purely oscillating trajectories. It was proven that if the inequalities

$$a > b, \quad gb > ah \tag{3}$$

are satisfied and the values of $a-b$ and $gb-ah$ are sufficiently small, then the system (2) is asymptotically stable.

In the present paper, we consider a nonlinear counterpart of (2):

$$\ddot{x}(t) + ax^\mu(t - h) - bx^\mu(t - g) = 0, \tag{4}$$

where $x \in \mathbb{R}$, a, b, g are positive constants and h is nonnegative, $\mu > 1$ is a rational number with an odd numerator and denominator.

Remark 1. Instead of x^μ, we can use $\lceil x \rfloor^\mu = |x|^\mu \operatorname{sign}(x)$, then it is not necessary to assume that μ is rational with an odd numerator and denominator. This restriction is used for brevity of notation.

The instantaneous value of the state vector of (4) is $[x(t)\ \dot{x}(t)]^\top \in \mathbb{R}^2$. Assume that initial functions for solutions of (4) belong to the space $C([-\tau, 0], \mathbb{R}^2)$, where $\tau = \max\{h, g\}$.

In the system (4), the term $ax^\mu(t - h)$ can be interpreted as a part of its own dynamics, while $bx^\mu(t - g)$ is the stabilizing part introduced by a control, which uses only delayed position measurements [25]. In such a case it is necessary to select the parameters b and g providing stability of the origin for (4).

In this work, we will prove that the asymptotic stability of the zero solution of (4) can be guaranteed under less conservative conditions than those for (2). In addition, we will extend our result obtained for constant delays to time-varying ones.

IV. Main Results

For the proof of the theorem below we will use the approaches proposed in [25], [26].

Theorem 2. *Let the inequalities* (3) *be fulfilled. Then the zero solution of* (4) *is locally asymptotically stable.*

Remark 2. Compared with the result of [20], [21], in Theorem 2 it is not assumed that the values of $a - b$ and $gb - ah$ are sufficiently small, the delay h can be zero.

Proof. By adding and subtracting the delay-free terms $ax^\mu(t)$ and $bx^\mu(t)$, using the Mean Value Theorem

$$x^\mu(t - h) = x^\mu(t) - h\mu x^{\mu-1}(t - \eta_1(t)h)\dot{x}(t - \eta_1(t)h),$$
$$x^\mu(t - g) = x^\mu(t) - g\mu x^{\mu-1}(t - \eta_2(t)g)\dot{x}(t - \eta_2(t)g)$$

for $\eta_1(t), \eta_2(t) \in (0, 1)$ (further for brevity of presentation the time dependence is omitted), and again by adding and subtracting the delay-free terms $\mu ahx^{\mu-1}(t)\dot{x}(t)$ and $\mu bgx^{\mu-1}(t)\dot{x}(t)$, the equation (4) can be rewritten in the form

$$\begin{aligned}\ddot{x}(t) + \alpha x^{\mu-1}(t)\dot{x}(t) + \beta x^\mu(t) + \mu ah(x^{\mu-1}(t)\dot{x}(t)\\ -x^{\mu-1}(t - \eta_1 h)\dot{x}(t - \eta_1 h))\\ -\mu bg(x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t - \eta_2 g)\dot{x}(t - \eta_2 g)) = 0,\end{aligned} \tag{5}$$

where $\alpha = \mu(bg - ah)$, $\beta = a - b$. Hence, in (5) the nominal part takes a form of the Liénard equation,

$$\ddot{x}(t) + \alpha x^{\mu-1}(t)\dot{x}(t) + \beta x^{\mu}(t) = 0, \tag{6}$$

which is asymptotically stable at the origin [27], [28], [29], and the remaining terms are considered as perturbations. Note that if the inequalities (3) are fulfilled, then $\alpha > 0$, $\beta > 0$, $\tau = g$. For the rest of the proof, all computations are done for the case $h > 0$, and if $h = 0$, then the arguments stay unchanged by imposing the respective terms to be zero.

