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Comparison between different continuous-time realizations based on the *tau* method of a nonlinear repetitive control scheme

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***Abstract*—This paper presents a new framework to study a continuous-time finite-dimensional approximation of a repetitive control scheme for minimum-phase nonlinear systems. Based on the *tau* method, three state matrices are proposed to approximate the transport equation, namely *tau*-Fourier, *tau*-Legendre and *tau*-Chebyshev models. Then, the forwarding approach allows us to set up three finite dimensional controllers whose structure is derived from these *tau* models. In all three cases, we prove that a smooth steady state is reached and that the output converges to zero as the order increases. We also show that, for a fixed order, the *tau*-Fourier controller leads to the smallest output. Lastly, numerical comparisons are discussed and emphasize our theoretical expectations.**

I. Introduction

In electronics, mechanics or robotics, the output regulation is a long-standing open problem [1]. Whenever a controller aims to track the same trajectory repeatedly, one may use a repetitive-control scheme [2] to adjust the control retroactively and achieve a decrease in tracking error. This repetitive control tracking periodic references is based on the internal model principle [3]. Here, the periodic model is based on a transport phenomenon, which is theoretically challenging [4], [5]. The problem of periodic tracking/rejection appears in the literature under many different keywords, such as output regulation [6], repetitive control [2] or iterative learning control [1]. The acknowledged solution relies on the implementation of an infinite-dimensional controller based on the use of a delay. Repetitive control schemes have been proposed for time-varying (or time-delay) linear systems [2], [7], [8], infinite-dimensional linear systems [9], [10], and nonlinear systems [11], [12]. More classical repetitive control schemes [2] include the following transfer function

$$H(s) = \frac{1}{\Delta(s)}, \qquad \Delta(s) = \exp(Ts) - 1, \tag{1}$$

in the closed-loop system. Such a strategy, however, is hard to apply in the context of nonlinear systems since transfer function analysis cannot be employed. Few results exists in the context of repetitive control for nonlinear systems, see, e.g. [10]–[13]. We highlight the results [10], [12] in which the transfer function (1) is represented via a partial differential equation (precisely a transport equation) and that will be the starting point of this article.

The major flow of the repetitive control approach is that the feedback law involves a model of infinite dimension. Indeed, the realization of the transfer function (1) involves a delay which is an object of infinite-dimensional nature (either in space or in time). In other words, the denominator Δ in (1) has an infinite number of roots. The question of implementation is still pending. How to obtain stability guarantees with few control parameters? What is the best way to approximate the internal periodic model by preserving the dominant modes while attenuating the high frequencies? The need for model reduction arises in many contexts where delays lie [14], [15]. The idea is to obtain finite dimensional control laws that preserve the stabilizing properties on the initial model, e.g. [14], [16]. Two main routes for discretizing the regulator (1) are possible. The first consists in applying a time discretization and realizing the regulator in discrete-time domain, e.g. [17]. The second one consists in approximating the transfer function (1) by an implementable transfer function, namely such that its equivalent realization in state space domain corresponds to a finite-dimensional ordinary differential equation. Following this second route, a common approach is to include a low-pass filter to the transfer function (1), e.g. [18]. The issue with this approach is that it is not clear how to analyze the properties of such a realization in the context of nonlinear systems. An other possible approach is to use a harmonic approximated model. This has been investigated also in the nonlinear context, e.g. [19]–[22]. Finally, a Padé approximated model has been studied in [13]. It is worth noticing however that in order to discretize a delay, other approaches exist. This includes the least-square method [23] or the *tau* method [24].

The objective of this work is to compare different continuous-time realizations of the repetitive control scheme (1) in the context of minimum-phase nonlinear systems developed in [12], [21]. In particular, we introduce and rigorously analyze three approximated models based on the *tau* method [24]. Among them, the Fourier approach [21] is recovered and two new approximated models appear, namely the *tau*-Legendre [25] and *tau*-Chebyshev [23] models. Thanks to the use of series expansion and convergence analysis, we characterize the behavior of the steady-state regulated output showing, under regularity assumptions, that output and its first d-th derivative are ultimately bounded. The ultimate bound decrease with n, where n is the dimension of the *tau* internal model. This shows that we can regulate the output error by increasing the dimension of the internal model, similarly to the context of harmonic regulation studied in [19], [21], [22]. For a given order, the Fourier realization

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allows to reach a better steady state than Legendre or Chebyshev ones. Finally, we compare the three different schemes on simulations.

