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Model Reduction in the Loewner Framework for Second-Order Network Systems On Graphs*

Alessio Moreschini[1], Joel D. Simard[2] and Alessandro Astolfi[3]

***Abstract*— This paper studies the model reduction problem in the Loewner framework for second-order network systems evolving on graphs. The selection of particular sets of tangential interpolation data allows constructing reduced order models which interpolate the underlying network system while preserving the second-order structure of the system. The conditions that the tangential interpolation data must satisfy are established on the basis of the block structure of the Loewner matrices. We use this result to link the Loewner matrices to the cluster matrix gained by partitioning the graph associated with the underlying model. Finally, we provide an illustrative example to validate the obtained results.**

***Index Terms*— Reduced order modeling; Networked control systems; Network analysis and control; Large-scale systems**

I. INTRODUCTION

Second-order network systems with diffusive couplings can be found in various scientific disciplines [1]. These systems usually give rise to a large number of differential equations which complicates the analysis, the simulation, and the control of the underlying network describing the diffusive couplings between subsystems. In light of this, model reduction for complex network systems has gained interest for the purpose of reducing the computational complexity of numerical simulations and facilitating analysis and synthesis of network systems [2]–[9]. The purpose of model order reduction is to construct a low-order model (the reduced order model) which approximates a high-order model by accurately capturing the essential behaviour of the high-order system, [10], [11]. The Loewner framework is a data-driven approach which generates reduced order models based on selected interpolation data (or measurements). The essential objects underpinning this framework are the Loewner matrices [12]. These are divided-difference matrices built from tangential interpolation data obtained by sampling the transfer function of the underlying system along particular directions and frequencies. These objects have been originally used to solve interpolation problems, and in the development of interpolants and reduced order models [13]–[17]. Yet, interpolants constructed using Loewner matrices may not possess the structure of the underlying system for arbitrary interpolation points since they are, by construction, independent of any particular state-space representation.

This paper aims to delve into the characterization of tangential interpolation data used to construct reduced order models. In particular, we set conditions upon the tangential data in order to retain the second-order structure and, possibly, the second-order structure of network systems evolving on graphs. To this end, and differently from [5] and [7], we propose a finite family of reduced-order models in the Loewner framework. These models which interpolate the large-scale system, are obtained by partitioning the underlying graph of the system and are parametrized by the cluster matrix.

This paper is organized as follows. In Section II the notion of model reduction in the Loewner framework is reviewed. In Section III the considered class of second-order network systems is presented and the problem of preservation of second-order network structure is formalized. In Section IV the main result of the paper is given. In particular, we consider the scenario in which the tangential data are generated by a second-order network system, and an interpolant retaining the underlying structure is constructed. Then, we specialize the presented conditions in the case in which a graph can be partitioned and a reduced order model is obtained by clustering neighbouring subsystems. In Section V an illustrative example is used to show the effectiveness of the proposed model reduction technique. Finally, in Section VI some concluding remarks are given.

Notation. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of vectors with n rows having complex entries is denoted by $\mathbb{C}^n$. The set of matrices with n rows and m columns having complex entries is denoted by $\mathbb{C}^{n\times m}$. The cardinality of a set $\mathcal{A}$ is denoted by $|\mathcal{A}|$. The spectrum of a square matrix $A \in \mathbb{C}^{n\times n}$ is denoted by $\sigma(A)$. The adjoint matrix of $A \in \mathbb{C}^{n\times m}$ is denoted by A^*. A matrix A is self-adjoint if $A = A^*$. The $n\times 1$ column vector of ones is denoted by $\mathbf{1}_n$. Given a list of n elements a_i, $\mathrm{diag}(a_1, \ldots, a_n)$ indicates a diagonal matrix with diagonal elements a_i's.

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[1]Alessio Moreschini is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K a.moreschini@imperial.ac.uk

[2]Joel D. Simard is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K j.simard18@imperial.ac.uk

[3]Alessandro Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K and with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico 1, 00133 Roma, Italy a.astolfi@imperial.ac.uk

II. PRELIMINARIES

Consider a linear, time-invariant, system of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t),\end{aligned} \tag{1}$$

with state $x(t) \in \mathbb{C}^n$, input $u(t) \in \mathbb{C}^m$, output $y(t) \in \mathbb{C}^p$, and matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, and $C \in \mathbb{C}^{p\times n}$ with constant entries[1]. The triple (A, B, C) is assumed to be a minimal realization of the system (1), *i.e.* the system (1) is reachable and observable.

Tangential interpolation data are central to the construction of a reduced order model which interpolates the transfer matrix $H(s) = C(sI - A)^{-1}B$ of the system (1). These data are represented by sets of right-tangential data, and left-tangential data, of the form

$$\{(\lambda_i, r_i, w_i) \mid \lambda_i \in \mathbb{C},\ r_i \in \mathbb{C}^m,\ w_i \in \mathbb{C}^p,\ i = 1, \dots, \rho\},$$
$$\{(\mu_j, \ell_j, v_j) \mid \mu_j \in \mathbb{C},\ \ell_j^* \in \mathbb{C}^p,\ v_j^* \in \mathbb{C}^m,\ j = 1, \dots, \nu\},$$

respectively. Tangential interpolation data generated by an underlying system are collected by sampling the transfer matrix H along specific directions and frequencies, *i.e.* the sets of data satisfy the so-called interpolation conditions

$$\begin{aligned} H(\lambda_i)r_i &= w_i, \qquad i = 1, \dots, \rho,\\ \ell_j H(\mu_j) &= v_j, \qquad j = 1, \dots, \nu. \end{aligned} \tag{2}$$

