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Propagation of Stubborn Opinions on Signed Graphs

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Abstract— **This paper addresses the problem of propagation of opinions in a Signed Friedkin-Johnsen (SFJ) model, i.e., an opinion dynamics model in which the agents are stubborn and the interaction graph is signed. We provide sufficient conditions for the stability of the SFJ model and for convergence to consensus of a concatenation of such SFJ models.**

I. Introduction

Opinion dynamics models try to describe the way in which the opinions and beliefs of individuals within a group or a community influence and are influenced by the opinions and beliefs of others [1], [2]. This process plays a crucial role in shaping societal attitudes and behaviors, and has been the subject of much research in fields such as sociology [3], psychology [4], and political science [5]. The state variables that represent the opinions of the agents are modified in response to information shared with nearby agents in a network. The most basic updating rule is given by the DeGroot model [6], in which the agents modify their states according to a weighted average of their neighbors' opinions and thereby achieve consensus.

The Friedkin-Johnsen (FJ) model [7] extends the DeGroot model by including the influence of the agents' stubbornness on the dynamics, based on their initial opinions [8]. Almost all existing works on the FJ model consider a collaborative behavior between the individuals in the social network [9], in which the opinions of the agents get closer to each other even though they do not reach a consensus. A recent research direction for these collaborative FJ models consists of concatenating a series of discussions events, each of them represented as a FJ model [5], [10], [11], using a two time-scale framework similar to the one introduced in [12]. Concatenation refers to the fact that the endpoint of a discussion becomes the initial condition of the next discussion. In this way, the opinions of the agents get progressively closer even in presence of a persistent stubbornness, and eventually converge to a consensus point.

The main contribution of this paper is to extend the FJ model, in both "single discussion" FJ and concatenated FJ versions, to signed graphs, i.e., to opinion dynamics models in which collaboration and antagonism among the agents are coexisting. Signed networks, i.e., graphs in which the edges between nodes are assigned a positive or negative weight according to the nature of the relationship between the corresponding agents, have gained significant attention in recent years, due to their potential to capture more nuanced social and information dynamics than traditional unsigned networks [13]–[17]. There are essentially two ways to construct Laplacian-based dynamics on a signed network. In [16], the two signed Laplacians are termed "opposing" and "repelling", and differ in the way the diagonal elements of the Laplacian are computed: in the former, we put on the diagonal the sum of the absolute values of the row elements, while in the latter we sum the row values with their signs [17]. Similar "opposing" and "repelling" adjacency matrices can be considered as replacements for stochastic matrices when dealing with discrete-time opinion dynamics models. In this study, we concentrate on the "repelling" Laplacian, which has always zero as an eigenvalue but may not be stable [18], or may be stable but may fail to converge [17]. The FJ model on signed graphs we introduce in this paper is henceforth denoted *Signed FJ* (SFJ), and its concatenated counterpart *concatenated SFJ*. Notice that the papers [19], [20] also consider a concatenated FJ-type model for signed graphs, but only for the "opposing" Laplacian and only in the structurally balanced case. Our model is much more challenging to analyze and its behavior is richer (the structurally balanced case becomes a special case).

To analyze the behavior of our SFJ models, we use the tools developed in [17], in particular the notions of Eventually Stochastic (ES) matrices (for discrete-time SFJ) and Eventually Exponentially Positive (EEP) Laplacians (for continuous-time SFJ). By itself, choosing an ES matrix (or an EEP Laplacian) is however not enough to guarantee stability of the SFJ, as easily shown in counterexamples. We show in the paper that if we add the assumption of normality of the ES signed interaction matrix (or normality of the EEP signed Laplacian) the SFJ model becomes globally asymptotically stable, and hence a unique equilibrium point always exists for it. Furthermore, the same condition is sufficient to guarantee that the concatenated SFJ model converges to consensus over an infinite number of discussion events.

The paper is organized as follows: Section II introduces preliminary concepts on signed graphs and matrix theory; Section III introduces our SFJ models in continuous and discrete time, and reviews the properties of ES and EEP matrices. Section IV provides sufficient conditions for asymptotic stability of the SFJ models, and Section V extends the results for reaching a consensus in the concatenated SFJ models.