Notation. The sets $\mathbb{N}$, $\mathbb{R}$, $\mathbf{i}\mathbb{R}$, $\mathbb{C}$ correspond to natural, real, imaginary and complex numbers. Note also that $\mathbf{i} = \sqrt{-1}$. $|\cdot|$ is the Euclidean norm. For any square matrix M, we denote its determinant $\det(M)$, its upper triangular part $\mathrm{triu}(M)$ and $\mathrm{He}(M) = M + M^\top$ with $M^\top$ the transpose of M. We also consider the identity matrix I_n of size n. Given a set of matrices $(M_1, \dots, M_n)$, we denote with $\mathrm{diag}(M_1, \dots, M_n)$ the block diagonal matrix containing the matrix M_i on the i-th block. The set C^∞ represents smooth functions. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of *class* $\mathcal{K}$ if α is continuous, increasing, and $\alpha(0) = 0$. If moreover $\lim_{s\to\infty} \alpha(s) = \infty$, we say that α is of *class* $\mathcal{K}_\infty$. Lastly, $L^2(0, 1; \mathbb{R})$ is the set of square integrable functions from $(0, 1)$ to $\mathbb{R}$ and $\mathrm{d}\mu$ is a localized measure satisfying $\int_0^1 \mathrm{d}\mu(x) = 1$.

II. Problem Statement

A. Problem Formulation

Following the framework in [12], consider the class of nonlinear minimum-phase systems with unitary relative degree of the form that can be written, possibly after a change of coordinates, as

$$\begin{cases} \dot{z} = f(t, z, e), \\ \dot{e} = q(t, z, e) + u, \end{cases} \tag{2}$$

where $(z, e) \in \mathbb{R}^m \times \mathbb{R}$ is the state, $e \in \mathbb{R}$ the regulated output and $u \in \mathbb{R}$ the controlled input. We suppose that functions f and q satisfy the following properties stated below.

Assumption 1. *The functions $f, q : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$ are T-periodic with respect to the first argument, globally Lipschitz with respect to the second and third components, and C^∞ in $[0, T] \times \mathbb{R}^m \times \mathbb{R}$. In particular, there exists $\ell_f, \ell_q > 0$ such that*

$$\begin{aligned} |f(t, z, e) - f(t, \hat{z}, \hat{e})| &\leq \ell_f |z - \hat{z}| + \ell_f |e - \hat{e}|, \\ |q(t, z, e) - q(t, \hat{z}, \hat{e})| &\leq \ell_q |z - \hat{z}| + \ell_q |e - \hat{e}|, \end{aligned}$$

for all $t \in [0, T]$, $z, \hat{z} \in \mathbb{R}^m$ and $e, \hat{e} \in \mathbb{R}$. Furthermore, for any compact set $Z \times E \subset \mathbb{R}^m \times \mathbb{R}$, f, q are exponentially-like functions. There exists $\ell > 0$ such that, for any $d \in \mathbb{N}$,

$$\sup_{(t,z,e)\in[0,T]\times Z\times E} \left| \frac{\partial^d f(t, z, e)}{\partial t^d} \right| + \left| \frac{\partial^d q(t, z, e)}{\partial t^d} \right| \leq \ell^d . \tag{3}$$

Assumption 2. *The zero-dynamics $\dot{z} = f(t, z, 0)$ admits a unique C^∞ T-periodic bounded solution $\bar{z}(t)$ which is globally uniformly stable. In particular, there exists a positive definite function $V : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ and class $\mathcal{K}_\infty$ functions $\underline{\alpha}, \bar{\alpha}$ and real numbers $\alpha, \gamma > 0$ satisfying*

$$\begin{aligned} &\underline{\alpha}(|z - \bar{z}(t)|) \leq V(t, z - \bar{z}(t)) \leq \bar{\alpha}(|z - \bar{z}(t)|), \\ &\left\langle \nabla V(t, z - \bar{z}(t)),\ f(t, z, e) - f(t, \bar{z}(t), 0) \right\rangle \leq \\ &\qquad\qquad -\alpha |z - \bar{z}(t)|^2 + \gamma |e|^2, \end{aligned}$$

for all $(t, z, e) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}$.

Remark 1. *In Assumption 2, it is supposed that system* (2) *is minimum-phase allowing to look for a pure output feedback control (i.e. that depends only on the measured output e). Furthermore, note that since q is not zero at the origin, the regulation problem at hand cannot be solved by a pure high-gain feedback controller of the form $u = -\sigma e$, $\sigma > 0$ [21].*