Using the interpolation data, we build the Loewner matrix

$$\mathbb{L} = \begin{bmatrix} \frac{v_1 r_1 - \ell_1 w_1}{\mu_1 - \lambda_1} & \cdots & \frac{v_1 r_\rho - \ell_1 w_\rho}{\mu_1 - \lambda_\rho} \\ \vdots & \ddots & \vdots \\ \frac{v_v r_1 - \ell_v w_1}{\mu_\nu - \lambda_1} & \cdots & \frac{v_v r_\rho - \ell_v w_\rho}{\mu_\nu - \lambda_\rho} \end{bmatrix}, \tag{3}$$

and the shifted Loewner matrix

$$\sigma\mathbb{L} = \begin{bmatrix} \frac{\mu_1 v_1 r_1 - \lambda_1 \ell_1 w_1}{\mu_1 - \lambda_1} & \cdots & \frac{\mu_1 v_1 r_\rho - \lambda_\rho \ell_1 w_\rho}{\mu_1 - \lambda_\rho} \\ \vdots & \ddots & \vdots \\ \frac{\mu_v v_v r_1 - \lambda_1 \ell_v w_1}{\mu_v - \lambda_1} & \cdots & \frac{\mu_v v_v r_\rho - \lambda_\rho \ell_v w_\rho}{\mu_v - \lambda_\rho} \end{bmatrix}. \tag{4}$$

Note that the Loewner matrix, $\mathbb{L}$, and the shifted Loewner matrix, $\sigma\mathbb{L}$, are independent of any particular state-space representation of the underlying system. The tangential data can be represented compactly using the matrices

$$\begin{aligned} \Lambda &= \operatorname{diag}(\lambda_1, \dots, \lambda_\rho), & M &= \operatorname{diag}(\mu_1, \dots, \mu_\nu),\\ R &= [r_1\ \cdots\ r_\rho], & L^* &= [\ell_1^*\ \cdots\ \ell_\nu^*],\\ W &= [w_1\ \cdots\ w_\rho], & V^* &= [v_1^*\ \cdots\ v_\nu^*], \end{aligned} \tag{5}$$

with $\Lambda \in \mathbb{C}^{\rho\times\rho}$, $R \in \mathbb{C}^{m\times\rho}$, $W \in \mathbb{C}^{p\times\rho}$, $M \in \mathbb{C}^{\nu\times\nu}$, $L^* \in \mathbb{C}^{p\times\nu}$, and $V^* \in \mathbb{C}^{m\times\nu}$. Throughout the paper we make the standing assumption that A, Λ, and M have no common eigenvalues, *i.e.* $\sigma(\Lambda) \cap \sigma(A) = \emptyset$, $\sigma(M) \cap \sigma(\Lambda) = \emptyset$, and $\sigma(A) \cap \sigma(M) = \emptyset$. It follows that the Loewner matrix (3) is the unique solution of the Sylvester equation

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR, \tag{6}$$

and the shifted Loewner matrix (4) is the unique solution of the Sylvester equation

$$\sigma\mathbb{L}\Lambda - M\sigma\mathbb{L} = LW\Lambda - MVR. \tag{7}$$

By uniqueness of solutions to the equations (6) and (7) it follows that the shifted Loewner matrix, $\sigma\mathbb{L}$, can be equivalently expressed as

$$\sigma\mathbb{L} = \mathbb{L}\Lambda + VR, \quad \sigma\mathbb{L} = M\mathbb{L} + LW.$$

Following [14], for given Λ and M, if the matrices $\mathbb{L}$, $\sigma\mathbb{L}$, V, and W are known, $\rho = \nu$, and $(s\mathbb{L} - \sigma\mathbb{L})$ is nonsingular for all $s \in \sigma(\Lambda) \cup \sigma(M)$, then a model satisfying conditions (2), with state $r(t) \in \mathbb{C}^\rho$, input $u_r(t) \in \mathbb{C}^m$, and output $y_r(t) \in \mathbb{C}^p$, is described by the equations

$$\begin{aligned} \mathbb{L}\dot{r}(t) &= \sigma\mathbb{L}r(t) - Vu_r(t),\\ y_r(t) &= Wr(t). \end{aligned} \tag{8}$$

The system (8) is by construction an interpolant of the system (2) at the tangential interpolation data (Λ, R, M, L), *i.e.* the transfer matrix $H_r(s) = -W(s\mathbb{L} - \sigma\mathbb{L})^{-1}V$ satisfies the interpolation conditions

$$\begin{aligned} H(\lambda_i)r_i = H_r(\lambda_i)r_i &= w_i, \qquad i = 1, \dots, \rho,\\ \ell_j H(\mu_j) = \ell_j H_r(\lambda_j) &= v_j, \qquad j = 1, \dots, \nu. \end{aligned} \tag{9}$$

The following definition is introduced to define a reduced order model in the Loewner framework.

Definition 1 (Reduced order model)**.** Let Σ and $\bar{\Sigma}$ be two systems of order n and v, respectively. $\bar{\Sigma}$ is a reduced order model of Σ in the Loewner sense if Σ and $\bar{\Sigma}$ satisfies the interpolation conditions (9) at (Λ, R, M, L) and $v < n$.