II. Preliminary material

Notations. $\mathbb{R}$, $\mathbb{R}_{\geq 0}$ corresponds to the real number set, and non-negative real number set, respectively. Real numbers are

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denoted by lowercase letters $a, b, c, \dots$ while bold lowercase letters $\mathbf{x}, \mathbf{y}, \mathbf{z}, \dots$ represent vectors of length n in $\mathbb{R}^n$. $\mathbf{0}$ and $\mathbb{1}$ are vectors of entries 0 and 1, respectively. Matrices are represented by capital Latin or Greek letters $X, Y, Z, \Theta, \dots$. Given a square matrix $Q = [Q_{ij}] \in \mathbb{R}^{n \times n}$, Q^T depicts its transpose, Q^k is its k-th power and $\ker(\mathrm{Q})$ its kernel. For a vector $\mathbf{x} \in \mathbb{R}^n$, the diagonal matrix $\mathrm{diag}(\mathbf{x}) \in \mathbb{R}^{n \times n}$ has the entries of $\mathbf{x}$ on the diagonal. The eigenvalues of the matrix Q are denoted by $\lambda(Q)$, and the spectrum as $\Lambda(Q) = \{\lambda_1(Q), \dots, \lambda_n(Q)\}$. The spectral radius and spectral abscissa of a matrix Q are defined as $\rho(Q) = \max_{i=1,\dots,n} |\lambda_i(Q)|$ and $\mu(Q) = \max_{i=1,\dots,n} \Re(\lambda_i(Q))$ where $\Re(\cdot)$ denotes the real part of the complex number. The matrix I represents an identity matrix of appropriate dimensions.

A. Signed Graphs

A directed graph (digraph) with n vertices $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ and an edge set $\mathcal{E}$ is represented by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ where an edge pair $(k, l) \in \mathcal{E}$ denotes a link from vertex v_k to v_l and the weighted adjacency matrix A represents the interaction pattern in the graph with $A_{kl} = 0$ if $(l, k) \notin \mathcal{E}$. For signed graphs, the values of A_{kl} can be positive or negative depending on the relationship between the vertices ("friends" or "enemies"). For an undirected graph, we have $A^T = A$, while the digraph is called weight balanced if $A\mathbb{1} = A^T\mathbb{1}$.

For a signed graph, the weighted in-degree and out-degree vectors are defined as $\sigma_{\text{in}} = \left[\sum_{j=1, i \neq j}^{n} A_{ij}\right]_{n \times 1} \in \mathbb{R}^n$ and $\sigma_{\text{out}} = \left[\sum_{i=1, i \neq j}^{n} A_{ij}\right]_{n \times 1} \in \mathbb{R}^n$. The "repelling Laplacian" [16] is defined as $L = \Sigma_{\text{in}} - A$ where $\Sigma_{\text{in}} = \mathrm{diag}(\sigma_{\text{in}})$. Notice that by construction $L\mathbb{1} = \mathbf{0}$.

A path in the graph is defined as pairs of edges $(v_k, v_1), (v_1, v_2), \dots, (v_{r-1}, v_r), (v_r, v_l)$ such that $A_{kr} A_{r(r-1)} \dots A_{21} A_{1k} \neq 0$. If vertex i can reach any other vertex in the graph via a directed path, then a directed spanning tree rooted at vertex i exists. If every pair of vertices in the graph are connected through a directed path, then the graph is said strongly connected.

B. Matrix Theory

If the eigenvalues of a square matrix Q have $\Re(\lambda(Q)) < 0$ (resp. $|\lambda(Q)| < 1$), then the matrix is called Hurwitz (resp. Schur) stable. It is marginally stable if $\Re(\lambda(Q)) \leq 0$ (resp. $|\lambda(Q)| \leq 1$), and the eigenvalues such that $\Re(\lambda(Q)) = 0$ (resp. $|\lambda(Q)| = 1$) are simple eigenvalues. If there exists a permutation matrix P such that $P^T A P = \begin{bmatrix} Q_1 & Q_2 \\ 0 & Q_3 \end{bmatrix}$ where Q_1 and Q_3 are non-trivial square matrices, then the matrix A is said to be reducible. It is irreducible if it is not reducible. A graph $\mathcal{G}$ is strongly connected if and only if the associated adjacency matrix A is irreducible.

A matrix Q which has all its entries $Q_{kl} > 0$ is said positive and denoted $Q > 0$; it is said nonnegative if $Q_{kl} \geq 0$ and it is denoted $Q \geq 0$. The corank of a matrix Q is defined as the dimension of the kernel space of Q, $\ker(\mathrm{Q})$. The matrix Q is normal if $QQ^T = Q^T Q$. Positive definiteness (resp. positive semi definiteness) of a matrix Q is defined as $\mathbf{x}^T Q_{sym} \mathbf{x} > 0$ (resp. $\mathbf{x}^T Q_{sym} \mathbf{x} \geq 0$) for all $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$ (resp. $\mathbf{x} \in \mathbb{R}^n$) where $Q_{sym} = (Q + Q^T)/2$, and it is denoted $Q \succ 0$ (resp. $Q \succeq 0$). Negative definiteness (resp. semi-definiteness) is denoted $Q \prec 0$ (resp. $Q \preceq 0$) and obviously corresponds to $-Q \succ 0$ (resp. $-Q \succeq 0$).