The problem of output regulation for system (2) consists in finding a dynamical regulator such that the output e is asymptotically regulated to zero, namely

$$\lim_{t\to\infty} e(t) = 0. \tag{4}$$

B. Highlights on a repetitive control scheme

Following [12], a robust[1] solution to this problem can be solved by employing a repetitive control scheme of the form

$$\begin{cases} \frac{\partial}{\partial t}\eta(t, x) = \frac{1}{T} \frac{\partial}{\partial x}\eta(t, x) & \forall\, (t, x) \in \mathbb{R}_+ \times [0, 1], \\ \eta(t, 1) = \eta(t, 0) + e(t) & \forall\, t \in \mathbb{R}_+, \\ \eta(0, x) = \eta^0(x) & \forall\, x \in [0, 1], \end{cases} \tag{5}$$

with initial conditions $\eta^0 \in L^2(0, 1)$, scalar $T > 0$ known, and the stabilizing feedback law

$$\begin{cases} u(t) = -\sigma e(t) + v(t) , \\ v(t) = \mu \displaystyle\int_0^1 M(x)(\eta(t, x) - M(x)e(t))dx , \end{cases} \tag{6}$$

where $\sigma, \mu > 0$ are design parameters and the function $M : [0, 1] \to \mathbb{R}$ in (6) is defined as the solution of the following two-point boundary value problem

$$\tfrac{\mathrm{d}}{\mathrm{d}x} M(x) = \sigma T M(x), \quad M(1) = M(0) + 1. \tag{7}$$

whose solution is given by

$$M(x) = \frac{\exp(\sigma T x)}{\exp(\sigma T) - 1}, \qquad x \in [0, 1]. \tag{8}$$

From a theoretical point of view, the scheme (5), (6) allows to achieve the regulation objective (4) for any σ large enough. Note that such a condition is needed to have a preliminary stabilizing effect for the (z, e)-dynamics.

C. Objectives of this article

Due to its infinite-dimensional nature, the proposed repetitive-control scheme (5)-(6) cannot be implemented and needs to be discretized. Such a discretization can be done in space and time. When both discretizations are realized, a pure digital controller is obtained. In this article, we are interested in space discretization, so that to obtain a finite-dimensional continuous-time controller described by an ODE of order n of the form

$$\begin{aligned} \dot{\eta}_n &= A_n \eta_n + B_n e \\ u &= -\sigma e + \mu M_n^\top (\eta_n - M_n e) \end{aligned} \tag{9}$$

with $\eta_n \in \mathbb{R}^n$ and matrices (A_n, B_n, M_n) to be defined later with respect to T. It can be noted that the structure

[1] Robust to model uncertainties, in the sense that the control law depends as little as possible by the knowledge of the functions f, q. In this case, we require only the knowledge of the constants ℓ_q, γ and the period T.

of the regulator (9) preserves that of (5)-(6). The objective of the next sections is to investigate a series of different methodologies to obtain such realizations. From the common *tau* method framework, we will present the Fourier realization explored in [21] as well as two new models based on Legendre and Chebyshev polynomials approximation [23].

Furthermore, since the controller (5), (6) is "truncated", the asymptotic regulation objective (4) is generically inevitably lost. For this reason, such an asymptotic goal is modified into an approximate regulation objective as follows

$$\lim_{t\to\infty} |e(t)| \leq \varepsilon,$$

where in particular we will show that the parameter $\varepsilon > 0$ can be tuned by increasing the number n defining the approximation of the finite-dimensional regulator (9).

III. Tau Method

The *tau* method is a pseudo-spectral approach introduced in [24]. As an extension of the Galerkin method [26], it approximates infinite-dimensional systems on canonical bases like Fourier, Legendre or Chebyshev polynomials, which are not necessarily the Riesz basis of the operator. The principle of the method is to solve the problem satisfied by the corresponding quasi-spectral truncated series.

For a given order $n \in \mathbb{N}$, consider a set of linearly and orthogonal independent functions $\{\varphi_k\}_{k\in\{0,\dots,n-1\}}$ defined on $L^2(0,1;\mathbb{R})$ with the measure $\mathrm{d}\mu$. We define $\Phi_n = (\varphi_0,\dots,\varphi_{n-1}) \in \mathbb{R}^n$ as well as the following matrices.

- *Gram-Schmidt matrix*: $\mathcal{I}_n = \displaystyle\int_0^1 \Phi_n(x)\Phi_n^\top(x)\mathrm{d}\mu(x).$
- *Derivation matrix*: $\mathcal{J}_n = \displaystyle\int_0^1 \Phi_n(x)\tfrac{\mathrm{d}}{\mathrm{d}x}\Phi_{n+1}^\top(x)\mathrm{d}\mu(x).$
- *Boundary matrix*: $\mathcal{K}_n = \begin{bmatrix} I_n \\ -\frac{\Phi_n^\top(1)-\Phi_n^\top(0)}{\varphi_n(1)-\varphi_n(0)} \end{bmatrix}.$