Consider now two auxiliary systems: a "signal generator" which is constructed from right-tangential interpolation data and it is described by the equations

$$\dot{\zeta}_r(t) = \Lambda\zeta_r(t), \quad z(t) = R\zeta_r(t), \tag{10}$$

with state $\zeta_r(t) \in \mathbb{C}^\rho$ and output $z(t) \in \mathbb{C}^m$, and a "filter" which is constructed from left-tangential interpolation data and it is described by the equations

$$\dot{\zeta}_\ell(t) = M\zeta_\ell(t) + Lv(t), \tag{11}$$

with state $\zeta_\ell(t) \in \mathbb{C}^v$ and input $v(t) \in \mathbb{C}^p$. The cascade interconnection of (10) and the plant (1) obtained setting $u = z$ possesses an invariant manifold $\mathcal{M}_X = \{(x, \zeta_r) \in \mathbb{C}^n \times \mathbb{C}^\rho \mid x = X\zeta_r\}$, in which the tangential generalized controllability matrix, X, is defined as the unique solution, by our standing assumption, of the Sylvester equation

$$AX + BR = X\Lambda. \tag{12}$$

Similarly, the cascade interconnection of the plant (1) and (11) obtained setting $v = y$ possesses an invariant manifold $\mathcal{M}_Y = \{(x, \zeta_\ell) \in \mathbb{C}^n \times \mathbb{C}^v \mid \zeta_\ell = -Yx\}$, in which the tangential generalized observability matrix, Y, is defined as the unique solution, by our standing assumption, of the Sylvester equation

$$YA + LC = MY. \tag{13}$$

[1]Throughout the paper, we consider complex-valued signals and matrices which are solely obtained through coordinates transformation of real-valued signals and matrices, as in [16].

Consequently, the Loewner matrix and the shifted Loewner matrix can be expressed as

$$\mathbb{L} = -YX, \qquad \sigma\mathbb{L} = -YAX, \tag{14}$$

and the right tangential data matrix, W, and the left tangential data matrix, V, can be expressed as

$$W = CX, \qquad V = YB, \tag{15}$$

respectively. Hence, considering the interconnection of the signal generator (10), the plant (1), and the filter (11), obtained setting $u = z$ and $v = y$, and the restriction to $\mathcal{M}_X$ and the projection on $\mathcal{M}_Y$ of the dynamics, yields

$$\begin{aligned} YX\dot{\zeta}_r(t) &= YAX\zeta_r(t) + YBu(t), \\ y(t) &= CX\zeta_r(t). \end{aligned} \tag{16}$$

The system (16), which is algebraically equivalent to the interpolant (8), is essential for constructing an interpolant which retains the structure of the underlying system.

III. Problem Formulation

In what follows we consider the model reduction problem for second-order network systems on graphs. A graph is a pair $\mathbb{G} := (\mathbb{V}, \mathbb{E})$ composed of a set of vertices, $\mathbb{V} = \{v_1, \ldots, v_{\bar{n}}\}$, and a set of edges, $\mathbb{E} \subset \mathbb{V} \times \mathbb{V}$. A graph is weighted if, for any pair $(v_i, v_j) \in \mathbb{E}$, the associated edge is assigned a $\mu_{i,j} \in \mathbb{R}\backslash 0$. For all $(v_i, v_j) \notin \mathbb{E}$ we consider $\mu_{i,j} = 0$. For any undirected weighted graph $\mathbb{G}$ we can define a $\bar{n} \times \bar{n}$ Laplacian matrix, $\mathcal{L}$, with (i,j)-th entry defined as

$$\mathcal{L}_{i,j} := \begin{cases} \sum_{k=1, k\neq j}^{n} \mu_{k,j} & i = j, \\ -\mu_{i,j} & \text{otherwise.} \end{cases}$$

A second-order network system on a graph $\mathbb{G}$ is a network system described by an inhomogeneous second-order differential equation of the form

$$\begin{aligned} Gu(t) &= P\ddot{q}(t) + D\dot{q}(t) + Kq(t), \\ y(t) &= Z_1 q(t) + Z_2\dot{q}(t), \end{aligned} \tag{17}$$

where $q(t) \in \mathbb{C}^{\bar{n}}$ describes the state, $y(t) \in \mathbb{C}^p$ describes the output, and $u(t) \in \mathbb{C}^m$ describes the input. The matrix $P \in \mathbb{C}^{\bar{n}\times\bar{n}}$ is assumed self-adjoint and positive definite. The matrices $K \in \mathbb{C}^{\bar{n}\times\bar{n}}$ and $D \in \mathbb{C}^{\bar{n}\times\bar{n}}$ are assumed self-adjoint and positive definite and are defined as

$$K = S_k + \alpha_k\mathcal{L}, \quad D = S_d + \alpha_d K,$$

respectively, where S_k and S_d are non-negative diagonal matrices; α_k and α_d are positive scalars; and $\mathcal{L}$ is the Laplacian matrix associated with the weighted graph $\mathbb{G}$. By its structure, the second-order network system (17) can be written in the form (1) with $n = 2\bar{n}$ and matrices

$$\begin{aligned} A &= \begin{bmatrix} 0 & P^{-1} \\ -K & -DP^{-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ G \end{bmatrix}, \\ C &= \begin{bmatrix} Z_1 & Z_2P^{-1} \end{bmatrix}. \end{aligned} \tag{18}$$

