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Learning controllers from data via kernel-based interpolation

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***Abstract*— We propose a data-driven control design method for nonlinear systems that builds on kernel-based interpolation. Under some assumptions on the system dynamics, kernel-based functions are built from data and a model of the system, along with deterministic model error bounds, is determined. Then, we derive a controller design method that aims at stabilizing the closed-loop system by cancelling out the system nonlinearities. The proposed method can be implemented using semidefinite programming and returns positively invariant sets for the closed-loop system.**

I. Introduction

Data-driven control is a cornerstone of automatic control. Starting from the pioneering work by Ziegler–Nichols [1], data-driven control has proved effective in contexts where finding a model of the system from first principles is difficult or time-consuming, and a controller is instead determined using experimental data. In the last years, there has been a renewed interest in data-driven control, and the reason is the growing complexity of the engineering systems for which first-principle laws are often difficult to determine.

The body of work on data-driven control is extremely vast, and it is not our goal to provide here any comprehensive review. We will focus on the basic problem of designing a feedback controller and consider batch (i.e., non-iterative) methods, that are methods in which a controller is computed once and for all using a finite set of data collected from the system. The interest for batch methods is related to the possibility of having *finite-sample* stability guarantees, as opposed to classic adaptive control schemes that usually only provide asymptotic guarantees.

Related work. Batch methods can be classified as *indirect* or *direct*. In the first case, data are used to build a model of the system (within a selected model class, e.g. linear models). In this process, explicit error bounds arising from noise in the data or a mismatch between system and model class can also be determined. Then, model-based control design techniques are applied. In contrast, direct method go directly from data to the controller. Also direct methods can involve notions of model class and uncertainty but the decision variables are directly the controller parameters, without any intermediate identification step.

Most of the existing works consider linear systems and assume that there are no unmodeled dynamics, which means that the plant-model mismatch is at most parametric. Recent contributions in this context are [2], [3] for what concerns indirect methods and [4], [5], [6] for what concerns direct methods. Dealing with nonlinear systems is arguably much more difficult. One main reason is that it becomes harder to compute finite-sample uncertainty bounds, even when the uncertainty is purely parametric. Another main reason is that controller design for nonlinear systems is itself much more complex. Recent contributions that consider parametric uncertainty tackle bilinear systems [7], [8], polynomial (and rational) systems [9], [10], [11], [12], and LPV systems [13]. For general nonlinear systems, but still in the context of parametric uncertainty, we find linearly parametrized models with known basis functions [14], [15]. The result in [15], in particular, introduces a controller design technique that provides, under rather mild conditions, finite-sample stability guarantees along with an estimate of regions of attraction and positive invariant sets for the closed-loop system.

Assuming the exact knowledge of the basis functions is reasonable in many practical cases such as with mechanical and electrical systems in which some prior information about the dynamics is available but the exact systems parameters may be unknown. In many other cases, however, this prior information may be unknown. Methods that consider this scenario include methods based on Gaussian process models [16], [17], methods based on linear [4], [18], [19], [20], [21] and polynomial approximations [22], [23], and methods based on linearly parametrized models with partially known basis functions [15]. Despite the differences, the common idea is to describe the system via a quantity which is known up to parametric uncertainty and treat unmodeled dynamics as an error term, i.e., a remainder. The challenge is thus twofold: (i) to derive finite-sample bounds for the remainder and (ii) to design a control law that is robust to the uncertainty that this remainder introduces.

Contribution and outline of the paper. In this paper, we consider the last scenario discussed above, that is the scenario where the system to control has general dynamics (e.g. not necessarily bilinear or polynomial) and there is no prior knowledge of the true basis functions. We propose a new method that combines ideas from kernel-based identification [24] and the controller design method introduced in [15]. Specifically, we consider an *indirect* method that consists of two steps: we first determine a kernel-based model of the system along with deterministic error bounds, in line with recent results on kernel learning [25]. Then, we consider a controller design method that explicitly accounts for the uncertainty around the nominal model. Since the nominal model is generally nonlinear and lacks a specific structure,

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we consider a method in which the control law is designed so as to render the dynamics in closed loop nearly linear (as much as possible) by cancelling the nonlinearities of the system. We show that the method returns positively invariant sets for the closed-loop system and can be implemented via semidefinite programming. For control design purpose, kernel-based methods have been previously considered mostly in connection with Gaussian processes [16], [17]. In a deterministic setting, contributions have been proposed in the realm of modelling and control [26], [27], [28], [29]. To the best of our knowledge, our work is the first work on kernel learning that gives deterministic guarantees in the context of feedback controller design. The proofs are omitted due to space limitations and can be found in [30].