C. Perron-Frobenious property and Eventual Positivity

The following definitions and properties can be found in [17].

Definition 1 ***(Perron-Frobenius property)*** *The matrix $Q \in \mathbb{R}^{n \times n}$ satisfies the Perron-Frobenius (PF) property (denoted $Q \in \mathcal{PF}$), if $\rho(Q)$ is a simple real positive eigenvalue of Q such that $\rho(Q) > |\lambda(Q)|$ for all $\lambda \in \Lambda(Q)$, $\lambda \neq \rho(Q)$, and the corresponding right eigenvector is positive.*

Definition 2 ***(Eventually Positive)*** *The matrix $Q \in \mathbb{R}^{n \times n}$ is said Eventually Positive (EP) if $\exists$ an integer k_o such that $Q^k > 0$ for $k \geq k_o$, and it is denoted $Q \overset{\vee}{>} 0$.*

Lemma 1 *[21] For a matrix $Q \in \mathbb{R}^{n \times n}$, the following properties are equivalent (i) Q, $Q^T \in \mathcal{PF}$; (ii) $Q \overset{\vee}{>} 0$; (iii) $Q^T \overset{\vee}{>} 0$.*

Definition 3 ***(Eventually Exponentially Positive)*** *The matrix $Q \in \mathbb{R}^{n \times n}$ is said Eventually Exponentially Positive (EEP), if $\exists\, d \in \mathbb{R}_{\geq 0}$, such that $Q + dI \overset{\vee}{>} 0$.*

An equivalent characterization of EEP is that $e^{Qt} > 0, \forall\, t > t_o$, for some $t_o \in \mathbb{R}_{\geq 0}$.

D. Eventually Stochastic Matrices

Definition 4 ***(Eventually Stochastic)*** *The matrix $Q \in \mathbb{R}^{n \times n}$ is said Eventually Stochastic (ES) if $Q \overset{\vee}{>} 0$ and $Q\mathbb{1} = \mathbb{1}$. If in addition it is $\mathbb{1}^T Q = \mathbb{1}^T$, then Q is said Eventually Doubly Stochastic (EDS).*

Lemma 2 *[22] For a ES matrix Q, $\rho(Q) = 1$ is a simple and strictly dominant eigenvalue, and the right and left eigenvectors $\mathbf{v}$ and $\mathbf{w}$ corresponding to the eigenvalue $\rho(Q)$ are positive.*

III. Problem Formulation

In this section, we formulate the SFJ model for both discrete and continuous time.

A. Discrete-time SFJ

According to [23], in discrete time (DT), the Friedkin-Johnsen model is defined as the convex combination:

$$\mathbf{x}(t+1) = ((I - \Theta) W) \mathbf{x}(t) + \Theta \mathbf{x}(0) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ defines the state ("opinion") of n agents, $\Theta = \mathrm{diag}(\theta_1, \dots, \theta_n)$ is the diagonal matrix containing the stubbornness coefficients $\theta_i \in [0, 1)$ of the agents (i.e. their "attachment" to their initial opinions $\mathbf{x}(0)$), and W is the weighted adjacency matrix representing the communication graph $\mathcal{G}$ of the agents. In this work we assume that the agents can have antagonistic interaction, and hence that W is a signed weighted adjacency matrix.

In the case that no agent is stubborn, i.e., when $\Theta = 0_{n\times n}$, the model (1) becomes

$$\mathbf{x}(t+1) = W\mathbf{x}(t) \tag{2}$$

and we want this special case to behave as a DeGroot model, namely to achieve consensus. In order to obtain this for signed graphs, we need to impose conditions on W. In particular, we will typically work with W which is a ES matrix, as per Definition 4. In fact, for ES matrices, we can recall the following properties, from [17].

Lemma 3 ([17], Lemma 7, Theorem 6, and Corollary 3) *Consider the DT system* (2)*, where the signed matrix W is such that $W\mathbb{1} = \mathbb{1}$. Consider the following conditions:*

(i) *The system* (2) *achieves consensus;*
(ii) *W is marginally Schur stable with a simple and strictly dominant eigenvalue $\rho(W) = 1$;*
(iii) *W is ES.*

Condition (i) and (ii) are equivalent. Conditions (iii) implies conditions (i) and (ii) but not viceversa. However, if in addition W is weight balanced, then all three conditions are equivalent (and W is EDS). Furthermore, if W is normal, then W is EDS and $I - W^TW \succeq 0$ of corank 1.

