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# Stabilizability of Game-based Control Systems

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*Abstract*— **We investigate the stabilizability of Nash equilibrium of the nonlinear game-based control system (GBCS), which has a similar decision-making structure as Stackelberg Games: one leader and multiple rational followers or agents. While the leader of the GBCS is a macro-regulator rather than a game player. The stabilizability problem is that whether the regulator can stabilize the system by regulating the Nash equilibrium formed by the agents at the lower level. The stabilizability of linear-quadratic GBCS has been investigated in [10]. In this paper, we will first formulate the stabilizability problem of the general nonlinear GBCS. Then the stabilizability relationships between the linear-quadratic and nonlinear GBCSs are given, by investigating the solvable relationship of the associated algebraic Riccati equations and nonlinear Hamilton-Jacobi equations.**

## I. INTRODUCTION

The game-based control system (GBCS) framework was first introduced in [7] to investigate the control problems of complex systems whose behaviours are not only driven by physical laws, but also by their own interests or willingness, such as, economic and the rapidly developing "intelligent" systems. In traditional control theoretical framework, the plants to be controlled are usually modelled by physical laws and do not have their own objective functions, such as cars, air planes, and industrial processes. The Lucas criticism and Braess's paradox show that the traditional control theory cannot be used directly to control such systems.

GBCS has a hierarchical decision-making structure: one regulator and multiple agents. The regulator is regarded as the global controller and makes decision first, and then the agents try to optimize their respective objective functions. Hence, for any given control of the regulator, the agents form a differential game and each agent solves an optimal control problem. In many cases, there is no natural terminal time $T$ when the decision process stops and any choice of the $T$ can always be contested [13, Page 5]. Hence, many economic models, such as optimal economic growth model and dynamic general equilibrium model, consider the optimization on an unbounded time interval. Sometimes the terminal time $T$ may be a random stopping time which represents the time of death or cease. Using the method in [15], we can reformulate equivalently this as an infinite-horizon decision problem under some weak assumptions. Because of these reasons, we will investigate the infinite-horizon GBCS in this paper.

For the infinite-horizon GBCS stability becomes an issue, which is an important concern for a dynamic system and is usually the first requirement to be considered. In game systems, there are many examples where individuals pursue their own interests and make the system unstable or even collapse, such as financial crises. Forcing rational agents to adopt cooperative behavior to stabilize the system is relatively difficult to achieve in reality. An easier way is to intervene in the game formed by the rational agents through a third party or upper level regulator, which makes the strategic behavior of individuals automatically stabilizes the system. For example, a government agency is in charge of designing an macrocontrol which would introduce the agents to satisfy globally the stability conditions. Therefore, whether there is such higher-level intervention and how to conduct it become a key issue. From the perspective of control theory, this is the stabilizability problem.

To the best of our knowledge, except for [10], there is no systematic study on the stability of this kind of system and only a few sporadic ad hoc methods have appeared in the literature. The authors of [10] have investigated the stabilizability of linear-quadratic GBCS and some explicit algebraic conditions have been given, where the dynamics of the system are linear and the payoff functions of the agents are quadratic. However, most real dynamic game systems are nonlinear, which makes the existing results in [10] have significant limitations. Inspired by the idea of Lyapunov indirect methods, we may apply the existing results of linear-quadratic GBCSs to nonlinear GBCSs by studying the relationship between the stabilizability of nonlinear systems and their linear-quadratic approximation systems. In this paper, we will first study the approximation problem of nonlinear optimal control, and then use the corresponding results to study the linear-quadratic approximation problem of GBCS. Using the tool of the symplectic geometry, [11] has given some results of the approximation problem of the optimal control, where there is no discount factor in the agents' payoff functions. For our general GBCS framework, the discount factor of the agents' payoff functions may not be zero, which makes essential differences in the Hamilton-Jacobi equations describing the optimal control, and so the method of [11] cannot be directly applied to our research on related problem. Fortunately, it can be handled through the methods of the contact geometry.

The rest of this paper is organized as follows. In Section II, some key concepts of the general nonlinear optimal control are given, and then the game-based control systems (GBCS) and the stabilizability problem are introduced. Section IV gives the main results and the proofs of them are given in

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this section. Section V will conclude the paper with some remarks.