The objective of this paper is to determine conditions on the tangential interpolation data such that an interpolant of the high-order system not only interpolates the system at desired frequencies and along desired directions, but also has the form (17). In particular, the interpolant must be a reduced second-order network system of the form

$$\begin{aligned} \bar{G}\bar{u}(t) &= \bar{P}\ddot{\bar{q}}(t) + \bar{D}\dot{\bar{q}}(t) + \bar{K}\bar{q}(t), \\ \bar{y}(t) &= \bar{Z}_1\bar{q}(t) + \bar{Z}_2\dot{\bar{q}}(t), \end{aligned} \tag{19}$$

with reduced state $\bar{q}(t) \in \mathbb{C}^{\bar{\rho}}$ (for $\bar{\rho} < \bar{n}$), self-adjoint and positive definite matrices $\bar{P} \in \mathbb{C}^{\bar{\rho}\times\bar{\rho}}$, $\bar{K} \in \mathbb{C}^{\bar{\rho}\times\bar{\rho}}$, and $\bar{D} \in \mathbb{C}^{\bar{\rho}\times\bar{\rho}}$ defined as

$$\bar{K} = \bar{S}_k + \bar{\alpha}_k\bar{\mathcal{L}}, \quad \bar{D} = \bar{S}_d + \bar{\alpha}_d\bar{K}, \tag{20}$$

where $\bar{S}_k$ and $\bar{S}_d$ are non-negative diagonal matrices, and $\bar{\alpha}_k$ and $\bar{\alpha}_d$ are positive scalars. In this respect, the matrix $\bar{\mathcal{L}}$ should be a reduced order Laplacian matrix associated with a weighted and undirected graph $\bar{\mathbb{G}}$.

IV. Main Result

To begin with, consider a nonsingular Loewner matrix, $\mathbb{L}$, and the right- and left-tangential data (5). With this in mind, the coordinates transformation $\omega := \mathbb{L}r$ yields

$$\begin{aligned} \dot{\omega}(t) &= \sigma\mathbb{L}\mathbb{L}^{-1}\omega(t) - Vu_r(t), \\ y_r(t) &= W\mathbb{L}^{-1}\omega(t). \end{aligned} \tag{21}$$

To discuss the second-order structure we set the order of the reduction $\rho = 2\bar{\rho}$ and, without loss of generality, we consider Loewner matrices as block-structured matrices of the form

$$\sigma\mathbb{L} := \begin{bmatrix} \sigma\mathbb{L}_{11} & \sigma\mathbb{L}_{12} \\ \sigma\mathbb{L}_{21} & \sigma\mathbb{L}_{22} \end{bmatrix}, \quad \mathbb{L} := \begin{bmatrix} \mathbb{L}_{11} & \mathbb{L}_{12} \\ \mathbb{L}_{21} & \mathbb{L}_{22} \end{bmatrix}, \tag{22}$$

with $\mathbb{L}_{ij} \in \mathbb{C}^{\bar{\rho}\times\bar{\rho}}$, $\sigma\mathbb{L}_{ij} \in \mathbb{C}^{\bar{\rho}\times\bar{\rho}}$. In a similar fashion, the right-tangential data can be assumed in the block structure

$$\Lambda := \begin{bmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{bmatrix}, \quad \begin{aligned} R &:= \begin{bmatrix} R_1 & R_2 \end{bmatrix}, \\ W &:= \begin{bmatrix} W_1 & W_2 \end{bmatrix}, \end{aligned} \tag{23}$$

with $\Lambda_{ij} \in \mathbb{C}^{\bar{\rho}\times\bar{\rho}}$, $R_i \in \mathbb{C}^{m\times\bar{\rho}}$, and $W_i \in \mathbb{C}^{p\times\bar{\rho}}$, and the left-tangential data

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \quad L := \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}, \quad V := \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}, \tag{24}$$

with $M_{ij} \in \mathbb{C}^{\bar{\rho}\times\bar{\rho}}$, $L_i \in \mathbb{C}^{\bar{\rho}\times p}$, and $V_i \in \mathbb{C}^{\bar{\rho}\times m}$.

The following is a list of lemmata concerning particular properties which can be achieved by selecting tangential interpolation data.

Lemma 1. Consider the right-tangential data, (Λ, R, W), and the left-tangential data, (M, L, V), and suppose that $\mathbb{L}$ and $\sigma\mathbb{L}$ in (22) are such that

$$\begin{aligned} \sigma\mathbb{L}_{11} &= 0, \quad \sigma\mathbb{L}_{12} = I, \quad \sigma\mathbb{L}_{21} = -I, \\ \mathbb{L}_{12} &= 0, \quad \mathbb{L}_{21} = 0, \quad V_1 = 0. \end{aligned} \tag{25}$$

Then the interpolant (21) is equivalent to a second-order inhomogeneous system of the form

$$\begin{aligned} -V_2\bar{u}(t) &= \mathbb{L}_{22}\ddot{\bar{q}}(t) - \sigma\mathbb{L}_{22}\dot{\bar{q}}(t) + \mathbb{L}_{11}^{-1}\bar{q}(t), \\ \bar{y}(t) &= W_1\mathbb{L}_{11}^{-1}\bar{q}(t) + W_2\dot{\bar{q}}(t), \end{aligned} \tag{26}$$

Proof: Consider $\mathbb{L}$ and $\sigma\mathbb{L}$ as in (22), W as in (23), and V as in (24). With (25) in mind we have that

$$\sigma\mathbb{L} = \begin{bmatrix} 0 & I \\ -I & \sigma\mathbb{L}_{22} \end{bmatrix}, \quad \mathbb{L} = \begin{bmatrix} \mathbb{L}_{11} & 0 \\ 0 & \mathbb{L}_{22} \end{bmatrix}, \quad V = \begin{bmatrix} 0 \\ V_2 \end{bmatrix}.$$