Let us define $\eta(t,x) = \Phi_{n+1}^\top(x)\left[\begin{smallmatrix}\eta_n\\ \nu\end{smallmatrix}\right](t)$, where the vector $\left[\begin{smallmatrix}\eta_n\\ \nu\end{smallmatrix}\right]$ can be seen as an approximation of the $n+1$ first projections of the state of system (5) on the selected basis. Assuming that this function η is solution of system (5), the following equations must be satisfied

$$\Phi_{n+1}^\top(x)\begin{bmatrix}\dot\eta_n\\ \dot\nu\end{bmatrix} = \frac{1}{T}\tfrac{\mathrm{d}}{\mathrm{d}x}\Phi_{n+1}^\top(x)\begin{bmatrix}\eta_n\\ \nu\end{bmatrix}, \tag{10}$$

$$(\varphi_n(1)-\varphi_n(0))\nu = -\left(\Phi_n^\top(1)-\Phi_n^\top(0)\right)\eta_n, \tag{11}$$

corresponding to the fulfillment of the transport dynamics and the boundary condition, respectively. The constraint (10) and the projection of the constraint (11) on $\Phi_n(x)$ yields

$$\nu = -\frac{\Phi_n^\top(1)-\Phi_n^\top(0)}{\varphi_n(1)-\varphi_n(0)}\eta_n, \qquad \mathcal{I}_n\dot\eta_n = \frac{1}{T}\mathcal{J}_n\mathcal{K}_n\eta_n,$$

which leads to system (9) with matrix

$$A_n = \frac{1}{T}\mathcal{I}_n^{-1}\mathcal{J}_n\mathcal{K}_n. \tag{12}$$

Thus, this method leads to several finite-dimensional realizations according to the base used.

A. Tau-Fourier model

For a given odd integer $n \in \mathbb{N}$, consider the trigonometric functions on the interval $[0,1]$ given by

$$\varphi_{k,F}(x) = \begin{cases}\cos(k\pi x), & \text{if } k \text{ even},\\ \sin((k+1)\pi x), & \text{if } k \text{ odd},\end{cases}$$

orthogonal with respect to the measure $\mathrm{d}\mu(x) = \mathrm{d}x$. The Fourier functions matrices $\mathcal{I}_n, \mathcal{J}_n, \mathcal{K}_n$ are stored in Table I. From (12), we obtain the approximated matrix

$$A_{n,F} = \frac{2\pi}{T}\mathrm{diag}\left(0, \left[\begin{smallmatrix}0 & 1\\ -1 & 0\end{smallmatrix}\right], \dots, \left[\begin{smallmatrix}0 & \frac{n-1}{2}\\ -\frac{n-1}{2} & 0\end{smallmatrix}\right]\right).$$

For instance, for $n \in \{1,3,5\}$, we obtain matrices

$$A_1 = 0,\ A_3 = \tfrac{2\pi}{T}\left[\begin{smallmatrix}0&0&0\\0&0&1\\0&-1&0\end{smallmatrix}\right],\ A_5 = \frac{2\pi}{T}\left[\begin{smallmatrix}0&0&0&0&0\\0&0&1&0&0\\0&-1&0&0&0\\0&0&0&0&2\\0&0&0&-2&0\end{smallmatrix}\right],$$

whose eigenvalues are $\{0\}$, $\{0, \pm\frac{2\pi}{T}\mathbf{i}\}$ and $\{0, \pm\frac{2\pi}{T}\mathbf{i} \pm \frac{4\pi}{T}\mathbf{i}\}$.

The denominator $\Delta(s) = \exp(Ts) - 1$ of the transfer function $H(s)$ in (1) is therefore approximated by

$$\Delta_{n,F}(s) = s\prod_{k=1}^{\frac{n-1}{2}}\left(s^2 + k^2\left(\frac{2\pi}{T}\right)^2\right) \underset{n\to\infty}{\longrightarrow} \Delta(s).$$

The *tau*-Fourier model allows to recover the first n modes. It can be remarked that it coincides with a truncation of the transport equation on the Riesz basis.

B. Tau-Legendre model

For a given odd integer $n \in \mathbb{N}$, consider the Legendre polynomials on the interval $[0,1]$ given by

$$\varphi_{k,L}(x) = \sum_{i=0}^{k}\frac{(k+i)!}{(i!)^2(k-i)!}(x-1)^i, \quad \forall k \in \mathbb{N},$$

orthogonal with respect to the measure $\mathrm{d}\mu(x) = \mathrm{d}x$. The Legendre polynomials matrices $\mathcal{I}_n, \mathcal{J}_n, \mathcal{K}_n$ are stored in Table I. From (12), we obtain the approximated matrix

$$A_{n,L} = \frac{1}{T}(2D_n + I_n)\left(\mathcal{U}_n - \frac{1}{2}(\mathbf{1}_n + \mathbf{1}_n^*)(\mathbf{1}_n - \mathbf{1}_n^*)^\top\right),$$

with

$$D_n = \mathrm{diag}(0,\dots,n-1), \quad \mathcal{U}_n = \mathrm{triu}(\mathbf{1}_n\mathbf{1}_n^\top - \mathbf{1}_n^*\mathbf{1}_n^{*\top}),$$
$$\mathbf{1}_n = \left[\begin{smallmatrix}1 & 1 & \cdots & 1\end{smallmatrix}\right]^\top, \qquad \mathbf{1}_n^* = \left[\begin{smallmatrix}1 & -1 & \cdots & 1\end{smallmatrix}\right]^\top.$$