## II. Preliminaries

### A. Nonlinear optimal control problem

Consider the following infinite-horizon nonlinear optimal control (INOC) problem: The system dynamics are described by

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)), \\ x(0) = x_0, \ t \in [0, \infty), \end{cases} \tag{1}$$

and the payoff function to be minimized is

$$J(x_0, u) = \int_0^\infty e^{-\alpha t} l(x(t), u(t)) dt. \tag{2}$$

where $f$ and $l$ are smooth functions and $\alpha \geq 0$ is the discount rate ($e^{-\alpha t}$ is the discount factor). We assume that there exists a unique solution of (1) for any $u(\cdot) \in \mathcal{U}$ and $x_0 \in \mathbb{R}^n$, where $\mathcal{U}$ is the admissible controls set. Here, we assume that

$$\mathcal{U} = \{u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^m) \mid u \text{ is piecewise continuous}\}. \tag{3}$$

Because the integral in (2) is over an infinite time interval and no priori assumptions of boundedness of $l$ are made, we must make clear what is the meaning of it. Let $x(\cdot;\ x_0, u)$ denote the state trajectory of (1) starting at $x_0$ when the control is $u \in \mathcal{U}$. Then for $T \geq 0$, the payoff function of finite time-horizon is

$$J_T(x_0, u) = \int_0^T e^{-\alpha t} l\big(x(t; x_0, u), u(t)\big) dt, \tag{4}$$

which is always well-defined for any $u \in \mathcal{U}$ under some weak assumptions. We define the set of the admissible control inputs to be

$$\mathcal{U}(x_0) = \{u \in \mathcal{U} : \lim_{T \to \infty} J_T(x_0, u) \text{ exists}\}. \tag{5}$$

For $u \in \mathcal{U}(x_0)$, the payoff function (2) is

$$J(x_0, u) = \lim_{T \to \infty} J_T(x_0, u). \tag{6}$$

### B. Local optimal feedback control

For simplicity, we will only consider the analytic systems, i.e., the function $f$ and $l$ are smooth, and so we only consider the smooth feedback controls. The following are the basic assumptions:

Assumption 1: The function $f$ and $l$ are smooth, which satisfy

$$f(0) = 0, \ l(0) = 0,$$
$$\frac{\partial l}{\partial x}(0, 0) = 0, \ \frac{\partial l}{\partial u}(0, 0) = 0.$$

Remark: The purpose of the above assumptions, in some sense, is to avoid the existence of a large number of controls near the origin whose cost are infinite. In this case, the overtaking optimization concepts may be a better option.

Definition 1: A smooth feedback control $u^*$ is called a local optimal feedback control of INOC (1)-(2) if for every smooth feedback control $u$ there exists a neighbourhood $N_u$ of the origin in $\mathbb{R}^n$ such that for every $x_0 \in N_u$

$$J(x_0, u^*) \leq J(x_0, u)$$

and $x(t; x_0, u) \in N_u$ and $x(t; x_0, u^*) \in N_u$ for all $t \geq 0$. If system (1) is locally exponentially stable when $u(t) = u^*(x(t))$, then we call it stable local optimal feedback control.

Remark: Just as [1], in order to assert the uniqueness of the optimal control, two controls are taken be to the same if they coincide on some neighborhood of the origin.

Remark: The infinite-horizon nonlinear optimal control which arise in many fields of economics has some specific and challenge mathematical problems. For example, the standard transversality conditions of the Pontryagin maximum principle may fail. Because the unbounded of the payoff, there may be no value function as in classic dynamic programming.

Remark: Without special specified, optimal control always means the minimization of payoff functions in this paper.

### C. Linear-quadratic approximation system

By Assumption 1, we know that the functions $f$ and $l$ can be rewritten as

$$f(x, u) = Ax + Bu + o(\|(x, u)\|),$$
$$l(x, u) = x^T Q x + 2x^T S u + u^T R u + o(\|(x, u)\|^2)$$

where

$$A = \frac{\partial f}{\partial x}(0, 0), \ B = \frac{\partial f}{\partial u}(0, 0)$$
$$Q = \frac{\partial^2 l}{\partial x^2}(0, 0), \ S = \frac{\partial^2 l}{\partial u \partial x}(0, 0)$$
$$R = \frac{\partial^2 l}{\partial u^2}(0, 0), \ M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}.$$

Hence, any analytic INOC (1)-(2) gives a linear-quadratic optimal control problem (LQCP) as follows: The system dynamic is

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ x(0) = x_0, \end{cases} \tag{7}$$

and the payoff function to be minimized is

$$\begin{aligned} J(x_0, u) &= \int_0^\infty \omega(x(t), u(t)) dt \\ &= \int_0^\infty e^{-\alpha t} [x^T(t), u^T(t)] M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt, \end{aligned} \tag{8}$$

where the matrices $A, B$ and $M$ are defined above.