By substituting the above matrices in (21), and setting $\omega = (\omega_1, \omega_2)$, we compute the second order equation

$$\mathbb{L}_{22}\ddot{\omega}_1(t) = \sigma\mathbb{L}_{22}\mathbb{L}_{22}^{-1}\omega_2(t) - \mathbb{L}_{11}^{-1}\omega_1(t) - V_2 u_r(t),$$
$$y_r(t) = W_1\mathbb{L}_{11}^{-1}\omega_1(t) + W_2\mathbb{L}_{22}^{-1}\omega_2(t).$$

Finally, setting $\omega_1 = \bar{q}$ and $\omega_2 = \mathbb{L}_{22}\dot{\bar{q}}$ yields the second-order inhomogeneous system (26). □

To address the structure preservation problem we assume the existence of an interpolant of the form (16) and an underlying system of the form (17) generating the tangential interpolation data. By the second-order structure (17) the tangential generalized controllability matrix, X, and the tangential generalized observability matrix, Y, yield block-structured matrices of the form

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}, \quad Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix},$$

with $X_{ij} \in \mathbb{C}^{\bar{n}\times\bar{\rho}}$ and $Y_{ij} \in \mathbb{C}^{\bar{\rho}\times\bar{n}}$.

Lemma 2. Consider the network system (17) and let (21) be its interpolant at (Λ, R, M, L). Suppose that (Λ, R, M, L) are such that X and Y satisfy (12) and (13) with

$$X_{12} = 0, \quad X_{21} = 0, \quad Y_{12} = 0, \quad Y_{21} = 0. \tag{27}$$

Then (21) yields a second-order system of the form (26) with matrices

$$\begin{aligned} \mathbb{L}_{12} &= 0, & \mathbb{L}_{11} &= -Y_{11}X_{11}, \\ \mathbb{L}_{21} &= 0, & \mathbb{L}_{22} &= -Y_{22}X_{22}, \\ V_1 &= 0, & W_1 &= Z_1X_{11}, \\ V_2 &= Y_{22}G, & W_2 &= Z_2P^{-1}X_{22}, \end{aligned} \tag{28}$$

and

$$\begin{aligned} \sigma\mathbb{L}_{11} &= 0, & \sigma\mathbb{L}_{12} &= -Y_{11}P^{-1}X_{22}, \\ \sigma\mathbb{L}_{21} &= Y_{22}KX_{11}, & \sigma\mathbb{L}_{22} &= Y_{22}DP^{-1}X_{22}. \end{aligned} \tag{29}$$

Proof: Recall that, in terms of X and Y, the structure of $\mathbb{L}$ and $\sigma\mathbb{L}$ is as in (14) and the structure of V and W is as in (15). Then, by construction, we have that the conditions (27) imply (28) and (29). □

Lemma 3. Consider the network system (17) and let (21) be its interpolant at (Λ, R, M, L). Suppose that

$$\begin{aligned} X_{11} &= -K^{-1}Y_{11}^*, & X_{12} &= 0, \\ X_{21} &= 0, & X_{22} &= -PY_{22}^*. \end{aligned} \tag{30}$$

If there exists a unique matrix, Y, satisfying (12), and such that (30) satisfies (13), with $Y_{12} = 0$ and $Y_{21} = 0$, then

$$\sigma\mathbb{L}_{11} = 0, \quad \sigma\mathbb{L}_{12} = -\sigma\mathbb{L}_{21}^*, \quad \sigma\mathbb{L}_{22} = \sigma\mathbb{L}_{22}^* < 0.$$

Moreover, if for some nonsingular matrix $\Phi \in \mathbb{C}^{\bar{n}\times\bar{n}}$

$$Y_{11} = (Y_{22}\Phi Y_{22}^*)^{-1}Y_{22}\Phi, \tag{31}$$

then $\sigma\mathbb{L}_{12} = -\sigma\mathbb{L}_{21} = I$.

Proof: Substituting (30) in (29) we have that

$$\sigma\mathbb{L}_{12} = Y_{11}Y_{22}^*, \quad \sigma\mathbb{L}_{21} = -Y_{22}Y_{11}^*, \quad \sigma\mathbb{L}_{22} = -Y_{22}DY_{22}^*, \tag{32}$$

and thus, by construction, $\sigma\mathbb{L}_{12} = -\sigma\mathbb{L}_{21}^*$ and $\sigma\mathbb{L}_{22} = \sigma\mathbb{L}_{22}^* < 0$ since $D = D^* > 0$. Finally, combining the structure (31) with the matrices in (32) we directly deduce that $\sigma\mathbb{L}_{12} = I$ and $\sigma\mathbb{L}_{21} = -I$. □

The lemmata lay the foundation for the solution of the considered problem. In particular, we present an interpolant for the second-order network system (17) which preserves the structure of (17) while accomplishing interpolation at (Λ, R, M, L).