The rest of the paper is organized as follows: preliminaries on kernels, RKHS and regularized interpolation are given in Section II. Section III provides the main result in which we derive a controller design method based on kernel models. Section IV presents simulation results on a nonlinear system. Conclusions and future work are discussed in Section V.

Notation. Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, and $\mathbb{N}_{>0}$ denotes the set of positive integers. $\mathbb{S}^{n\times n}$ denotes the set of real-valued symmetric matrices. Given a matrix M, $M \succ 0$ ($M \succeq 0$) means that M is positive definite (positive semidefinite), while $M \prec 0$ ($M \preceq 0$) means that M is negative definite (negative semidefinite). Finally, we denote by $|x|$ the 2-norm of a vector x, and by $\|M\|$ the induced 2-norm of a matrix M. Other, less standard, notions are introduced throughout the paper.

II. Preliminaries

A. Kernels and their RKHS

Given a non-empty set $\Omega \subseteq \mathbb{R}^n$, a continuous function $K : \Omega \times \Omega \to \mathbb{R}$ is called a positive definite kernel on Ω if $\sum_{i,j} \alpha_i \alpha_j K(x_i, x_j) > 0$ for all $N \in \mathbb{N}_{>0}$, all sets of pairwise distinct points $x_1, \ldots, x_N \subseteq \Omega$, and all nonzero vectors $\alpha \in \mathbb{R}^N$. K is called positive semidefinite if $\sum_{i,j} \alpha_i \alpha_j K(x_i, x_j) \geq 0$ for all $N \in \mathbb{N}_{>0}$, all points $x_1, \ldots, x_N \subseteq \Omega$, and all vectors $\alpha \in \mathbb{R}^N$. K is called symmetric if $K(x, y) = K(y, x)$ for all $x, y \in \Omega$ [31].

Definition 1: ([32, Def. 10.1]) Let $\mathcal{H}$ be a real Hilbert space of functions $f : \Omega \to \mathbb{R}$. The function $K : \Omega \times \Omega \to \mathbb{R}$ is a reproducing kernel of $\mathcal{H}$ if

1) For every $y \in \Omega$, the function $K(\cdot, y)$ belongs to $\mathcal{H}$.
2) (Reproducing property) For every $y \in \Omega$ and every $f \in \mathcal{H}$, it holds that

$$f(y) = \langle f(\cdot), K(\cdot, y)\rangle_{\mathcal{H}},$$

where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the inner product in $\mathcal{H}$.

Fact 1: [31] To every positive semidefinite and symmetric kernel K, there corresponds a unique Hilbert space admitting K as a reproducing kernel. ■

A Hilbert space that admits a reproducing kernel is called a reproducing kernel Hilbert space (RKHS). By Definition 1, the kernel centred at a point $a \in \Omega$, i.e., $K(\cdot, a)$, belongs to $\mathcal{H}$. For a function of the form $f(\cdot) = \sum_{i=1}^{N} \alpha_i K(\cdot, x_i)$ where $N \in \mathbb{N}_{>0}$, $\alpha_i \in \mathbb{R}$ and $x_i \in \Omega$, we have that $f \in \mathcal{H}$ and its RKHS function norm is $\|f\|_{\mathcal{H}} := \sqrt{\langle f, f \rangle_{\mathcal{H}}}$. Further,

$$\|f\|_{\mathcal{H}}^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha_i \alpha_j K(x_i, x_j). \tag{1}$$

B. Regularized interpolation and its error bound

Consider a positive semidefinite and symmetric reproducing kernel $K : \Omega \times \Omega \to \mathbb{R}$ and the associated RKHS $\mathcal{H}$. Consider an unknown function $f : \Omega \to \mathbb{R}$ belonging to $\mathcal{H}$, and let f generate the data points $(y_i, x_i), i = 0, \ldots, T-1$, where $y_i = f(x_i)$. Our objective is to find a function $s_f \in \mathcal{H}$ that minimizes the cost function

$$\sum_{i=0}^{T-1} |y_i - s_f(x_i)|^2 + \lambda \|s_f\|_{\mathcal{H}}^2, \tag{2}$$

where $\lambda > 0$ is the regularization parameter. By the representer theorem [33], the minimizer takes the form

$$s_f(x) = \alpha \mathbf{k}(x) \tag{3}$$

where $\alpha \in \mathbb{R}^{1\times T}$ and

$$\mathbf{k}(x) := \begin{bmatrix} K(x, x_0) & K(x, x_1) & \cdots & K(x, x_{T-1}) \end{bmatrix}^\top. \tag{4}$$