For instance, for $n \in \{1,3,5\}$, we obtain matrices

$$A_1 = 0,\ A_3 = \tfrac{2D_3+I_3}{T}\left[\begin{smallmatrix}0&0&0\\0&0&2\\0&-2&0\end{smallmatrix}\right],\ A_5 = \tfrac{2D_5+I_5}{T}\left[\begin{smallmatrix}0&0&0&0&0\\0&0&2&0&2\\0&-2&0&0&0\\0&0&0&0&2\\0&-2&0&-2&0\end{smallmatrix}\right].$$

Note that the eigenvalues of matrices A_1, A_3 and A_5 are respectively $\{0\}$, $\{0, \pm\frac{7.75}{T}\mathbf{i}\}$ and $\{0, \pm\frac{6.30}{T}\mathbf{i} \pm \frac{19.50}{T}\mathbf{i}\}$ and that they tend to $\pm k\frac{2\pi}{T}\mathbf{i}$ as n goes to infinity. As a matter of fact, as shown in [25, Proposition 3], the realization

$$H_n(s) = \tfrac{1}{2T}(\mathbf{1}_n + \mathbf{1}_n^*)^\top(sI_n - A_{n,L})^{-1}\mathcal{I}_n^{-1}\mathbf{1}_n - \tfrac{1}{2},$$

is the $(n|n)$ Padé approximant of $H(s)$ in (1). The denominator $\Delta(s) = \exp(Ts) - 1$ of the transfer function $H(s)$ in (1) is then approximated by $\Delta_{n,L}(s) = \det(sI_n - A_{n,L})$.

Matrices	Fourier	Legendre	Chebyshev
Gram-Schmidt ($\mathcal{I}_n$)	$\mathrm{diag}\Big(1, \frac{1}{\sqrt{2}}, \dots \frac{1}{\sqrt{2}}\Big)$	$(2D_n + I_n)^{-1}$	$\pi \,\mathrm{diag}\left(1, \frac{1}{2}, \dots, \frac{1}{2}\right)$
Derivation ($\mathcal{J}_n$)	$\left[\sqrt{2}\pi \mathrm{diag}\Big(0, \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], \dots, \begin{bmatrix} 0 & \frac{n-1}{2} \\ -\frac{n-1}{2} & 0 \end{bmatrix}\Big) \quad 0\right]$	$\left[\mathcal{U}_n \quad (\mathbf{1}_n + \mathbf{1}_n^*)\right]$	$\pi \left[\mathcal{U}_n D_n \quad (\mathbf{1}_n + \mathbf{1}_n^*) n\right]$
Boundary ($\mathcal{K}_n$)	$\begin{bmatrix} I_n \\ - \end{bmatrix}$	$\begin{bmatrix} I_n \\ -\frac{1}{2}(\mathbf{1}_n - \mathbf{1}_n^*)^\top \end{bmatrix}$	$\begin{bmatrix} I_n \\ -\frac{1}{2}(\mathbf{1}_n - \mathbf{1}_n^*)^\top \end{bmatrix}$

TABLE I: Properties satisfied by Fourier, Legendre and Chebyshev functions.

According to the explicit forms of Padé approximations provided by [27, page 436], we have

$$\Delta_{n,L}(s) = \sum_{p=0}^{\frac{n-1}{2}} \frac{(2(n-p)-1)!}{(2p+1)!(n-2p-1)!} s^{2p+1} \underset{n\to\infty}{\longrightarrow} \Delta(s),$$

The *tau*-Legendre model is the best rational approximation of the transfer function H when s tends to zero, for low frequencies.