In fact, the LQCP (7)-(8) is a truncated system of INOC (1)-(2) (see [1] for the case $\alpha = 0$), we call it the linear-quadratic approximation system.

Assumption 2: The LQCP (7)-(8) is regular, i.e., $R > 0$.

## III. Problem formulation

In this section, we will give a general framework of game-based control system (GBCS) and then formulate the stabilizable problem.

### A. Game-based Control Systems

Consider the following nonlinear time-invariant affine GBCS: The system dynamics are described by

$$\begin{cases} \dot{x}(t) = f(x(t), u_1(t), \cdots, u_L(t), u(t)), \\ x(0) = x_0, \end{cases} \tag{9}$$

with $f(0) = 0$, and the payoff function to be minimized by $u_i(\cdot)$ of the agent $i (i \in [L])$ is

$$J_i(x_0, u_i, u_{-i}, u) = \int_0^{\infty} e^{-\alpha t} l_i(x(t), u_1(t), \cdots, u_L(t), u(t)) dt, \tag{10}$$

where $x \in \mathbb{R}^n, u_i \in \mathbb{R}^{m_i} (i \in [L]), u \in \mathbb{R}^m$, and $\alpha \geq 0$ is the common discount rate. In GBCS (9)-(10), $u(\cdot) \in \mathcal{U}$ and $u_i(\cdot) \in \mathcal{U}_i (i \in [L])$ are the inputs or strategies of the regulator and agent $i (i \in [L])$, respectively, where $\mathcal{U}$ and $\mathcal{U}_i (i \in [L])$ are their respective admissible control sets.

Remark: The cost function (10) can be written as (see, e.g., [2, Page 101] or [3])

$$J_i = \frac{1}{L} \sum_{j=1}^{L} J_j + \frac{1}{L} \sum_{j=1}^{L} (J_i - J_j) = J_{cp} + J_{sf,i},\ i \in [L], \tag{11}$$

where $J_{cp}$ is an overall cooperative team cost and $J_{sf,i}$ a noncooperative or selfish cost for agent $i$. This becomes a zero-sum game when $J_{cp} = 0$.

It is well known that the information structure and the timing of actions play a crucial role in differential games [4], [5]. The basic information structures in GBCS are as follows:

- The data $\{f, \alpha, l_i(x),\ i \in [L]\}$ and the initial system state $x(0)$ are "common knowledge", which is a terminology used for information in game theory [6, Chapter 14].
- Agents use stationary feedback strategies, i.e., $u_i(t) = k_i(x(t))$ for some function $k_i(x)$ $(i \in [L])$. In this case, all rational agents can measure the system states.
- The regulator will first make and announce its macrodecision, and then each agent makes a decision to minimize its own payoff function. The lower level agents have access to information about the regulator's input but do not know other agents' inputs when making their own decisions.

Hence, for any given decision of the regulator, the agents in GBCS will form a noncooperative differential game and the actions space of agent $i$ $(i \in [L])$ is

$$\mathcal{U}_i = \{u_i(\cdot) = k_i(x(\cdot)) \mid k_i \text{ is smooth}\}. \tag{12}$$

Remark: Depending on the information available to the agents, there are two major kinds of strategies in noncooperative differential games: open-loop and feedback [8]. Feedback strategy is more reasonable in the infinite-planning horizon case. More details about the GBCS can be found in [7], [8].

We also limit the regulator's admissible controls to the stationary feedback strategies, i.e.,

$$\mathcal{U} = \{u(\cdot) = k(x(\cdot)) \mid k \text{ is smooth}\}. \tag{13}$$

Definition 2: For any given regulator's strategy $u \in \mathcal{U}$, we call the game formed by agents in GBCS (9)-(10) has a local Nash equilibrium $u^{F*} = (u_1^*, \cdots, u_L^*)$, if the following conditions hold:

1) For any $i \in [L]$ $u_i^* \in \mathcal{U}_i$.
2) For any $i \in [L]$, $u_i^*$ is a local optimal control of the optimal control problem for the agent $i$ when $u_{-i} = u_{-i}^*$ in GBCS (9)-(10), where $u_{-i}$ represents the strategy profile of all agents except for agent $i$.

If $u_i^*$ is a global optimal control for all $i \in [L]$ in Condition 2), then we call it global Nash equilibrium or Nash equilibrium.