The functions $K(x, x_i)$ are called kernel-based basis functions that are the kernels centered at the data points $x_i, i = 0, \ldots, T-1$. The number of kernel-based basis functions is equal to the number of data points, and when the dataset is fixed, determining the model s_f is equivalent to computing the coefficients α. By [24, Th. 2], we have

$$s_f(x) = y_X (\lambda I_T + K_X)^{-1} \mathbf{k}(x) \tag{5}$$

where

$$y_X := \begin{bmatrix} y_0 & y_1 & \cdots & y_{T-1} \end{bmatrix}, \tag{6}$$

and

$$K_X := \begin{bmatrix} K(x_0, x_0) & K(x_1, x_0) & \cdots & K(x_{T-1}, x_0) \\ K(x_0, x_1) & K(x_1, x_1) & \cdots & K(x_{T-1}, x_1) \\ \vdots & \vdots & \ddots & \vdots \\ K(x_0, x_{T-1}) & K(x_1, x_{T-1}) & \cdots & K(x_{T-1}, x_{T-1}) \end{bmatrix}. \tag{7}$$

The following result gives a deterministic finite-sample error bound associated with (5).

Theorem 1: Consider a positive semidefinite symmetric reproducing kernel $K : \Omega \times \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}^n$ along with the associated RKHS $\mathcal{H}$. Let $f \in \mathcal{H}$ generate the data points (y_i, x_i), $i = 1, \ldots, T-1$, where $y_i = f(x_i)$. Then the interpolating function $s_f(x)$ in (5) provides an estimate of the function $f(x)$ for $x \in \Omega$ with interpolation error satisfying

$$|f(x) - s_f(x)| \leq \|f\|_{\mathcal{H}} \sqrt{K(x, x) - \mathbf{k}(x)^\top \hat{K}_X^{-1} \mathbf{k}(x)}, \quad \forall x \in \Omega. \tag{8}$$

where $\hat{K}_X := (\lambda I_T + K_X)(2\lambda I_T + K_X)^{-1}(\lambda I_T + K_X)$. □

III. MAIN RESULTS

Consider a discrete-time affine-input nonlinear system

$$x^{+} = f(x) + Bu \tag{9}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ is the control input, f is the drift vector field, and B is a constant matrix. Both f and B are considered unknown. We instead assume that $(x_e, u_e) = (0, 0)$ is a known unstable equilibrium point of the system, and the set Ω contains the origin. The objective is to design a feedback controller that stabilizes the dynamics around the origin.

As anticipated in the Introduction, we will consider an *indirect* method that consists of two steps: we first construct a kernel-based model of the system along with deterministic error bounds (Theorem 1). Then, we will derive a controller design method that explicitly accounts for the uncertainty around the nominal model. This method is inspired by [15] but presents some differences that will be discussed later on in the paper.

A. Kernel-based functions and error bounds

To derive a model of the system, we proceed in two steps. As a first step, we set the control input $u = 0$ and collect from the system a dataset

$$\mathbb{D} := \{x(k)\}_{k=0}^{T} \tag{10}$$

of samples satisfying $x(k+1) = f(x(k))$, $k = 0, \ldots, T-1$, with $T > 0$. We note that the samples can be computed from a single trajectory or from multiple trajectories of the system.

$$X_0 := \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix} \tag{11}$$

$$X_1 := \begin{bmatrix} x(1) & x(2) & \cdots & x(T) \end{bmatrix}. \tag{12}$$

Let now K denote a kernel function chosen by the designer. Given K and the dataset $\mathbb{D}$, let

$$\mathbf{k}(x) = \begin{bmatrix} K(x, x(0)) & K(x, x(1)) & \cdots & K(x, x(T-1)) \end{bmatrix}^{\top}. \tag{13}$$

The function $\mathbf{k}(x)$ represents the vector of basis functions that will generate the interpolation function $s_f(x)$.

To use Theorem 1, we need the following assumption.

Assumption 1: All the n components of f in (9) belong to the RKHS $\mathcal{H}$ associated to K. Moreover, an upper bound Γ_i for $\|f_i\|_{\mathcal{H}}$, $i = 1, \ldots, n$, is known. ■