C. Chebyshev realization

For a given odd integer $n \in \mathbb{N}$, consider the Chebyshev polynomials of the first kind on the interval $[0, 1]$ given by

$$\varphi_{k,C}(x) = \cos(k \arccos(2x-1)), \quad \forall k \in \mathbb{N},$$

orthogonal with respect to the measure $\mathrm{d}\mu(x) = \frac{\mathrm{d}x}{\pi\sqrt{x(1-x)}}$. With Chebyshev polynomials matrices $\mathcal{I}_n, \mathcal{J}_n, \mathcal{K}_n$ are stored in Table I. From (12), we obtain the approximated matrix

$$A_{n,C} = \frac{\mathrm{diag}\,(1, 2, \dots, 2)}{T}\Big(\mathcal{U}_n \mathcal{I}_n - \frac{n}{2}(\mathbf{1}_n + \mathbf{1}_n^*)(\mathbf{1}_n - \mathbf{1}_n^*)^\top\Big).$$

For instance, for $n \in \{1, 3, 5\}$, we obtain

$$A_1 = 0,\ A_3 = \frac{1}{T}\begin{bmatrix} 0 & -4 & 0 \\ 0 & 0 & 8 \\ 0 & -12 & 0 \end{bmatrix},\ A_5 = \frac{1}{T}\begin{bmatrix} 0 & -8 & 0 & -4 & 0 \\ 0 & 0 & 8 & 0 & 16 \\ 0 & -20 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 & 16 \\ 0 & -20 & 0 & -20 & 0 \end{bmatrix}.$$

Note that the eigenvalues of matrices A_1, A_3 and A_5 are respectively $\{0\}$, $\{0, \pm\frac{9.80}{T}\mathbf{i}\}$ and $\{0, \pm\frac{6.36}{T}\mathbf{i} \pm \frac{27.56}{T}\mathbf{i}\}$. Though no explicit forms can be provided, we know that $\Delta_{n,C}(s) = \det(sI_n - A_{n,C})$ approximates $\Delta(s)$ in (1).

IV. Convergence properties of Tau Method

Consider a function $f \in C^\infty(0, T)$ and, for a given order n, consider orthogonal independent functions $\{\varphi_k\}_{k\in\mathbb{N}}$ defined on $(L^2(0, 1; \mathbb{R}), \mathrm{d}\mu)$. Define the truncated series approximation of the function f as

$$f_n(t) = \sum_{k=0}^{n} \varphi_k(\tfrac{t}{T}) \frac{\int_0^T \varphi_k(\frac{t}{T}) f(t) \mathrm{d}\mu(t)}{\int_0^T \varphi_k(\frac{t}{T})\varphi_k(\frac{t}{T}) \mathrm{d}\mu(t)},$$

and the corresponding remainder

$$\tilde{f}_n(t) = f(t) - f_n(t) = \sum_{k=n+1}^{\infty} \varphi_k(\tfrac{t}{T}) \frac{\int_0^T \varphi_k\left(\frac{t}{T}\right) f(t) \mathrm{d}\mu(t)}{\int_0^T \varphi_k\left(\frac{t}{T}\right) \varphi_k\left(\frac{t}{T}\right) \mathrm{d}\mu(t)}.$$

A. Convergence of Fourier series

Lemma 1. *Assume that there exists $\ell > 0$ such that $\sup_{t\in[0,T]} |f^{(d)}(t)| \leq \ell^d$ holds for all $d \in \mathbb{N}$. Then, for $n > d+1$,*

$$\exists \bar{\ell} > 0 \text{ s.t. } \sup_{t\in[0,T]} \left|\tilde{f}^{(d)}_{n,F}(t)\right| \leq \frac{\bar{\ell}^n}{n^{n-d-1}}.$$

Proof. See [28, Theorems 4 and 9]. □

B. Convergence of Legendre polynomials series

Lemma 2. *Assume that there exists $\ell > 0$ such that $\sup_{t\in[0,T]} |f^{(d)}(t)| \leq \ell^d$ holds for all $d \in \mathbb{N}$. Then, , for $n \geq d+2$,*

$$\exists \bar{\ell} > 0 \text{ s.t. } \sup_{t\in[0,T]} \left|\tilde{f}^{(d)}_{n,L}(t)\right| \leq \frac{\bar{\ell}^n}{(n-d-2)!}.$$

Proof. See [29, Theorem 2.1]. □

C. Convergence of Chebyshev polynomials series

Lemma 3. *Assume that there exists $\ell > 0$ such that $\sup_{t\in[0,T]} |f^{(d)}(t)| \leq \ell^d$ holds for all $d \in \mathbb{N}$. Then, for $n \geq d+2$,*

$$\exists \bar{\ell} > 0 \text{ s.t. } \sup_{t\in[0,T]} \left|\tilde{f}^{(d)}_{n,C}(t)\right| \leq \frac{\bar{\ell}^n}{\sqrt{n}(n-d-2)!},$$

Proof. See [28, Theorems 10]. □

V. Main Result

A. Preliminaries

For a given odd integer n and positive scalars μ and σ, consider the finite-dimensional controller (9) where the state matrix A_n is designed as explained in Section III and formatting in the Jordan real form being then a skew-symmetric matrix, the input matrix B_n is selected in order to have a controllable pair (A_n, B_n) and where the output matrix is given by

$$M_n = -(A_n + \sigma I_n)^{-1} B_n. \tag{13}$$

It can be seen as an approximation of the infinite-dimensional repetitive control law (6). This feedback is inspired by the recent series of works [12], [21], [22] and derived from the forwarding approach.