B. Continuous-time SFJ

A signed Friedkin-Johnsen model can be built also in continuous-time (CT), for instance as follows:

$$\dot{\mathbf{x}}(t) = -\left((I-\Theta)L + \Theta\right)\mathbf{x}(t) + \Theta\mathbf{x}(0) \tag{3}$$

where $L = \Sigma_{\text{in}} - A$ is the signed Laplacian associated with the adjacency matrix A of the signed graph $\mathcal{G}$ and $\Theta = \text{diag}(\theta_1, \dots, \theta_n)$ represents again the stubbornness matrix. Also in this case, the requirement on L is that when $\Theta = 0$ the resulting system

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t) \tag{4}$$

achieves consensus. For signed graphs, this is achieved when $-L$ is an EEP matrix. The equivalent of Lemma 3 for the CT case can be found in [17] (in particular, see Lemma 4, Theorem 4, and Corollary 1 of [17]).

IV. Convergence in SFJ models

In this section we study the asymptotic stability of both models (1) and (3). In both cases, we need the assumption that all agents are partially stubborn, i.e., that their stubbornness coefficients are positive (perhaps small) and always less than 1. In other words, we exclude both cases of completely non-stubborn and of totally stubborn agents.

Assumption 1 *Every agent in the network is partially stubborn, i.e. $\theta_i \in (0,1)\ \forall i = 1,\dots,n$.*

The assumption leads to a stubbornness matrix Θ which is invertible.

A. Discrete Time Case

Theorem 1 *Consider the DT SFJ model* (1)*, with W a ES and normal matrix. Under Assumption 1, then it is $\rho\left((I-\Theta)W\right) < 1$, and the system* (1) *converges to the equilibrium point $\mathbf{x}^* = P_W\mathbf{x}(0)$, where $P_W = \left(I - (I-\Theta)W\right)^{-1}\Theta$, with $P_W\mathbb{1} = \mathbb{1}$.*

Proof. The stability analysis of the affine system (1) and of the linear system obtained from (1) applying the change of basis $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{x}^*$ are the same whenever $\mathbf{x}^*$ exists uniquely, i.e, when $\rho((I-\Theta)W) < 1$. Hence this condition can be checked on the system $\mathbf{z}(t+1) = (I-\Theta)W\mathbf{z}(t)$, and for it the Lyapunov function $V(\mathbf{z}(t)) = \mathbf{z}^T(t)(I-\Theta)^{-2}\mathbf{z}(t)$, leads to

$$\Delta V = \mathbf{z}^T(t)\left(W^TW - (I-\Theta)^{-2}\right)\mathbf{z}(t).$$

When $\theta_i < 1\,\forall i$, the following Neumann series is converging

$$(I-\Theta)^{-2} = \left(\sum_{k=0}^{\infty}\Theta^k\right)^2 = I + \sum_{k=1}^{\infty}\Theta^k\sum_{k=0}^{\infty}\Theta^k + \sum_{k=1}^{\infty}\Theta^k.$$

So, we get

$$\begin{aligned}\Delta V = &-\mathbf{z}^T(t)\left(I - W^TW\right)\mathbf{z}(t)\\ &-\mathbf{z}^T(t)\left(\sum_{k=1}^{\infty}\Theta^k\sum_{k=0}^{\infty}\Theta^k + \sum_{k=1}^{\infty}\Theta^k\right)\mathbf{z}(t).\end{aligned}$$

From Lemma 3, normality of W implies $\left(I - W^TW\right) \succeq 0$, which, together with $\sum_{k=1}^{\infty}\Theta^k \succ 0$, leads to $\Delta V < 0$, i.e., $\rho((I-\Theta)W) < 1$. Consequently, $\mathbf{z}(t) \xrightarrow{t\to\infty} \mathbf{0}$ and $\mathbf{x}(t) \xrightarrow{t\to\infty} \mathbf{x}^*$. The rest of the proof is straightforward. ■

Remark 1 The normality condition on W is also needed in Theorem 1, as the following counterexample (Example 1) shows. In the proof of Theorem 1, positive semidefinitness of the symmetric part of W (i.e., $\left(I - W^TW\right) \succeq 0$ for $\rho(W) \leq 1$), which requires normality of W, is used in the calculation of ΔV. Normality of W is a sufficient but not necessary condition for stability of the SFJ model, see Example 2.