Methods for estimating Γ_i are discussed in [25]. Here we just point out that the bound can be loose, although this may render the control design step more difficult. By solving (5), the interpolation function of $f(x)$ takes the form

$$s_f(x) = A\mathbf{k}(x) \tag{14}$$

where $A := X_1(\lambda I_T + K_{X_0})^{-1}$ and where the matrix K_{X_0} is as in (7) with X replaced by X_0. Let

$$d(x) := f(x) - s_f(x). \tag{15}$$

By (8), each component of the vector d thus satisfies

$$|d_i(x)| \le \|f_i\|_{\mathcal{H}} \sqrt{K(x, x) - \mathbf{k}(x)^{\top} \hat{K}_{X_0}^{-1} \mathbf{k}(x)}, \quad i = 1, \ldots, n, \quad \forall x \in \Omega, \tag{16}$$

with $\hat{K}_{X_0}$ as in Theorem 1 with X replaced by X_0. Hence, by letting $\Gamma := \begin{bmatrix} \Gamma_1 & \Gamma_2 & \cdots & \Gamma_n \end{bmatrix}^{\top}$ and defining

$$\delta(x) := |\Gamma| \sqrt{K(x, x) - \mathbf{k}(x)^{\top} \hat{K}_{X_0}^{-1} \mathbf{k}(x)}, \tag{17}$$

it follows from Assumption 1 that the interpolation error on the function f satisfies the deterministic bound

$$|d(x)| \le \delta(x), \quad \forall x \in \Omega. \tag{18}$$

B. Controller design method based on approximate nonlinearity cancellation

As a second step, we derive a control design method that exploits the bound on the interpolation error. By previous analysis, the dynamics (9) can be written as

$$x^{+} = A\mathbf{k}(x) + Bu + d(x) \tag{19}$$

where $A \in \mathbb{R}^{n \times T}$ is known and $B \in \mathbb{R}^{n \times m}$ is still unknown.

To determine the feedback controller, we make a second experiment on the system where we apply a nonzero input sequence u and collect a new dataset

$$\overline{\mathbb{D}} := \{\overline{x}(k), u(k)\}_{k=0}^{\overline{T}} \tag{20}$$

of samples satisfying $\overline{x}(k+1) = f(\overline{x}(k)) + Bu(k)$, where $k = 1, \cdots, \overline{T}$ and $\overline{T} > 0$. These data are grouped in the data matrices

$$\overline{X}_0 := \begin{bmatrix} \overline{x}(0) & \overline{x}(1) & \ldots & \overline{x}(\overline{T}-1) \end{bmatrix} \in \mathbb{R}^{n \times \overline{T}} \tag{21a}$$

$$\overline{X}_1 := \begin{bmatrix} \overline{x}(1) & \overline{x}(2) & \ldots & \overline{x}(\overline{T}) \end{bmatrix} \in \mathbb{R}^{n \times \overline{T}} \tag{21b}$$

$$U_0 := \begin{bmatrix} u(0) & u(1) & \ldots & u(\overline{T}-1) \end{bmatrix} \in \mathbb{R}^{m \times \overline{T}} \tag{21c}$$

$$K_0 := \begin{bmatrix} \mathbf{k}(\overline{x}(0)) & \mathbf{k}(\overline{x}(1)) & \ldots & \mathbf{k}(\overline{x}(\overline{T}-1)) \end{bmatrix} \in \mathbb{R}^{T \times \overline{T}} \tag{21d}$$

which satisfy the identity

$$\overline{X}_1 = AK_0 + BU_0 + D_0 \tag{22}$$

where $D_0 := \begin{bmatrix} d(\overline{x}(0)) & d(\overline{x}(1)) & \cdots & d(\overline{x}(\overline{T}-1)) \end{bmatrix}$ is the (unknown) data matrix of samples of d.

We assume that this second experiment is carried out with an input such that the corresponding matrix U_0 has full row rank. This can be interpreted as an excitation condition on the experiment. We will write this condition as an assumption but it is indeed a *design* condition.

Assumption 2: U_0 has full row rank. ■

By letting $\hat{X}_1 := \overline{X}_1 - AK_0$, we have $BU_0 = \hat{X}_1 - D_0$. Assumption 2 thus implies

$$B = (\hat{X}_1 - D_0) \underbrace{U_0^{\top}(U_0 U_0^{\top})^{-1}}_{=:U_0^{\dagger}} \tag{23}$$

and the dynamics can be written as

$$x^{+} = A\mathbf{k}(x) + (\hat{X}_1 - D_0)U_0^{\dagger} u + d(x). \tag{24}$$

Arrived at this point, note that the dynamics of $\mathbf{k}(x)$ depend on the selected kernel. We will consider the general case in which $\mathbf{k}(x)$ consists of both linear and nonlinear functions, so that $A\mathbf{k}(x)$ can be decomposed as $A\mathbf{k}(x) = \overline{A}x + \hat{A}\hat{\mathbf{k}}(x)$ with $\hat{\mathbf{k}} : \mathbb{R}^n \to \mathbb{R}^S$ that contains only nonlinear functions. The special case $\mathbf{k}(x) = x$, gives $\hat{A} = 0_{n\times S}$. In contrast, $\overline{A} = 0_{n\times n}$ when $\mathbf{k}(x)$ contains only nonlinear functions. Note that for a fixed $\mathbf{k}(x)$, the choice of $\hat{\mathbf{k}}(x)$ is not unique, and different choices of $\hat{\mathbf{k}}(x)$ generate different matrices $\overline{A}$ and $\hat{A}$. With this decomposition, (24) reads equivalently as

$$x^+ = \overline{A}x + \hat{A}\hat{\mathbf{k}}(x) + (\hat{X}_1 - D_0)U_0^\dagger u + d(x). \tag{25}$$