Proposition 1 (Structure preservation)**.** *Consider the network system* (17) *and let* (21) *be its interpolant at* (Λ, R, M, L). *Suppose that there exist* X *and* Y, *unique solutions of* (12) *and* (13), *respectively, of the form*

$$\begin{aligned} Y &= \begin{bmatrix} (Y_{22}KY_{22}^*)^{-1}Y_{22}K & 0 \\ 0 & Y_{22} \end{bmatrix}, \\ X &= \begin{bmatrix} -Y_{22}^*(Y_{22}KY_{22}^*)^{-1} & 0 \\ 0 & -PY_{22}^* \end{bmatrix}. \end{aligned} \tag{33}$$

Then the interpolant (21) *is equivalent to the reduced order network system* (19) *with*

$$\begin{aligned} \bar{K} &= Y_{22}KY_{22}^*, & \bar{P} &= Y_{22}PY_{22}^*, & \bar{D} &= Y_{22}DY_{22}^*, \\ \bar{G} &= -Y_{22}G, & \bar{Z}_1 &= -Z_1Y_{22}^*, & \bar{Z}_2 &= -Z_2Y_{22}^*. \end{aligned} \tag{34}$$

Proof: The proof is constructive and follows by substituting the matrices (28) and (33) into the system (26). □

The construction of an interpolant for the second-order network systems (17), which preserves the structure at (Λ, R, M, L), yields a reduced Laplacian matrix, $\bar{\mathcal{L}}$, which is not directly obtained by partitioning the vertices of the underlying graph $\mathbb{G}$. The obtained reduced Laplacian matrix in (20) is expressed in terms of the bottom-right block of the tangential generalized observability matrix, Y, *i.e.* $\bar{\mathcal{L}} = Y_{22}\mathcal{L}Y_{22}^*$, where the matrix Y_{22} is such that the Laplacian structure of the matrix $\mathcal{L}$ is retained in (19). In this respect, we proceed by associating to Y_{22} the properties of a cluster matrix, with the aim of retaining not only the Laplacian structure of $\bar{\mathcal{L}}$ but also the structure of the underlying graph of the reduced order model.

Recall that a cluster of a graph $\mathbb{G}$ is a nonempty index subset of $\mathbb{V}$, and a cluster matrix associated to a clustering $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_{\bar{\rho}}\}$ is defined by $\mathcal{T} := (\kappa(\mathcal{C}_1), \kappa(\mathcal{C}_2), \cdots, \kappa(\mathcal{C}_{\bar{\rho}}))$, where $\kappa(\mathcal{C}_i) \in \mathbb{R}^{\bar{n}}$ is the characteristic vector of $\mathcal{C}_i$. The cluster matrix is such that: the s-th element of $\kappa(\mathcal{C}_i)$ is 1 if $v_s \in \mathcal{C}_i$ and 0 otherwise; the clustering $\mathcal{C} = \{\mathcal{C}_1, \ldots, \mathcal{C}_{\bar{\rho}}\}$ is a partition of $\mathbb{G}$, that is $\mathcal{C}_i \cap \mathcal{C}_j = \emptyset$ and $\bigcup_{i=1}^{\bar{\rho}} \mathcal{C}_i = \mathbb{V}$; the characteristic vector $\kappa(\mathcal{C}_i) \in \mathbb{R}^{\bar{n}}$ is such that $\mathbf{1}_{\bar{n}}^*\kappa(\mathcal{C}_i) = |\mathcal{C}_i|$ with $\mathcal{C}_i \neq \emptyset$. The finite set $\mathbb{T} = \{\mathcal{T}_1, \ldots, \mathcal{T}_{\varphi}\}$ is the set of all $\bar{n} \times \bar{\rho}$ cluster matrices obtained partitioning the set $\mathbb{V}$.

With conditions (28) in mind, we can force the matrix Y_{22} to define a clustering $\mathcal{C}$, *i.e.* $Y_{22}^* = \mathcal{T} \in \mathbb{T}$. The selection of the clustering implies that the matrices in (34) - characterizing the reduced order second-order equation (19) - are solely composed of a cluster matrix, $\mathcal{T}$, which is obtained by partitioning the set $\mathbb{V}$, and of the matrices P, D, and K, which are matrices defining the underlying system, namely

$$\bar{K} = \mathcal{T}^* K \mathcal{T}, \quad \bar{P} = \mathcal{T}^* P \mathcal{T}, \quad \bar{D} = \mathcal{T}^* D \mathcal{T}.$$

Theorem 1. *Consider a system of the form* (1) *with matrices* (18)*. Then for any $\bar{n}\times\bar{\rho}$ cluster matrix $\mathcal{T} \in \mathbb{T}$ there exists a reduced order network of the form* (21) *with matrices*

$$\begin{aligned} \mathbb{L} &= \begin{bmatrix} \bar{K}^{-1} & 0 \\ 0 & \bar{P} \end{bmatrix}, \quad \sigma\mathbb{L} = \begin{bmatrix} 0 & I \\ -I & -\bar{D} \end{bmatrix}, \\ V &= \begin{bmatrix} 0 \\ \mathcal{T}^* G \end{bmatrix}, \qquad W = -\begin{bmatrix} Z_1 \mathcal{T} \bar{K}^{-1} & Z_2 \mathcal{T} \end{bmatrix}, \end{aligned} \tag{35}$$

which is an interpolant at

$$\begin{aligned} \Lambda &= \begin{bmatrix} 0 & \bar{K} \\ -\bar{P}^{-1}(I + \mathcal{T}^* G R_1) & -\bar{P}^{-1}(\bar{D} + \mathcal{T}^* G R_2) \end{bmatrix}, \\ M &= \begin{bmatrix} L_1 Z_1 \mathcal{T} & (I + L_1 Z_2 \mathcal{T})\bar{P}^{-1} \\ L_2 Z_1 \mathcal{T} - \bar{K} & (\sigma\mathbb{L}_{22} + L_2 Z_2 \mathcal{T})\bar{P}^{-1} \end{bmatrix}, \end{aligned} \tag{36}$$

for any pair (R, L) such that $\sigma(\Lambda) \cap \sigma(A) = \emptyset$, $\sigma(M) \cap \sigma(\Lambda) = \emptyset$, and $\sigma(A) \cap \sigma(M) = \emptyset$.