Let a Lyapunov-Razumikhin function candidate for (5) be given by the formula

$$V(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{\beta}{\mu+1}x^{\mu+1} - \delta x\dot{x}|\dot{x}|^{\lambda-1} + \varepsilon x\dot{x}|x|^{\sigma-1},$$

where δ and ε are positive coefficients, $\lambda \geq 1$, $\sigma \geq 1$. With the aid of Lemma 1 and Young's inequality, we obtain that if

$$\lambda \geq \frac{2\mu}{\mu+1}, \quad \sigma \geq \frac{\mu+1}{2}, \tag{7}$$

and in the case where $\lambda = 2\mu/(\mu+1)$ or $\sigma = (\mu+1)/2$ the values of parameters δ or ε are sufficiently small, respectively, there exist positive numbers a_1, a_2, D_1 such that

$$a_1(\dot{x}^2 + x^{\mu+1}) \leq V(x, \dot{x}) \leq a_2(\dot{x}^2 + x^{\mu+1}) \tag{8}$$

for all $\dot{x}^2 + x^2 < D_1^2$. Note that the full energy $E(x, \dot{x}) = \frac{1}{2}\dot{x}^2 + \frac{\beta}{\mu+1}x^{\mu+1}$ can be used to establish global asymptotic stability of the Liénard equation (6), however, E is not a strict Lyapunov function in this case, then V was proposed in [29] being strict and guaranteeing asymptotic stability only locally.

Consider the derivative of the function V calculated on the solutions of the equation (5):

$$\begin{aligned}
\dot{V} &= -\alpha x^{\mu-1}(t)\dot{x}^2(t) - \delta|\dot{x}(t)|^{\lambda+1} - \varepsilon\beta|x(t)|^{\sigma+\mu} \\
&\quad + \varepsilon\sigma\dot{x}^2(t)|x(t)|^{\sigma-1} \\
&\quad - \mu ah\dot{x}(t)(x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_1 h)\dot{x}(t-\eta_1 h)) \\
&\quad + \mu gb\dot{x}(t)(x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_2 g)\dot{x}(t-\eta_2 g)) \\
&\quad + \varepsilon x(t)|x(t)|^{\sigma-1}\Big(-\alpha x^{\mu-1}(t)\dot{x}(t) \\
&\quad - \mu ah(x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_1 h)\dot{x}(t-\eta_1 h)) \\
&\quad + \mu bg(x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_2 g)\dot{x}(t-\eta_2 g))\Big) \\
&\quad + \delta x(t)\lambda|\dot{x}(t)|^{\lambda-1}\Big(\alpha x^{\mu-1}(t)\dot{x}(t) + \beta x^{\mu}(t) \\
&\quad + \mu ah(x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_1 h)\dot{x}(t-\eta_1 h)) \\
&\quad - \mu bg(x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_2 g)\dot{x}(t-\eta_2 g))\Big) \\
&\leq -\alpha x^{\mu-1}(t)\dot{x}^2(t) - \delta|\dot{x}(t)|^{\lambda+1} - \varepsilon\beta|x(t)|^{\sigma+\mu} \\
&\quad + \varepsilon\sigma\dot{x}^2(t)|x(t)|^{\sigma-1} + \alpha\varepsilon|x(t)|^{\sigma+\mu-1}|\dot{x}(t)| \\
&\quad + \alpha\lambda\delta|x(t)|^{\mu}|\dot{x}(t)|^{\lambda} + \beta\delta\lambda|x(t)|^{\mu+1}|\dot{x}(t)|^{\lambda-1} \\
&\quad + \Big(|\dot{x}(t)| + \varepsilon|x(t)|^{\sigma} + \delta\lambda|x(t)||\dot{x}(t)|^{\lambda-1}\Big) \\
&\quad \times\Big(\mu ah|x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_1 h)\dot{x}(t-\eta_1 h)| \\
&\quad + \mu bg|x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_2 g)\dot{x}(t-\eta_2 g)|\Big).
\end{aligned}$$

Using Lemma 2, it is easy to show that if

$$\sigma > \mu, \quad \lambda > \frac{3\sigma - 1}{\sigma + 1}, \tag{9}$$

then there exists a positive number D_2 such that

$$\begin{aligned}
\dot{V} &\leq -\frac{1}{2}\left(\alpha x^{\mu-1}(t)\dot{x}^2(t) + \delta|\dot{x}(t)|^{\lambda+1} + \varepsilon\beta|x(t)|^{\sigma+\mu}\right) \\
&\quad + \Big(|\dot{x}(t)| + \varepsilon|x(t)|^{\sigma} + \delta\lambda|x(t)||\dot{x}(t)|^{\lambda-1}\Big) \\
&\quad \times\Big(\mu ah|x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_1 h)\dot{x}(t-\eta_1 h)| \\
&\quad + \mu bg|x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_2 g)\dot{x}(t-\eta_2 g)|\Big)
\end{aligned}$$

for all $\dot{x}^2(t) + x^2(t) < D_2^2$. It should be noted that if the inequalities (9) are fulfilled, then (7) are also verified.