This decomposition suggests a control law in the form

$$u = \overline{K}x + \hat{K}\hat{\mathbf{k}}(x) \tag{26}$$

which gives the closed-loop dynamics

$$\begin{aligned} x^+ = &(\overline{A} + (\hat{X}_1 - D_0)U_0^\dagger \overline{K})x \\ &+(\hat{A} + (\hat{X}_1 - D_0)U_0^\dagger \hat{K})\hat{\mathbf{k}}(x) + d(x). \end{aligned} \tag{27}$$

A natural way to design the control law is then to design $\overline{K}$ so as to stabilize the linear part of the dynamics, and to design $\hat{K}$ so as to try to cancel out the nonlinear terms. This approach has been originally proposed in [15], and we refer the reader to it for a discussion regarding the connections between this approach and the classic feedback linearization. By Lyapunov theory, a necessary and sufficient condition for the linear dynamics $\dot{\xi} = (\overline{A} + (\hat{X}_1 - D_0)U_0^\dagger \overline{K})\xi$ to be stable is that for any $Q \succ 0$ there exists a matrix $S \succ 0$ that solves the Lyapunov equation

$$\begin{aligned} (\overline{A} + (\hat{X}_1 - D_0)U_0^\dagger \overline{K})^\top S(\overline{A} + (\hat{X}_1 - D_0)U_0^\dagger \overline{K}) \\ -S + SQS \preceq 0. \end{aligned} \tag{28}$$

Letting $P = S^{-1}$ and multiplying both sides by P, this turns out to be equivalent to

$$\begin{aligned} (\overline{A}P + (\hat{X}_1 - D_0)U_0^\dagger Y)^\top P^{-1}(\overline{A}P + (\hat{X}_1 - D_0)U_0^\dagger Y) \\ -P + Q \preceq 0 \end{aligned} \tag{29}$$

having set $Y = \overline{K}P$. As we will see, this form is particularly convenient because it can be expressed as a linear matrix inequality (LMI) constraint. However, we cannot implement directly (29) because D_0 is unknown. The idea is thus to ensure that the constraint is satisfied for all the matrices D in a given set $\mathcal{D}$ to which D_0 is known to belong, i.e.,

$$\begin{aligned} (\overline{A}P + (\hat{X}_1 - D)U_0^\dagger Y)^\top P^{-1}(\overline{A}P + (\hat{X}_1 - D)U_0^\dagger Y) \\ -P + Q \preceq 0 \quad \forall D \in \mathcal{D}. \end{aligned} \tag{30}$$

Let

$$\Delta := \left(\sum_{k=0}^{\overline{T}-1} \delta(\overline{x}(k))^2 I_n \right)^{1/2}. \tag{31}$$

Since $D_0 D_0^\top \preceq \Delta^2$, we can therefore solve (30) with respect to the set

$$\mathcal{D} := \{D \in \mathbb{R}^{n\times \overline{T}} : DD^\top \preceq \Delta^2\}. \tag{32}$$

Condition (30) cannot be implemented directly because it involves infinitely many constraints. The next result provides a tractable (and convex) condition for (30).

Lemma 1: Given $Q \succ 0$ and Δ defined in (31), if there exist $P \in \mathbb{S}^{n\times n}$, $Y \in \mathbb{R}^{m\times n}$ and a scalar $\epsilon > 0$ such that

$$\begin{bmatrix} P - Q & (\overline{A}P + \hat{X}_1 U_0^\dagger Y)^\top & (U_0^\dagger Y)^\top \\ \overline{A}P + \hat{X}_1 U_0^\dagger Y & P - \epsilon\Delta^2 & 0_{n\times\overline{T}} \\ U_0^\dagger Y & 0_{\overline{T}\times n} & \epsilon I_{\overline{T}} \end{bmatrix} \succeq 0 \tag{33}$$

then (30) holds. □

Condition (33) guarantees stability of the linear dynamics $\dot{\xi} = (\overline{A} + (\hat{X}_1 - D_0)U_0^\dagger \overline{K})\xi$ with $\overline{K} = YP^{-1}$. The remaining part of the controller, i.e., the matrix $\hat{K}$, can be determined so as to minimize the effect of the nonlinearities in the closed loop. Including the design of $\overline{K}$, a prototypical formulation is the following:

$$\text{minimize}_{P,Y,\hat{K},\epsilon} \quad \|\hat{A} + \hat{X}_1 U_0^\dagger \hat{K}\| + \alpha \|P\| \tag{34a}$$

$$\text{subject to} \quad (33) \tag{34b}$$

where $\alpha \geq 0$ is a design parameter. As shown, (33) ensures stability of the linear dynamics $\dot{\xi} = (\overline{A} + (\hat{X}_1 - D_0)U_0^\dagger \overline{K})\xi$. Instead, minimizing $\|\hat{A} + \hat{X}_1 U_0^\dagger \hat{K}\|$ tries to reduce as much as possible the effect of the nonlinearities in the closed loop. In this context, the term $\alpha\|P\|$ acts as a regularization term that permits to enlarge the estimate of the positive invariant set for the closed-loop dynamics, as detailed in the sequel. Before proceeding, we remark that (34) should be viewed as an example. An alternative is to explicitly account for D_0 for the nonlinear term as well:

$$\text{minimize}_{P,Y,\hat{K},\epsilon} \quad \|\hat{A} + (\hat{X}_1 - D)U_0^\dagger \hat{K}\| + \alpha \|P\| \tag{35a}$$

$$\text{subject to} \quad (33),\ D \in \mathcal{D}. \tag{35b}$$

Also this problem can be cast as a semidefinite program.

The rest of this section is devoted to show that this method guarantees the existence of a positively invariant set for the closed loop if the modelling error is sufficiently small.

Definition 2: For the system $x^+ = f(x)$, if for every $x(0) \in \mathcal{S}$, it holds that $x(t) \in \mathcal{S}$ for $t > 0$, then $\mathcal{S}$ is called a positively invariant (PI) set. ■

Let $V(x) = x^\top P^{-1} x$, which acts as a Lyapunov function for the linear part of the dynamics, and define for brevity $\Psi = \overline{A} + (\hat{X}_1 - D_0)U_0^\dagger \overline{K}$ and $\Xi = \hat{A} + (\hat{X}_1 - D_0)U_0^\dagger \hat{K}$. Then, the Lyapunov function satisfies

$$\begin{aligned} &V(x^+) - V(x) \\ &= (\Psi x + \Xi\,\hat{\mathbf{k}}(x) + d(x))^\top P^{-1}(\Psi x + \Xi\,\hat{\mathbf{k}}(x) + d(x)) \\ &\quad - x^\top P^{-1} x \end{aligned}$$

Bearing in mind the expressions of Ψ and Ξ, the fact that $D_0 D_0^\top \preceq \Delta^2$, and $|d(x)| \leq \delta(x)$, simple (although tedious) calculations give

$$V(x^+) - V(x) \leq l(x) + g(x, \delta(x)) \tag{36}$$

where

$$
\begin{aligned}
l(x) &:= -x^\top P^{-1} Q P^{-1} x + l_1(x) + l_2(x) + l_3(x) + l_4(x) \\
l_1(x) &:= (2(\overline{A} + \hat{X}_1 U_0^\dagger \overline{K})x \\
&\quad + (\hat{A} + \hat{X}_1 U_0^\dagger \hat{K})\hat{\mathbf{k}}(x))^\top P^{-1} (\hat{A} + \hat{X}_1 U_0^\dagger \hat{K})\hat{\mathbf{k}}(x) \\
l_2(x) &:= \|\Delta\| |(2(\overline{A} + \hat{X}_1 U_0^\dagger \overline{K})x \\
&\quad + (\hat{A} + \hat{X}_1 U_0^\dagger \hat{K})\hat{\mathbf{k}}(x))^\top P^{-1} || U_0^\dagger \hat{K} \hat{\mathbf{k}}(x)| \\
l_3(x) &:= \|\Delta\| |2 U_0^\dagger \overline{K} x + U_0^\dagger \hat{K} \hat{\mathbf{k}}(x)| | P^{-1} (\hat{A} + \hat{X}_1 U_0^\dagger \hat{K}) \hat{\mathbf{k}}(x)| \\
l_4(x) &:= \|\Delta\|^2 \|P^{-1}\| |2 U_0^\dagger \overline{K} x + U_0^\dagger \hat{K} \hat{\mathbf{k}}(x)| | U_0^\dagger \hat{K} \hat{\mathbf{k}}(x)| \\
g(x, \delta(x)) &:= r_1(x)\delta(x) + r_2(x)\delta(x) + r_3 \delta(x)^2 \\
r_1(x) &:= 2|((\overline{A} + \hat{X}_1 U_0^\dagger \overline{K})x + (\hat{A} + \hat{X}_1 U_0^\dagger \hat{K})\hat{\mathbf{k}}(x))^\top P^{-1}| \\
r_2(x) &:= 2\|\Delta\| \|P^{-1}\| |U_0^\dagger \overline{K} x + U_0^\dagger \hat{K} \hat{\mathbf{k}}(x)| \\
r_3 &:= \|P^{-1}\|.
\end{aligned}
$$