Example 1 The following signed matrix

$$W = \begin{bmatrix} 0.6683 & -0.5264 & 0.2627 & 0.5954\\ 0.0580 & 0 & 0.3073 & 0.6347\\ 0.1093 & 0.1965 & 0.9058 & -0.2115\\ 0.5888 & -0.4422 & 0.1360 & 0.7173\end{bmatrix}$$

is ES but not normal. If we choose the stubbornness values $\Theta = \text{diag}(0.0587, 0.4962, 0.3003, 0.0877)$, it results in $\rho\left((I-\Theta)W\right) = 1.0261 > 1$, i.e., the system (1) becomes unstable.

Example 2 The matrix

$$W = \begin{bmatrix} 0 & 0.6154 & -0.1787 & 0.5633\\ -0.1603 & 0.7406 & 0.4073 & 0.0124\\ 0.7207 & 0.0273 & 0.4964 & -0.2443\\ 0.7157 & -0.0338 & -0.3664 & 0.6845\end{bmatrix}$$

is ES but not normal. With stubbornness e.g. $\Theta = \mathrm{diag}(0.6355, 0.7138, 0.9971, 0.1395)$, it is $\rho\left((I-\Theta)\,W\right) = 0.7543 < 1$. Hence the SFJ model is stable.

Remark 2 One of the properties of a FJ model on $\mathcal{G}$ nonnegative is that $\mathbf{x}^*$ belongs to the convex hull of the initial conditions: $\mathbf{x}_i^* \in \mathrm{co}(\mathbf{x}(0)) = [\min_i \mathbf{x}_i(0), \max_i \mathbf{x}_i(0)]$. The following example shows that this is no longer true for SFJ models, even when convergence is guaranteed, i.e., when $\rho\left((I-\Theta)\,W\right) < 1$.

Example 3 The following signed matrix

$$W = \begin{bmatrix} 0.8219 & -0.0173 & -0.0862 & 0.2816 \\ -0.0173 & 0.9242 & -0.0147 & 0.1079 \\ 0.2816 & 0.1079 & 0.7708 & -0.1603 \\ -0.0862 & -0.0147 & 0.3301 & 0.7708 \end{bmatrix}$$

is ES and normal. Choosing the stubbornness values $\Theta = \mathrm{diag}(0.9575, 0.9649, 0.1576, 0.9706)$ results in $\rho\left((I-\Theta)\,W\right) = 0.6458 < 1$, i.e., the system (1) is stable. Also any other combination of Θ results in $\rho\left((I-\Theta)\,W\right) < 1$ (Theorem 1 applies). Figure 1 shows a simulation for the system dynamics with initial conditions $\mathbf{x}(0) = [0.4640, 0.1014, 0.5177, -0.8688]$, where we can notice that the state of agent $\mathbf{x}_3$ leaves the convex hull.

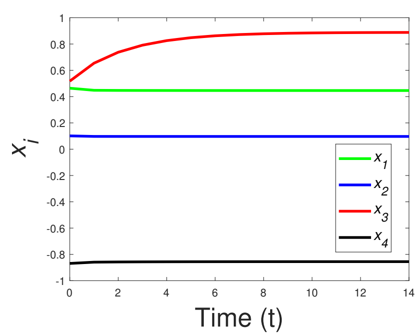

Fig. 1. A simulation for the SFJ model of Example 3.

When normality is missing, convergence of the SFJ model, i.e., $\rho\left((I-\Theta)\,W\right) < 1$, does not guarantee that $P_W = \left(I-(I-\Theta)\,W\right)^{-1}\Theta$ has $\rho(P_W) \leq 1$. For instance, in Example 2, it is $\rho(P_W) = 1.0041$. Also in this case, obviously, $\mathbf{x}^* \notin \mathrm{co}(\mathbf{x}(0))$, i.e., the SFJ model is not contracting in opinion space. More importantly, $\rho(P_W) > 1$ implies that P_W cannot be a ES matrix. The following theorem shows that normality of W is a sufficient condition for $\rho(P_W) = 1$.

Theorem 2 *Consider the DT SFJ model* (1)*, with W a ES and normal matrix. Under Assumption 1, $P_W = \left(I-(I-\Theta)\,W\right)^{-1}\Theta$ is a ES matrix. The left eigenvector $\mathbf{w} > 0$ associated to $\rho(P_W) = 1$ has entries $\mathbf{w}_i = \theta_i/\left(1-\theta_i\right)$.*

In order to prove this theorem, we need the following lemma.