Let, for a solution of the equation (5), the inequality $\dot{x}^2(t) + x^2(t) < D_3^2$ and the Razumikhin condition

$$V(x(\xi), \dot{x}(\xi)) \leq 2V(x(t), \dot{x}(t))$$

be satisfied for all $\xi \in [t - 2\tau, t]$, where $D_3 = \min\{D_1, D_2\}$. Then, recalling the estimates (8), one can choose positive constants c_1 and c_2 such that

$$|x(\xi)| \leq c_1(|x(t)| + |\dot{x}(t)|^{\frac{2}{\mu+1}}), |\dot{x}(\xi)| \leq c_2(|x(t)|^{\frac{\mu+1}{2}} + |\dot{x}(t)|)$$

for $\xi \in [t - 2\tau, t]$. Therefore, we obtain

$$\begin{aligned}
&|x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_1 h)\dot{x}(t-\eta_1 h)| \\
&= |x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_1 h)(\dot{x}(t-\eta_1 h) + \dot{x}(t) - \dot{x}(t))| \\
&\leq |\dot{x}(t)||x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| \\
&\quad + |x^{\mu-1}(t-\eta_1 h)||\dot{x}(t) - \dot{x}(t-\eta_1 h)| \\
&= |\dot{x}(t)||x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| \\
&\quad + \eta_1 h|x^{\mu-1}(t-\eta_1 h)||\ddot{x}(t-\eta_3 h)| \\
&\leq |\dot{x}(t)||x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| \\
&\quad + c_3(|x(t)| + |\dot{x}(t)|^{\frac{2}{\mu+1}})^{2\mu-1}
\end{aligned}$$

where $\eta_3 \in (0, 1)$ is calculated through the Mean Value Theorem, $c_3 = \text{const} > 0$. A similar estimate can be derived for the term $|x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_2 g)\dot{x}(t-\eta_2 g)|$:

$$\begin{aligned}
&|x^{\mu-1}(t)\dot{x}(t) - x^{\mu-1}(t-\eta_2 g)\dot{x}(t-\eta_2 g)| \\
&\leq |\dot{x}(t)||x^{\mu-1}(t) - x^{\mu-1}(t-\eta_2 g)| + c_3(|x(t)| \\
&\quad + |\dot{x}(t)|^{\frac{2}{\mu+1}})^{2\mu-1}.
\end{aligned}$$

Applying Lemma 2 and Jensen's inequality, it can be shown that if

$$\sigma < 2\mu - 1, \quad \lambda < \frac{4\mu - 2}{\mu + 1}, \tag{10}$$

the inequalities (9) are fulfilled and the value of D_3 is sufficiently small, then

$$\dot{V}(t) \leq -\frac{1}{3}\left(\alpha x^{\mu-1}(t)\dot{x}^2(t) + \delta|\dot{x}(t)|^{\lambda+1} + \varepsilon\beta|x(t)|^{\sigma+\mu}\right) + |\dot{x}(t)|\Big(|\dot{x}(t)| + \varepsilon|x(t)|^{\sigma} + \delta\lambda|x(t)||\dot{x}(t)|^{\lambda-1}\Big) \left(\mu a h|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| + \mu b g|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_2 g)|\right).$$

Finally, consider the terms $|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)|$ and $|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_2 g)|$. Assuming that $x(t) \neq 0$ the former term can be rewritten as follows:

$$|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| = |x^{\mu-1}(t) - (x(t) + x(t-\eta_1 h) - x(t))^{\mu-1}|$$
$$= |x(t)|^{\mu-1}\left|1 - \left(1 + \frac{x(t-\eta_1 h) - x(t)}{x(t)}\right)^{\mu-1}\right|$$
$$= |x(t)|^{\mu-1}\left|1 - \left(1 - \frac{\eta_1 h \dot{x}(t-\eta_4 h)}{x(t)}\right)^{\mu-1}\right|,$$

where $\eta_4 \in (0,1)$, and applying the Razumikhin condition as above we get an upper bound:

$$\frac{|\dot{x}(t-\eta_4 h)|}{|x(t)|} \leq c_2\left(|x(t)|^{\frac{\mu-1}{2}} + \frac{|\dot{x}(t)|}{|x(t)|}\right).$$