Let $\mathcal{X} := \{x : l(x) + g(x, \delta(x)) \leq 0\}$ and let $\mathcal{X}^c$ be its complement. Let $\mathcal{R}_\gamma := \{x : V(x) \leq \gamma\}$, where $\gamma > 0$ is arbitrary, and define $\mathcal{Z} := \mathcal{R}_\gamma \cap \mathcal{X}^c$, which characterizes all the points in $\mathcal{R}_\gamma$ for which the Lyapunov difference $V(x^+) - V(x)$ can be positive. Then the following main result holds.

Theorem 2: Consider a nonlinear system as in (19), and suppose that (34) is feasible with a given $Q \succ 0$ and where Δ is defined in (31). Consider the closed-loop system with the controller (26) obtained from (34). If

$$
V(x) + l(x) + g(x, \delta(x)) \leq \gamma \quad \forall x \in \mathcal{Z} \tag{38}
$$

then $\mathcal{R}_\gamma$ is a PI set for the closed-loop system. □

We close this section with remarks regarding the comparison with [15]. In [15, Th. 8], a similar result is given that takes unmodeled dynamics into account. In this respect, the results presented here give a systematic principled method for bounding modelling errors. [15, Th. 7] also shows that asymptotic stability follows when the error bound $\delta(x)$ satisfies $\lim_{|x|\to 0} \frac{\delta(x)}{|x|} = 0$, e.g. when $\delta(x)$ acts as remainder in a power series expansion of f about 0. The same result holds also here but we have to bear in mind that the condition $\lim_{|x|\to 0} \frac{\delta(x)}{|x|} = 0$ may fail to hold depending on the choice of the kernel function. In any case, invariance sets provide a safe region where we can perform additional experiments to estimate regions of attraction.

Another remark concerns the experimental conditions. We have assumed noise-free data, but bounds similar to the one in (8) can be given also in case of noisy data [25] and, combined with robust control design tools (*cf.* [15, Sec. VI]), can be used to extend the results of this paper.

IV. Numerical example

Consider the nonlinear system

$$
x_1^+ = x_2 + x_1^3 + u \tag{39a}
$$

$$
x_2^+ = 0.5x_1 + 0.2x_2^2. \tag{39b}
$$

We consider a polynomial kernel of the degree 3:

$$
K(x, y) := x^\top y + (x^\top y)^2 + (x^\top y)^3 \tag{40}
$$

on the domain $\Omega = [-20, 20] \times [-20, 20] \subset \mathbb{R}^2$. We set $u = 0$ and collect a dataset $\mathbb{D}$ containing $T = 10$ samples by performing multiple one-step experiments with initial states uniformly distributed in $[-2, 2]$. With these data we construct the vector $\mathbf{k}(x)$ of basis functions. The kernel $K(x, y)$ is symmetric positive semidefinite and there exists a unique RKHS $\mathcal{H}$ that admits $K(x, y)$ as a reproducing kernel by Fact 1. We just need to show that the nonlinear dynamics $f_1(x) = x_2 + x_1^3$ and $f_2(x) = 0.5x_1 + 0.2x_2^2$ in (39) are members of $\mathcal{H}$. By Definition 1, all of the components of $\mathbf{k}(x)$ belong to $\mathcal{H}$. Then, it is sufficient to show that $f_1(x)$ and $f_2(x)$ are linear combinations of $\mathbf{k}(x)$. Denote by $M(x)$ the vector of all monomials up to degree 3. We can write $f_1(x) = c_1 M(x)$, $f_2(x) = c_2 M(x)$ and $\mathbf{k}(x) = M_\mathbf{k} M(x)$. Note that when the matrix $M_\mathbf{k}$ has full column rank, there exists α_i such that $c_i = \alpha_i M_\mathbf{k}$, $i = 1, 2$, and this implies that $f_1(x)$ and $f_2(x)$ can be written as the linear combinations of $\mathbf{k}(x)$. Hence, the collected data in $\mathbb{D}$ should satisfy the condition that the corresponding matrix $M_\mathbf{k}$ is full column rank, and this condition is indeed satisfied for the collected samples. Finally, in order to find an upper bound Γ on $\|f\|_\mathcal{H}$ as in Assumption 1, we compute $\|f\|_\mathcal{H}$ explicitly. By (1), we have $\|f_1\|_\mathcal{H} = \alpha_1 K_{X_0} \alpha_1^\top = 2$ and $\|f_2\|_\mathcal{H} = \alpha_2 K_{X_0} \alpha_2^\top = 0.29$. For controller design we select $\Gamma_1 = 3$ and $\Gamma_2 = 0.4$, which over-approximate the true values by more than 30%. Finally, we select $\lambda = 10^{-7}$. We note that large values of λ results in large bounds $\delta(x)$ (Theorem 1), and this may eventually render the controller design program infeasible.