Before going any further, let us show that these matrices have a stable and bounded pattern.

Lemma 4. *For any $\mu > 0$, the matrix $A_n - \mu M_n M_n^\top$ is Hurwitz. In particular, there exists a positive matrix Π_n and a positive real scalar κ such that*

$$\mathrm{He}[(I_n + \kappa \Pi_n)(A_n - \mu M_n M_n^\top)] \preceq -\mu M_n M_n^\top - \kappa I_n.$$

Proof. See [21, Lemma 1]. □

Lemma 5. *Consider the linear system*

$$\dot{\zeta}_n = (A_n - \mu M_n M_n^\top)\zeta_n - M_n q.$$

There exist scalars $\kappa_0, \kappa_1 > 0$ independent of n such that the transfer function between q and ζ satisfies

$$\zeta_n^{*\top}(\mathbf{i}\omega)\zeta_n(\mathbf{i}\omega) \leq (\kappa_0 + \kappa_1 \omega^2) q^*(\mathbf{i}\omega) q(\mathbf{i}\omega), \quad \forall \omega \in \mathbb{R}. \quad (14)$$

Proof. See [21, Lemma 2]. □

B. Stability and convergence analysis

With the change of coordinate $\zeta = \eta - M_n e$, the closed-loop system (2)-(9) is rewritten as follows

$$\begin{cases} \dot{z} = f(t, z, e), \\ \dot{e} = q(t, z, e) - \sigma e + \mu M_n^\top \zeta, \\ \dot{\zeta} = (A_n - \mu M_n M_n^\top)\zeta - M_n q(t, z, e). \end{cases} \quad (15)$$

First, we show that system (15) admits a T-periodic steady state (z_p, e_p, ζ_p) in $\mathcal{C}^\infty(0, T)$ which is exponentially stable.

Theorem 1. *Under Assumptions 1-2, for any $\mu > 0$ and sufficiently large $\sigma > 0$, there exists a periodic solution (z_p, e_p, ζ_p) to system* (15) *which satisfies*

$$\exists \ell > 0 \text{ s.t. } \sup_{t \in [0,T]} \left| z_p^{(d)}(t) \right| + \left| e_p^{(d)}(t) \right| + \left| \zeta_p^{(d)}(t) \right| \leq \ell^d \quad (16)$$

and which is globally exponentially stable.

Proof. First, we know that the existence of a T-periodic C^∞ solution (z_p, e_p, ζ_p) satisfying (16) is issued from [30, Lemma 5.1] and the Banach fixed point theorem.

With the change of coordinates $(\tilde{z}, \tilde{e}, \tilde{\zeta}) = (z, e, \zeta) - (z_p, e_p, \zeta_p)$, we introduce the function $V = \tilde{z}^\top P \tilde{z} + \tilde{e}^2 + \tilde{\zeta}^\top (I_n + \kappa \Pi_n)\tilde{\zeta}$ where P is defined as in Assumption 2 and matrix Π_n as in Lemma 4. Then, we follow the calculations provided in [21, Proposition 2 (108)-(109)]. At the end, for any $\mu > 0$ and sufficiently large $\sigma > 0$, we obtain that there exist $\nu_1, \nu_2 > 0$ such that along the trajectories of system (15)

$$\dot{V} \leq -\nu_1 \left(|\tilde{x}|^2 + |\tilde{e}|^2 + |M_n^\top \tilde{\zeta}|^2 \right) - \kappa \nu_2 |\tilde{\zeta}|^2.$$

Applying the Lyapunov theorem, we conclude that the periodic solution is exponentially stable. □

Finally, we show that the steady state e_p as well as its derivatives is vanishing as the order n goes to infinity.

Theorem 2. *Considering the closed-loop system* (15)*, there exists $\bar{\ell} > 0$ such that the periodic steady state e_p satisfies*

$$\sup_{t \in [0,T]} |e_p^{(d)}(t)| \leq \frac{\bar{\ell}^n}{n^{n-d-1}}, \qquad \forall n > d + 1, \quad (17a)$$

$$\sup_{t \in [0,T]} |e_p^{(d)}(t)| \leq \frac{\bar{\ell}^n}{(n - 2d - 2)!}, \qquad \forall n > 2d + 2, \quad (17b)$$

$$\sup_{t \in [0,T]} |e_p^{(d)}(t)| \leq \frac{\bar{\ell}^n}{\sqrt{n}(n - 2d - 2)!}, \quad \forall n > 2d + 2, \quad (17c)$$

where (17a)*,* (17b) *and* (17c) *apply respectively for Fourier, Legendre and Chebyshev approximations.*

Proof. The proof can be found in the extended version. □

Whether it is for Fourier, Legendre and Chebyshev, the convergence is exponential with respect to the order n. Still, Fourier leads to the lowest steady state's upper bound.