Lemma 4 *Consider W ES and Θ such that Assumption 1 is obeyed. Whenever $\left(I-(I-\Theta)\,W\right)^{-1}$ exists, then the condition $\rho(P_W) \leq 1$, where $P_W = \left(I-(I-\Theta)\,W\right)^{-1}\Theta$, is equivalent to $(I-\Theta^{-1})(I-W)$ having all eigenvalues outside (or on the boundary of) the unit disk centered at 1:*

$$\left|\lambda\left(I-(I-\Theta^{-1})(I-W)\right)\right| \geq 1$$

Proof. Under the assumptions of the Lemma, Θ is invertible and $(I-(I-\Theta)W)^{-1}$ exists, and we can write

$$\begin{aligned} P_W &= \left(I-(I-\Theta)\,W\right)^{-1}\Theta \\ &= \left(\Theta^{-1}\left(I-(I-\Theta)W\right)\right)^{-1} \\ &= \left(\Theta^{-1}-\Theta^{-1}W+W\right)^{-1} \\ &= \left(W+\Theta^{-1}(I-W)\right)^{-1}. \end{aligned}$$

Hence $\rho(P_W) \leq 1$ is equivalent to $\left|\lambda\left(\left(W+\Theta^{-1}(I-W)\right)^{-1}\right)\right| \leq 1$ for all eigenvalues of $\left(W+\Theta^{-1}(I-W)\right)^{-1}$, which corresponds to say that for all eigenvalues of $W+\Theta^{-1}(I-W)$ it is $\left|\lambda(W+\Theta^{-1}(I-W))\right| \geq 1$. Some further manipulations give the following eigenvalue localization condition: $\left|\lambda(I-(I-\Theta^{-1})(I-W))\right| \geq 1$ which corresponds to say that the eigenvalues of $(I-\Theta^{-1})(I-W)$ cannot be in the interior of the unit disc centered at 1 in $\mathbb{C}$. ∎

Example 4 For Example 1, we see that $H = (I-\Theta^{-1})(I-W)$ has the eigenvalue $\lambda(H) = 0.4154$ inside the unit disk centered at 1. Example 2 instead has $\lambda(H) = 0.0042 \pm 0.0165i$. Lastly, for Example 3 it is $\Re\left(\lambda(H)\right) \leq 0$.

Proof of Theorem 2 Let us first show that under the assumptions of the theorem $I-(I-\Theta)W$ is an invertible matrix. From Theorem 1, $\rho((I-\Theta)W) < 1$, hence $I-(I-\Theta)W$ has all eigenvalues in the open right half plane: $\Re(\lambda(I-(I-\Theta)W)) > 0$. Therefore, Lemma 4 is applicable. From Lemma 4, showing that $\rho(P_W) \leq 1$ is equivalent to showing that $(I-\Theta^{-1})(I-W)$ has all eigenvalues outside (or on the boundary of) the unit disk centered at 1. This can be achieved for instance by imposing the more conservative criterion that $(I-\Theta^{-1})(I-W)$ has all eigenvalues in the left half plane (possibly on the imaginary axis). From W ES and normal, it follows that also $I-W \succeq 0$ is normal. Denote $D = -(I-\Theta^{-1})$ and $F = I-W$. Note that since $0 < \theta_i < 1$ for all i, it is $D \geq 0$ and $D \succ 0$. Then we can write $F = F_s + F_a$ where F_s and F_a are the symmetric and antisymmetric part of F: $F_s = (F+F^T)/2$ and $F_a = (F-F^T)/2$. We need to show that the following quadratic form is nonnegative:

$$\mathbf{x}^T D F \mathbf{x} \geq 0 \qquad \forall \mathbf{x} \neq 0. \tag{5}$$

This quadratic form can be rewritten as

$$\begin{aligned} &\mathbf{x}^T D^{1/2} D^{1/2} (F_s + F_a) D^{-1/2} D^{1/2} \mathbf{x} \\ &= \mathbf{x}^T D^{1/2} (D^{1/2} F_s D^{-1/2} + D^{1/2} F_a D^{-1/2}) D^{1/2} \mathbf{x} \\ &= \mathbf{y}^T E_s \mathbf{y} + \mathbf{y}^T E_a \mathbf{y} \end{aligned}$$