Next, we collect a dataset $\overline{\mathbb{D}}$ containing $\overline{T} = 10$ samples by performing again multiple one-step experiments with input uniformly distributed in $[-0.5, 0.5]$, and with initial states within $[-2, 2]$. With these data, we compute the two matrices K_0 and $U_0^\dagger$ as in (21d) and (23), respectively. Note that the first term of $K(x, y)$, i.e. $x^\top y$, produces the linear part of $A\mathbf{k}(x)$, and gives

$$
\begin{aligned}
\overline{A}x &= A \begin{bmatrix} x^\top x(0) & x^\top x(1) & \cdots & x^\top x(T-1) \end{bmatrix}^\top \\
&= A \begin{bmatrix} x(0) & x(1) & \cdots & x(T-1) \end{bmatrix}^\top x \\
&= A X_0^\top x,
\end{aligned}
$$

and thus $\overline{A} = A X_0^\top$. In addition, we set

$$
\begin{aligned}
\hat{\mathbf{k}}(x) := [&(x^\top x(0))^2 + (x^\top x(0))^3 \;\; (x^\top x(1))^2 + (x^\top x(1))^3 \\
&\cdots \;\; (x^\top x(T-1))^2 + (x^\top x(T-1))^3]^\top
\end{aligned}
$$

and thus $\hat{A} = A$. We solve (34) with $Q = I_2$, and $\alpha = 1$. For the dynamics not depending on D_0 in (27), we obtain

$$
(\overline{A} + \hat{X}_1 U_0^\dagger \overline{K})x + (\hat{A} + \hat{X}_1 U_0^\dagger \hat{K})\hat{\mathbf{k}}(x) = \begin{bmatrix} 0.2481x_2 \\ 0.5x_1 + 0.2x_2^2 \end{bmatrix}
$$

We note that the program (34) correctly forces u to cancel out the nonlinearity in (39a).

For the obtained controller, we numerically determine the set $\mathcal{X} = \{\xi : l(\xi) + g(\xi, \delta(\xi)) \leq 0\}$. Any sub-level set $\mathcal{R}_\gamma$ of the Lyapunov function $V(x) = x^\top P^{-1} x$ contained in $\mathcal{X} \cup \{0\}$ and satisfying (38) gives an estimate of the PI set for the closed-loop system. The set $\mathcal{X}$ and a sublevel set of V are shown in Figure 1. We can numerically verify that the

PI set in Figure 1 is also a region of attraction (ROA), and one possible reason is that both $\hat{\mathbf{k}}(x)$ and $\delta(x)$ converge to 0 when x converges to 0 since we use the polynomial kernel $K(x, y)$. Remarkably, the obtained estimate of the ROA is almost the same as the one obtained in [15] with knowledge of the true basis functions.

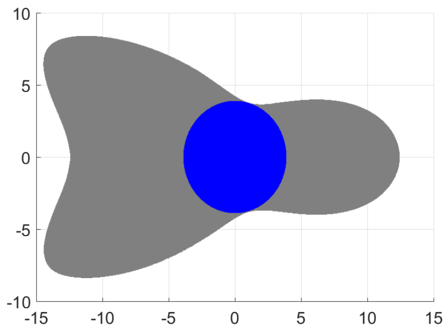

Fig. 1. The grey set represents the set $\mathcal{X}$, while the blue set is the PI set $\mathcal{R}_\gamma$; here, $P = \begin{bmatrix} 1.3350 & 0 \\ 0 & 1.3350 \end{bmatrix}$ and $\gamma = 11.5$. We observe that the set $\mathcal{Z}$ is empty and hence the PI set also provides an estimate of the ROA for the closed-loop system.

V. Conclusions

We have investigated the problem of designing feedback controllers for affine-input nonlinear systems from data using kernel learning techniques. We have considered a method in which a nominal model of the system is determined using kernel-based functions, along with an explicit upper bound on the modelling error. Then, a controller design method is proposed that involves the solution of a semidefinite program. We have shown that the method ensures, despite the presence of unmodeled dynamics, the existence of positively invariant sets for the closed-loop dynamics. An important venue for future research is the problem of understanding what kernels are more suited for control goals.

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