Proof: For a given system with matrices (18), we set the tangential interpolation data (Λ, M) as in (36) for any pair (R, L). Then, plugging (18) and (36) into the Sylvester equations (12) and (13) we have that the matrices

$$Y = \begin{bmatrix} (\mathcal{T}^* K \mathcal{T})^{-1} \mathcal{T}^* K & 0 \\ 0 & \mathcal{T}^* \end{bmatrix}, \quad X = -\begin{bmatrix} \mathcal{T}(\mathcal{T}^* K \mathcal{T})^{-1} & 0 \\ 0 & P\mathcal{T} \end{bmatrix},$$

are solutions of (12) and (13), respectively. However, since (Λ, M) in (36) are parametrized by R and L, respectively, then to ensure uniqueness of the solutions of (12) and (13) the matrices R and L must be such that $\sigma(\Lambda) \cap \sigma(A) = \emptyset$ and $\sigma(A) \cap \sigma(M) = \emptyset$. Hence, from the computed X and Y and from the equations (14) and (15) we have that the matrices $\mathbb{L}$, $\sigma\mathbb{L}$, V, and W yield the form (35). Finally, by construction, $\mathbb{L}$ and $\sigma\mathbb{L}$ are solutions of (7) and (6), respectively, and uniqueness is achieved since $\sigma(M) \cap \sigma(\Lambda) = \emptyset$. □

Theorem 1 reveals that there exists a finite family of reduced order model, parametrized by the cluster matrix $\mathcal{T} \in \mathbb{T}$, which preserves the network system structure and interpolates the system (17) at (36). The advantage of using the result of Theorem 1 is that, since the reduced order model meets the interpolation conditions by construction, we can always build a reduced order model which preserves the network system structure without solving any Sylvester equations for any arbitrary $\bar{n} \times \bar{\rho}$ cluster matrix $\mathcal{T} \in \mathbb{T}$. Hence, for each cluster matrix $\mathcal{T}_i \in \mathbb{T}$ there exists a transfer function for the system (21) of the form

$$H_{r_i}(s) = -W_i (s\mathbb{L}_i - \sigma\mathbb{L}_i)^{-1} V_i \tag{37}$$

Algorithm 1 Minimum $\mathcal{H}_2$-norm Loewner interpolant

Input: A, B, C, as in (18)

Initialize the order of the reduction $\bar{\rho}$ such that $\bar{\rho} < \bar{n}$

Compute the set $\mathbb{T} = \{\mathcal{T}_1, \dots, \mathcal{T}_\varphi\}$ containing all the $\bar{n} \times \bar{\rho}$ cluster matrices partitioning the underlying graph

Set H as the transfer function with matrices (A, B, C)

for $i = 1 : \varphi$ **do**

  Set H_{r_i} as the i-th transfer function in (37)

  Set $\mathbf{e}_i = ||H - H_{r_i}||_{\mathcal{H}_2}$ as the $\mathcal{H}_2$-norm of the error

end for

Select k such that $\mathbf{e}_k = \min(\mathbf{e}_1, \cdots, \mathbf{e}_\varphi)$

Select $\mathcal{T}_k$ from $\mathbb{T}$

return $\mathbb{L}_k$, $\sigma\mathbb{L}_k$, V_k, W_k from (38)

with matrices as in (35), *i.e.*

$$\begin{aligned} \mathbb{L}_i &= \begin{bmatrix} (\mathcal{T}_i^* K \mathcal{T}_i)^{-1} & 0 \\ 0 & \mathcal{T}_i^* P \mathcal{T}_i \end{bmatrix}, \ \sigma\mathbb{L}_i = \begin{bmatrix} 0 & I \\ -I & -\mathcal{T}_i^* D \mathcal{T}_i \end{bmatrix}, \\ V_i &= \begin{bmatrix} 0 \\ \mathcal{T}_i^* G \end{bmatrix}, \ W_i = -\begin{bmatrix} Z_1 \mathcal{T}_i \bar{K}^{-1} & Z_2 \mathcal{T}_i \end{bmatrix}. \end{aligned} \tag{38}$$

In this respect, among all the possible interpolants associated with the clustering in (37), we can formulate an optimization problem in terms of the $\mathcal{H}_2$-norm of the error between the transfer function of (17) and the reduced order interpolant (37) as follows:

$$k = \arg\min_i ||H(j\omega) - H_{r_i}(j\omega)||_{\mathcal{H}_2}. \tag{39}$$

The $\mathcal{H}_2$-norm of the error is reported within the pseudocode in Algorithm 1. We point out that the complexity of the algorithm is related to the cardinality of $\mathbb{T}$, *i.e.* the number of disjoint partitions of the underlying graph.