The Nash equilibrium defined here is the sub-game perfect Nash equilibrium or feedback Nash equilibrium which has the time-consistence property.

From the assumption of the rational agents, the system dynamics are essentially determined by equation

$$\begin{cases} \dot{x}(t) = f(x(t), u_1^*(t), \cdots, u_L^*(t), u(t)), \\ x(0) = x_0. \end{cases} \tag{14}$$

Hence, GBCS (9)-(10) is a control system rather than a noncooperative differential game. Its control input is the regulator's strategy which regulate the system by intervening in the game formed by the agents.

### B. Stabilizability Problem

As a control system, there are many interesting problems to be investigated. The controllability problem has been investigate by [7], [8]. Here, we are interested in whether or not the system can be stabilized by the intervention of the regulator, which can be captured by the concept of stabilizability defined in this section.

We note that the system may not be stable when the Nash equilibrium exists and the agents use the Nash equilibrium control $u^{F*}$. In [9], the author restrict the admissible controls of the agents as

$$\mathcal{F} = \{F = (F_1, \cdots, F_L) :\ A + \sum_{i=1}^{L} B_i F_i \text{ is stable}\}, \tag{15}$$

which means that the strategy spaces of the agents are interdependent. Because all the lower level agents are rational and they only concern about their own payoff, just as the author of [9] says, the constraint presupposes that there are some coordination between the agents and is a bit unwieldy. More practically, the stability of the system is guaranteed by the control of a macro regulator, a third party, or a syndicate.

Definition 3. GBCS (9)-(10) is called locally stabilizable, if there exists a control $u(\cdot) = F(x(\cdot)) \in \mathcal{U}$ of the regulator such that the noncooperative differential game formed by the agents has a local Nash equilibrium $u^{F*} = (F_1^*, \cdots, F_L^*)$ and the system (9) is locally exponentially stable when

$u(\cdot) = F(x(\cdot))$ and $u_i(\cdot) = F_i^*(x(\cdot))$ ($i \in [L]$). GBCS (9)-(10) is called (globally) stabilizable if the Nash equilibrium is global and the system is global asymptotic stable.

Remark: The macrocontrol of the regulator has two goals. The first is to make the lower-level differential game has at least one (local) Nash equilibrium through some proper macrocontrol. Another goal is to make the system stable at Nash equilibrium.

Remark: The third party (macroregulator) coordination approach is very different from the way the admissible controls of the agents are restricted in (15). For the former, there is no unwieldy dependence between the strategy spaces of the agents and once all agents adopt the selected stable equilibrium strategies, no individual has the motivation to deviate form the equilibrium. But for the latter, in some cases, one agent can further minimize its payoff function unilaterally by choosing a linear feedback control for which the system will be unstable [9, Example 8.7].

## IV. Main Results

In [10], the authors have investigated the stabilizability of linear-quadratic GBCS. Just like the Lyapunov inderect method, we will study the stabilizability relationship between the linear-quadratic GBCS and the nonlinear GBCS.

### A. Nonlinear and linear-quadratic optimal control

The following theorem is the main result of this section. Using symplectic geometry method, the author of [11] has investigated the solvability relationship between the non-linear Hamilton-Jacobi equation and the algebraic Riccati equation, using which we can get the solvability relationship between the nonlinear and linear optimal control problems when $\alpha = 0$.

Through the corresponding results can not deal with the general case ($\alpha \geq 0$), the method in [11] can be adapted to solve the general case by using contact geometry which can extend the Hamilton-Jacobi theory for contact Hamilton systems.

Here we first introduce some mathematical background of the contact geometry.

Let $M$ be a $n$-dimensional smooth manifold and $T^*M$ be the cotangent vector bundle. Then $T^*M \times \mathbb{R}$ equipped with the canonical contact form $\eta = dz - \eta_M$ is a contact manifold, where $z$ is a global coordinate in $\mathbb{R}$ and $\eta_M$ the Liouville form on $T^*M$. A submanifold $L \subseteq T^*M \times \mathbb{R}$ is called a Legendre submanifold is $\mathrm{Dim}(L) = n$ and furthermore $\eta|_L = 0$, i.e., the restriction of $\eta$ on $L$ is zero (see e.g., [14] for more details). The Reeb vector field is