Hence, for any $\tilde{\varepsilon} > 0$, there exists $\tilde{\delta} > 0$ such that if $|x(t)| < \tilde{\delta}$, $|\dot{x}(t)| < \tilde{\delta}|x(t)| < \tilde{\delta}^2$, then

$$|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| \leq \tilde{\varepsilon}|x(t)|^{\mu-1},$$

while inversely, when $\tilde{\delta}|x(t)| \leq |\dot{x}(t)| < \tilde{\delta}^2$, we obtain

$$|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| \leq \frac{1}{\tilde{\delta}^{\mu-1}}|\dot{x}(t)|^{\mu-1} + |x(t) + x(t-\eta_1 h) - x(t)|^{\mu-1}$$
$$\leq \frac{c_4}{\tilde{\delta}^{\mu-1}}|\dot{x}(t)|^{\mu-1} + c_5|\dot{x}(t-\eta_5 h)|^{\mu-1}$$
$$\leq \frac{c_4}{\tilde{\delta}^{\mu-1}}|\dot{x}(t)|^{\mu-1} + c_6(|x(t)|^{\frac{\mu+1}{2}} + |\dot{x}(t)|)^{\mu-1}$$
$$\leq |\dot{x}(t)|^{\mu-1}\left(\frac{c_4}{\tilde{\delta}^{\mu-1}} + c_6\left(\frac{|\dot{x}(t)|^{\frac{\mu-1}{2}}}{\tilde{\delta}^{\frac{\mu+1}{2}}} + 1\right)^{\mu-1}\right),$$

where $\eta_5 \in (0,1)$, and c_4, c_5, c_6 are positive constants. Therefore, the following upper estimate has been derived:

$$|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_1 h)| \leq \tilde{\varepsilon}|x(t)|^{\mu-1} + c_7|\dot{x}(t)|^{\mu-1}$$

for all $|x(t)| < \tilde{\delta}$ and $|\dot{x}(t)| < \tilde{\delta}^2$, where $c_7 = \frac{c_4}{\tilde{\delta}^{\mu-1}} + c_6\left(\tilde{\delta}^{\frac{\mu-3}{2}} + 1\right)^{\mu-1}$. The same approach can be used to derive an estimate for the term $|x^{\mu-1}(t) - x^{\mu-1}(t-\eta_2 g)|$.

As a result, we obtain that if values of D_3 and $\tilde{\delta}$ are sufficiently small, the conditions (9) and (10) are fulfilled and

$$\lambda < \mu, \tag{11}$$

then

$$\mu(ah + bg)|\dot{x}(t)|\Big(|\dot{x}(t)| + \varepsilon|x(t)|^{\sigma} + \delta\lambda|x(t)||\dot{x}(t)|^{\lambda-1}\Big) \times \left(\tilde{\varepsilon}|x(t)|^{\mu-1} + c_7|\dot{x}(t)|^{\mu-1})\right)$$
$$\leq \frac{1}{12}\left(\alpha x^{\mu-1}(t)\dot{x}^2(t) + \delta|\dot{x}(t)|^{\lambda+1} + \varepsilon\beta|x(t)|^{\sigma+\mu}\right)$$

implying that

$$\dot{V}(t) \leq -\frac{1}{4}\left(\alpha x^{\mu-1}(t)\dot{x}^2(t) + \delta|\dot{x}(t)|^{\lambda+1} + \varepsilon\beta|x(t)|^{\sigma+\mu}\right)$$

and the conditions of the Lyapunov-Razumikhin theorem [4] are satisfied. It is easy to verify that the domain of admissible values of parameters σ and λ defined by the inequalities (9), (10), (11) is not empty for any $\mu > 1$.

This completes the proof. □

Note that (4) is $\mathbf{r}$-homogeneous for $\mathbf{r} = [1 \ \frac{\mu+1}{2}]$ of degree $\frac{\mu-1}{2}$ in the sense of [30], where it has been proven that if a homogeneous system of positive degree is locally asymptotically stable in a given vicinity of the origin for any $\tau > 0$, then it is globally asymptotically stable independently of the delay value. In Theorem 2 the estimate on the domain of convergence depends on τ (as well as on other parameters a, b and μ), hence, the global outcome from [30] cannot be applied. Note that the delay free model (4),

$$\ddot{x}(t) + (a-b)x^{\mu}(t) = 0,$$

has purely oscillating trajectories, therefore, it does not confirm the asymptotically stable behavior of the system for small delays, whose appearance changes qualitatively the kind of stability in (4).