where $\mathbf{y} = D^{1/2}\mathbf{x}$, $E_s = D^{1/2}F_sD^{-1/2}$ and $E_a = D^{1/2}F_aD^{-1/2}$. By construction E_s is symmetric and has the same eigenvalues as F_s, while E_a is antisymmetric and isospectral with F_a. Furthermore, since $F_s \succeq 0$, so is E_s. Instead F_a (and hence E_a) has eigenvalues on the imaginary axis. Since $D \succ 0$, then (5) follows.
Since W is ES, for it $\rho(W) = 1$ is a simple, strictly dominant eigenvalue. This means that F has $\mu(F) = 0$ which is simple, strictly, dominant eigenvalue, i.e., the matrix DF in the quadratic from (5) has corank 1.
In other words $\left(I - \Theta^{-1}\right)(I - W)$ has corank 1 and all eigenvalues strictly outside the unit disk centered at 1 except for one at the origin. Lemma 4 then tells us that $\rho(P_W) \leq 1$ and P_W has at most one eigenvalue on the unit circle.
From Theorem 1, we know that $P_W\mathbb{1} = \mathbb{1}$. Hence, it follows that $\rho(P_W) = 1$ is an eigenvalue of P_W and that it is strictly dominating all other eigenvalues of P_W. To complete the proof we need to show that the left eigenvector of P_W associated to $\rho(P_W) = 1$, call it $\mathbf{w}$, is positive.
For $\mathbf{w}$, it is $\mathbf{w}^T\left((I - (I - \Theta)W)^{-1}\Theta\right) = \mathbf{w}^T$, or equivalently $\mathbf{w}^T\left(I - \left(I - \Theta^{-1}\right)(I - W)\right)^{-1} = \mathbf{w}^T$ or $\mathbf{w}^T\left(I - \left(I - \Theta^{-1}\right)(I - W)\right) = \mathbf{w}^T$, which leads to $\mathbf{w}^T\left(\Theta^{-1} - I\right)(I - W) = \mathbf{0}^T$. Using the notation above, this expression becomes $\mathbf{w}^TD(I - W) = \mathbf{0}^T$ or $\mathbf{w}^TDW = \mathbf{w}^TD$, i.e., the vector $D\mathbf{w}$ is a left eigenvector of W associated to the eigenvalue $\rho(W) = 1$. However, since W is ES and normal, it is EDS, and therefore $\mathbb{1}^TW = \mathbb{1}^T$. Combining the two expressions, it must be $D\mathbf{w} = \mathbb{1}$, or $\mathbf{w} = D^{-1}\mathbb{1}$. Since D is diagonal, in components we get $\mathbf{w}_i = (1/\theta_i - 1)^{-1} = \theta_i/(1-\theta_i)$. This completes the proof. ■

B. Continuous Time Case

The CT case can be treated in a similar way to the DT case (all proofs are therefore omitted).

Theorem 3 *Consider the CT SFJ model* (3)*, with* $-L$ *marginally stable of corank* 1 *and normal. Under Assumption 1, then* $-((I - \Theta)L + \Theta)$ *is Hurwitz and the system* (3) *converges to the equilibrium point* $\mathbf{x}^* = P_L\mathbf{x}(0)$*, where* $P_L = ((I - \Theta)L + \Theta)^{-1}\Theta$*, with* $P_L\mathbb{1} = \mathbb{1}$.

Similarly, the normality condition on L is also needed in Theorem 3, as L not a normal matrix may result in $-((I - \Theta)L + \Theta)$ which is not Hurwitz, as shown in the following counterexample (Example 5). Example 6, instead, shows that the normality of L is a sufficient but not necessary condition for the stability of the SFJ model.

Example 5 Consider the Laplacian

$$L = \begin{bmatrix} 0.7286 & -0.7384 & -0.3274 & 0.3373 \\ 0.8639 & 0.4008 & -0.4472 & -0.8175 \\ -0.0755 & -0.0585 & 0.5208 & -0.3868 \\ -0.3806 & -0.7311 & 0.4908 & 0.6208 \end{bmatrix}.$$

The matrix $-L$ is EEP but not normal. Choosing the stubbornness values $\Theta = \operatorname{diag}\{0.9635,\, 0.0160,\, 0.5799,\, 0.3105\}$ results in $\mu(-((I - \Theta)L + \Theta)) = 0.0847$, i.e. the system (3) is unstable.

Example 6 For the Laplacian

$$L = \begin{bmatrix} 0.0418 & 0.1131 & -0.1398 & -0.0152 \\ -0.7470 & 0.1598 & -0.0359 & 0.6232 \\ 0.4106 & 0.0892 & 0.3206 & -0.8205 \\ 0.6004 & -0.9164 & -0.9666 & 1.2827 \end{bmatrix},$$

we have that $-L$ is EEP but not normal. The stubbornness values $\Theta = \operatorname{diag}\{0.9940,\, 0.9514,\, 0.1993,\, 0.5663\}$ lead to $-((I - \Theta)L + \Theta)$ which is Hurwitz. Hence, the SFJ model (3) is stable.

The following is an example in which $-L$ marginally stable of corank 1 and normal yields $-((I - \Theta)L + \Theta)$ Hurwitz for all possible choices of $0 < \theta_i < 1$.

Example 7 For the Laplacian

$$L = \begin{bmatrix} 0.2199 & -0.2464 & 0.0557 & -0.0293 \\ -0.2464 & 0.4507 & -0.0880 & -0.1163 \\ -0.0293 & -0.1163 & 0.1045 & 0.0410 \\ 0.0557 & -0.0880 & -0.0723 & 0.1045 \end{bmatrix},$$

$-L$ is EEP and normal.