$$\xi = \frac{\partial}{\partial z}.$$

For any smooth function $H : T^*M \times \mathbb{R} \to \mathbb{R}$ there is a corresponding contact vector field $X_H$ on $T^*M \times \mathbb{R}$. By contact Darboux's theorem (see e.g. [12]), there exists local coordinates $(x_1, \cdots, x_n, p_1, \cdots, p_n, z)$ for $T^*M \times \mathbb{R}$ such that

$$\eta = dz - \sum_{i=1}^{n} x_i dp_i,$$

and so the contact vector field of $H$ is

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial x_i} - \sum_{i=1}^{n}\Big(\frac{\partial H}{\partial x_i} + p_i\frac{\partial H}{\partial z}\Big)\frac{\partial}{\partial p_i} + \sum_{i=1}^{n}\Big(p_i\frac{\partial H}{\partial p_i} - H\Big)\frac{\partial}{\partial z}.$$

Consider the special smooth function $\overline{H}(x,p,z) = H(x,p) - \alpha z$ with $H(0,0) = 0$. Then the flow of the contact vector field $X_{\overline{H}}$ is described by the following contact Hamilton equations:

$$\begin{cases} \dot{x}_i(t) = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i(t) = -\dfrac{\partial H}{\partial x_i} + \alpha p_i, \\ \dot{z}(t) = p_i\dfrac{\partial H}{\partial p_i} - H + \alpha z, \end{cases} \tag{16}$$

where the origin 0 is an equilibrium. We should note that the evolution of $z$ does not affect $x_i$ and $p_i$ for all $i = 1, \cdots, n$, and the differential equations of $x$ and $p$ are the same form of the necessary conditions of the maximum principle for discounted nonlinear optimal control [13, Chapter 6.2].

Consider the Jacobian matrix $DX_{\overline{H}}(0)$ at origin. In the Darboux's coordinates, it is given as

$$DX_{\overline{H}}(0) = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial p} & \frac{\partial^2 H}{\partial p^2} & 0 \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial p \partial x} + \alpha I_n & 0 \\ \frac{\partial H}{\partial x} & 0 & \alpha \end{bmatrix}.$$

Now assume that $\alpha > 0$, then 0 is a hyperbolic equilibrium if and only if the matrix

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x \partial p} & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial p \partial x} + \alpha I_n \end{bmatrix}$$

has no imaginary eigenvalues. If $\frac{\partial^2 H}{\partial p^2} > 0$, then it can imply that 0 is a hyperbolic equilibrium if and only if the matrix

$$H_\alpha = \begin{bmatrix} \frac{\partial^2 H}{\partial x \partial p} - \frac{\alpha}{2} I_n & \frac{\partial^2 H}{\partial p^2} \\ -\frac{\partial^2 H}{\partial x^2} & -\frac{\partial^2 H}{\partial p \partial x} + \frac{\alpha}{2} I_n \end{bmatrix}$$

has no imaginary eigenvalues, which is a Hamilton matrix.

Hence, if 0 is a hyperbolic equilibrium, then the matrix $DX_{\overline{H}}(0)$ has $n$ eigenvalues whose real parts are negative and $n+1$ eigenvalues whose real parts are positive. By the stable manifold theorem there exists a global invariant stable submanifold $S$ of $T^*M \times \mathbb{R}$ through 0 which is tangent at 0 to the generalized stable eigenspace of $DX_{\overline{H}}(0)$, i.e.,

$$T_0 S = X^-(DX_{\overline{H}}(0))$$

where $X^-(DX_{\overline{H}}(0))$ is the $n$-dimensional eigenspace of the matrix $DX_{\overline{H}}(0)$ corresponding to it $n$ eigenvalues with negative real part.

Lemma 1: Let $(T^*M \times \mathbb{R}, \eta)$ be the contact manifold mentioned above and $\overline{H}(x,p,z) = H(x,p) - \alpha z$ be a smooth function with $H(0,0) = 0$. If 0 is a hyperbolic equilibrium for the contact vector field $X_{\overline{H}}$, then the global stable invariant submanifold $S$ of $X_{\overline{H}}$ through 0 is a Legendre submanifold of $T^*M \times \mathbb{R}$.

Proof: By the argumentation above, we know that $\mathrm{Dim}(S) = n$.

Because $X_{\overline{H}}$ is the contact vector field corresponding to contact Hamilton $\overline{H}$, then we have

$$L_{X_{\overline{H}}}\eta = -\xi(\overline{H})\eta = \alpha\eta,$$

where $L_X$ denotes the Lie derivative with respected to vector field $X$.