V. Illustrative Example

In this section, the theoretical results discussed in Section IV and the effectiveness of Algorithm 1 are illustrated. The considered example is a second-order network system of the form (17) of order $\bar{n} = 8$ with $P = \mathrm{diag}(10, 2, 1, 1, 2, 3, 6, 8)$, $S_k = I$, $S_d = 0$, $Z_1 = 0$, $Z_2 = G^* = (0, 0, 0, 0, 0, 1, 0, 0)$, $\alpha_k = 1$, $\alpha_d = 0.15$, and

$$\mathcal{L} = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 5 & 0 & 0 & -3 & -2 & 0 & 0 \\ 0 & 0 & 6 & -1 & -2 & -3 & 0 & 0 \\ 0 & 0 & -1 & 6 & -5 & 0 & 0 & 0 \\ 0 & -3 & -2 & -5 & 25 & -2 & -6 & -7 \\ -5 & -2 & -3 & 0 & -2 & 25 & -6 & -7 \\ 0 & 0 & 0 & 0 & -6 & -6 & 13 & -1 \\ 0 & 0 & 0 & 0 & -7 & -7 & -1 & 15 \end{bmatrix}. \tag{40}$$

The second-order network system yields a system of the form (1) for $n = 16$ and matrices (18). Suppose that we want to compute a reduced order model which retains the network structure with a desired reduced order $\bar{\rho} = 3$. Applying Algorithm 1 we find that the number of disjoint partitions associated with (40) is $\varphi = 966$. We then construct the reduced order network system of the form (21) of order $\rho = 6$ with matrices as in (38). The Bode diagram of the

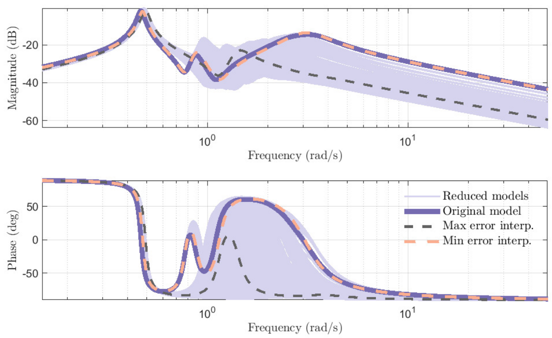

Fig. 1: Bode plots of the original system with $n = 16$, and of all the reduced order models with $\rho = 6$.

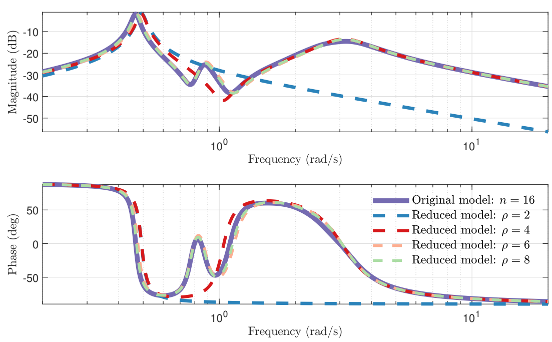

Fig. 2: Bode plots of the original system with $n = 16$, and the minimum error interpolant for each $\rho \in \{2, 4, 6, 8\}$.

minimum error interpolant is plotted in Figure 1 highlighting the difference with the other 965 reduced order models. Accordingly, for the sake of illustration, Figure 2 illustrates the Bode diagrams associated with the reduced order models obtained by applying the proposed algorithm for $\rho \in \{2, 4, 6, 8\}$. The minimum error for each $\rho \in \{2, 4, 6, 8\}$ is approximately 0.2, 0.09, 0.04, and 0.02, respectively. Finally, due to the interpolation matrices given in (36), we have that for each $\rho \in \{2, 4, 6, 8\}$ the minimum error model solving the optimization problem (39) interpolates the high-order model at the points reported in Table I.

VI. Conclusion

Second-order network systems which evolve on graphs are usually described by a large number of differential equations. In this paper, we have addressed the model reduction problem in the Loewner framework with the objective of preserving the second-order network structure of the underlying large-scale system. In this respect, a family of reduced order models which not only interpolates the underlying system but also retains the network structure has been presented. It has been shown that by partitioning the underlying graph of the system the associated reduced order model meets, by construction, the interpolation conditions at particular tangential interpolation points which are parametrized by the cluster matrix. Additionally, to select the most accurate reduced order model among all the possible interpolants, we have formulated an optimization problem in terms of the

Table I: Interpolation points for each order ρ

2	4	6	8
$-\frac{1}{30} \pm j\frac{217}{312}$	$-\frac{479}{562} \pm i\frac{1479}{497}$ $-\frac{196}{4419} \pm j\frac{2028}{2933}$	$-\frac{1053}{1246} \pm j\frac{2179}{730}$ $-\frac{310}{5973} \pm j\frac{1055}{1666}$ $-\frac{954}{15347} \pm j\frac{774}{835}$	$-\frac{230}{279} \pm j\frac{2380}{797}$ $-\frac{1328}{3713} \pm j\frac{3412}{1587}$ $-\frac{244}{4081} \pm j\frac{436}{695}$ $-\frac{151}{2162} \pm j\frac{1838}{2013}$
$-\frac{1}{330} \pm j\frac{697}{1001}$	$-\frac{454}{795} \pm j\frac{3218}{1059}$ $\frac{337}{43565} \pm j\frac{729}{1051}$	$-\frac{743}{1260} \pm j\frac{6598}{2169}$ $\frac{148}{11319} \pm j\frac{1934}{3033}$ $-\frac{859}{17450} \pm j\frac{1053}{1138}$	$-\frac{709}{1153} \pm j\frac{1695}{559}$ $-\frac{299}{867} \pm j\frac{2503}{1163}$ $-\frac{107}{2603} \pm j\frac{566}{621}$ $\frac{130}{5763} \pm j\frac{281}{444}$

$\mathcal{H}_2$-norm of the error between the transfer function of the underlying network system and the family of interpolants. Finally, the theoretical results have been illustrated by means of an example comparing the Bode plot of the underlying model with the proposed minimum error interpolant and showing the effectiveness of the proposed approach.

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