Convergence of the FJ model, i.e., $-((I - \Theta)L + \Theta)$ Hurwitz, does not guarantee that $P_L = (\Theta - (I - \Theta)L)^{-1}\Theta$ has $\rho(P_L) \leq 1$. For instance in Example 6, it is $\rho(P_L) = 1.6993$. However, if we add the normality condition on L, then we have that P_L is a ES matrix.

Theorem 4 *Consider the CT SFJ model* (3)*, with* $-L$ *marginally stable of corank* 1 *and normal. Under Assumption 1,* $P_L = ((I - \Theta)L + \Theta)^{-1}\Theta$ *is a ES matrix, and its left eigenvector* $\mathbf{w} > 0$ *associated to* $\rho(P_L) = 1$ *has components* $\mathbf{w}_i = \theta_i/(1 - \theta_i)$.

V. Extension to Concatenated SFJ Models

A concatenated SFJ model is a two-time scale model representing a sequence of discussion events, each of which is represented by a (DT or CT) SFJ model. The main motivation for studying concatenated SFJ models is that complex decisions typically require a series of intermediate negotiation steps (here our discussion events) rather than being resolved in a single meeting by the participants. Using the setting we developed in [11], a second index $s = 1, 2, \ldots$ provides the clock for the sequence of discussion events, hence the opinion vector becomes $\mathbf{x}(s, t)$, and the SFJ dynamics at event s is

$$\mathbf{x}(s, \infty) = P\,\mathbf{x}(s, 0), \tag{6}$$

where $P \in \{P_W, P_L\}$ depending on whether we are dealing with DT or CT dynamics (recall from the previous section that $P_W = (I - (I - \Theta)W)^{-1}\Theta$ and $P_L = ((I - \Theta)L + \Theta)^{-1}\Theta$). Concatenation refers to the endpoint of the $s - 1$ meeting becoming the initial condition of the

s meeting: $\mathbf{x}(s,\,0) = \mathbf{x}(s-1,\,\infty)$, with the understanding that $\mathbf{x}(0,\,\infty) = \mathbf{x}(1,\,0)$ is the initial opinion of the agents at the begin of the first meeting. This implies that (6) can be rewritten as

$$\mathbf{x}(s,\,\infty) = P\,\mathbf{x}(s-1,\infty), \quad P \in \{P_W,\,P_L\} \qquad (7)$$

i.e., a DT dynamics in s in which P is a signed matrix. Notice that for simplicity in this paper we assume that P is the same on each discussion event. Under the assumptions of Theorems 2 and 4, P has special properties, and we have that the concatenated SFJ system (7) converges to consensus.

Theorem 5 *Under Assumption 1, if*

(i) *[DT case:] W is ES and normal,*

(ii) *[CT case:] $-L$ is EEP and normal,*

then the concatenated SFJ model (7) *converges to consensus.*

Proof. In the DT case, from Theorem 2, W ES and normal means that P_W is ES. Using Lemma 3, the concatenated FJ model (7) achieves consensus. A similar argument holds for the CT case, using Theorem 4 and the equivalent of Lemma 3 for CT. This completes the proof. ∎

Example 8 Figure 2 shows a simulation plot of the concatenated SFJ case for the DT Example 3. Notice how the convex hull violation observed in Fig. 1 in the fast time scale of the single discussion gets reabsorbed as the discussion events concatenate. Notice further that unlike standard consensus problems (i.e., with P_W row stochastic) which are always contracting, our concatenated SFJ model is not contracting, even though it is converging asymptotically.

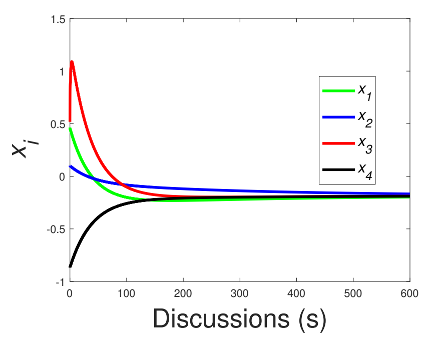

Fig. 2. A simulation for a concatenated SFJ model built on Example 3.

VI. Conclusion

In this paper, we give sufficient conditions for existence of a stable equilibrium point in the SFJ model. Any value of stubbornness leading to an invertible Θ suffices to get a stable equilibrium point if W is ES and normal for the DT case, and $-L$ is EEP and normal for the CT case. We show that the same conditions are required also for the convergence to consensus in the concatenated case.

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