The obtained result can be extended to the time-varying case:

$$\ddot{x}(t) + ax^{\mu}(t-h(t)) - bx^{\mu}(t-g(t)) = 0 \tag{12}$$

with positive, continuous and bounded for $t \geq 0$ delays $h(t)$ and $g(t)$ such that $\tau = \max_{t\geq 0}\{h(t), g(t)\}$ is well defined (the delay $h(t)$ can take zero values).

Corollary 1. *Let the inequalities*

$$a > b, \quad g(t)b - ah(t) \geq \varsigma \quad \forall t \geq 0,$$

where ς is a positive constant, be fulfilled. Then the zero solution of (12) *is locally asymptotically stable.*

Proof. The proof repeats all steps of Theorem 2. □

These results can also be extended to a more complicated model:

$$\ddot{x}(t) + ax^{\nu}(t)x^{\mu}(t-h(t)) - bx^{\nu}(t)x^{\mu}(t-g(t)) = 0,$$

where $a, b > 0$, $g(t) > 0$ and $h(t) \geq 0$ for all $t \geq 0$ as before, $\mu + \nu > 1$ is a rational number with an odd numerator and denominator, $\mu > 1$ and ν is nonnegative.

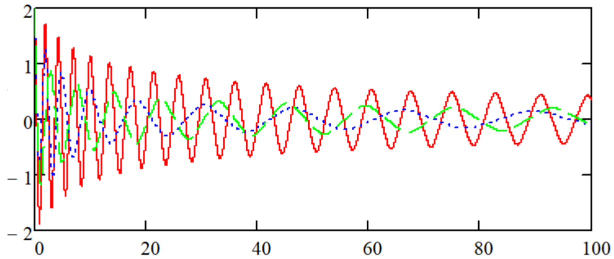

Figure 1. Behavior of $x(t)$ for different values of delay g and $h = 0$, $t \in [0, 100]$

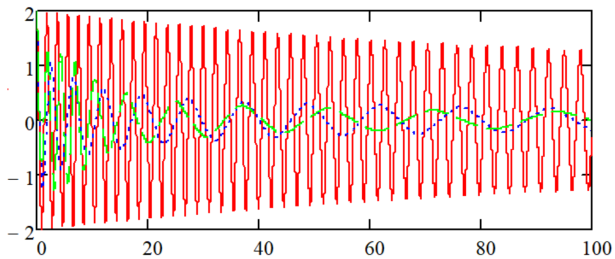

Figure 2. Behavior of $x(t)$ for different values of delay g and $h = 0.01$, $t \in [0, 100]$

V. Example

For an illustration consider the case with $h \in \{0, 0.01\}$, sufficiently big gains

$$a = 5, \; b = 1,$$

$\mu = 3$ and different values of the delay $g \in \{0.1, 0.5, 1\}$. In the case $g = 0$ (or $b = 0$) the system (4) takes the form of a nonlinear mechanical oscillator. The appearance of delayed part for $g \neq 0$ can be interpreted as a stabilizing control that uses delayed position measurements.

The results of simulation of $x(t)$ for the initial conditions $x(s) = 2$ and $\dot{x}(s) = 0$ for all $s \in [-\tau, 0]$ are shown in figures 1 and 2 for $h = 0$ and $h = 0.01$, respectively, where the fastest convergence (blue dot line) is observed for the biggest value of delay $g = 1$, while the slowest and most oscillatory (red solid line) for the smallest delay $g = 0.1$ (hence, green dash line corresponds to $g = 0.5$). These results can be explained by the fact that for $g = 0$ and $h = 0$ the system is purely oscillating, then introducing the delay stabilizes it with the domain of stability dependent on g (and also the values of a, b and μ).

VI. Conclusion

For a second-order system with time-varying delays and smooth power nonlinearity, without a velocity damping term, the conditions of local asymptotic stability at the origin are formulated. These conditions are very simple for checking. It is seen from the proof and the example that the domain of attraction depends on the value of delays and parameters, but there is no restriction on admissible maximal value of delays.

Extensions to the case of vector position variable x, non-Lipschitz nonlinearity with $\mu \in (0, 1)$ and input-to-state stability analysis can be considered as directions for future research.

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