C. Discussion of the main result

The result of Theorem 2 ensures the convergence of e to zero as $t \to \infty$ and as the dimension n of the *tau* internal model regulator (9) tends to infinity. It highlights the efficiency of the proposed regulation method for sufficiently large orders. Furthermore, we show that this regulator has a smoothing effect on the regulated output.

VI. Numerical results

Consider the simple numerical example proposed in [12]

$$\begin{cases} \dot{z} = -z^3 + \cos(2\pi t) - 0.5 + e, \\ \dot{e} = (2t[1] - 1)^2 + 2\arctan(z)(1 + e) + u. \end{cases} \quad (18)$$

where $t[1]$ stands for the modulus and the control law (9) with parameters $\sigma = 2$, $\mu = 10$ and $T = 1$ and with matrices

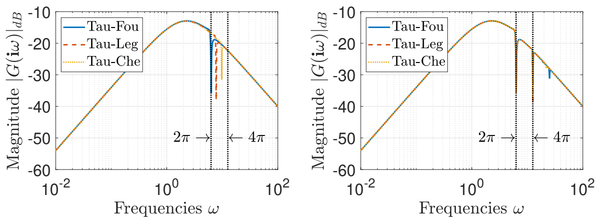
$$A_n \in \{A_{n,F}, A_{n,L}, A_{n,C}\}, \quad B_n = \mathbf{1}_n. \quad (19)$$

The transfer function from q to e is given by

$$G(s) = \begin{bmatrix} 0 & 1 \end{bmatrix} \left(sI_{n+1} - \begin{bmatrix} A_n & B_n \\ \mu M_n^\top & -\sigma - \mu M_n^\top M_n \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

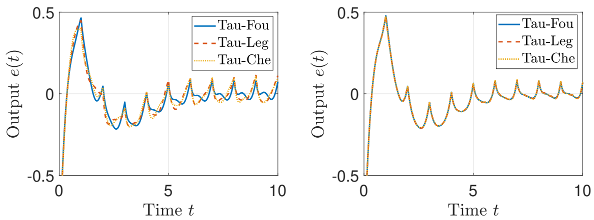
In the case of the three *tau* models, the bode diagram is depicted on Fig. 1 for orders $n \in \{3, 9\}$. It is a pass-band filter. We note that the cutoff frequencies correspond to the poles of the approximated model, fitting with the $\{k\frac{2\pi}{T}\}_{k \in \mathbb{N}}$ harmonics as the order n increases. As expected, we fall exactly on the n first harmonics $\{k\frac{2\pi}{T}\}_{k \in \{1,\ldots,n\}}$ only for the *tau*-Fourier model (spectral decomposition).

From the initial condition $(z(0), e(0), \zeta(0)) = (2, -1, 0)$, the output of the closed-loop system and its sup norm are plotted on Fig. 2 and Fig. 3 for orders $n \in \{3, 9\}$. As proved in Theorem 1, the output e is converging towards a periodic solution for any initial conditions and any orders n. Moreover, Theorem 2 adds that the steady state is vanishing as the order n increases, with a geometric rate. Lastly, when comparing the plots for a fixed order n, we confirm that *tau*-Fourier model seems to reach the smallest asymptotic output.



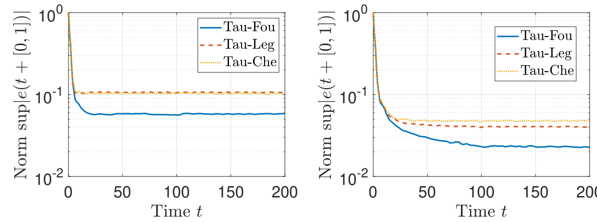
(a) Order $n = 3$. (b) Order $n = 9$.

Fig. 1: Magnitude of G with respect to the frequencies.



(a) Order $n = 3$. (b) Order $n = 9$.

Fig. 2: Output e with respect to the time.



(a) Order $n = 3$. (b) Order $n = 9$.

Fig. 3: The $\sup$ norm of e with respect to the time.

VII. Conclusions

In this article we have introduced *tau*-Fourier, *tau*-Legendre and *tau*-Chebychev finite-dimensional models to mimic the infinite-dimensional output regulator structure of a repetitive control scheme. The proposed approximations have the ability to reach a periodic steady state output, whose norm decreases exponentially fast as the order increases. We proved and showed in simulation that the Fourier model seems to be the most appropriate and accurate model.

For future works, we would like to investigate generic *tau* models that could be more easily adapted to internal models which are not governed by the transport equation.

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