Let $\phi_t$ be the flow of the vector field $X_{\overline{H}}$ on $S$. Then we have

$$\frac{d}{dt}(\phi_t^*\eta) = \phi_t^*(L_{X_{\overline{H}}}\eta) = \phi_t^*(\alpha\eta) = \alpha(\phi_t^*\eta),$$

which implies that

$$\phi_t^*\eta = e^{\alpha t}\eta.$$

Then for any $p \in S$ and $v \in T_pS$, we have

$$\eta_p(v) = e^{-\alpha t}\eta_{\phi_t(p)}(\phi_t(p)_*v).$$

Because $S$ is the stable manifold, then

$$\phi_t(p)_*v \to 0, \ t \to \infty,$$

and so $\eta_p(v) = 0$.

Lemma 2. Let $(T^*M \times \mathbb{R}, \eta)$ be the contact manifold mentioned above and $\overline{H}(x,p,z) = H(x,p) - \alpha z$ be a smooth function with $H(0,0) = 0$. If 0 is a hyperbolic equilibrium for the contact vector field $X_{\overline{H}}$ and there exists canonical coordinates $(x_1, \cdots, x_n, p_1, \cdots, p_n, z)$ around 0 such that $S$ is locally given as

$$\{(x_1, \cdots, x_n, p_1 = \frac{\partial V}{\partial x_1}(x), \cdots, p_n = \frac{\partial V}{\partial x_n}(x), V(x)) : x \text{ around } 0\}.$$

Then $V$ is a local solution of the Hamilton-Jacobi equation

$$\alpha V(x) - H(x, \frac{\partial V}{\partial x}(x)) = 0, \ x \text{ around } 0.$$

Proof: Using Proposition 6.7.16 in [12] and the fact that $\overline{H}$ is preserved by the flow of $X_{\overline{H}}$

$$\overline{H}(x,p,z) = \overline{H}(\phi_t(x,p,z)) \to \overline{H}(0,0,0) = 0,$$

we can prove the result.

Theorem 1: Suppose that Assumptions 1 and 2 hold. If LQCP (7)-(8) has a stable optimal control which is linear feedback form, then INOC (1)-(2) has a stable local optimal feedback control.

Proof: The results in [11] imply the theorem when $\alpha = 0$. Now we assume that $\alpha > 0$.

Consider the INOC (1)-(2), the pseudo-Hamilton is

$$H(x,p,u) = p^T f(x,u) + l(x,u)$$

If the LQCP (7)-(8) is regular, i.e., $R > 0$, then there is a unique smooth function $u = u^*(x,p)$ such that in a neighbourhood of the origin we have

$$H(x,p) \triangleq H(x,p,u^*(x,p)) \geq H(x,p,u).$$

The space $(\mathbb{R}^{2n+1}, \eta)$ with the global coordinate $(x_1, \cdots, x_n, p_1, \cdots, p_n, z)$ and

$$\eta = dz - \sum_{i=1}^{n} x_i dp_i$$

is a standard contact manifold. Consider the Hamilton $\overline{H}(x,p,z) = H(x,p) - \alpha z$.

For notations simplicity and without loss generality, we assume that $S = 0$. The case $S \neq 0$ can be transformed to the special case. Because LQCP (7)-(8) has a stable optimal control which is linear feedback form, we have that there exists a solution $K$ to the following algebraic Riccati equaiton

$$(A - \frac{\alpha}{2}I_n)^T K + K(A - \frac{\alpha}{2}I_n) + Q - KBR^{-1}B^TK = 0$$

satisfying

$$\sigma(A - \frac{\alpha}{2}I_n - BR^{-1}B^TK) \subseteq \mathbb{C}_<.$$

Equivalently,

$$\mathrm{Span}\begin{bmatrix} I_n \\ K \end{bmatrix}$$

is the generalized stable eigenspace of the Hamiltonian matrix

$$\mathrm{Ham} = \begin{bmatrix} A - \frac{\alpha}{2}I_n & -BR^{-1}B^T \\ -Q & -(A - \frac{\alpha}{2}I_n)^T \end{bmatrix}.$$

Hence, Ham has no imaginary eigenvalues, which is precisely the matrix $H_\alpha$ corresponding to the Hamiltonian $\overline{H}$. Denote the global stable manifold of $\overline{H}$ as $S$, by Lemma 1, we know that $S$ is a Legendre submanifold of $(\mathbb{R}^{2n+1}, \eta)$. Furthermore,

$$T_0S = \mathrm{Span}\begin{bmatrix} I_n \\ K \end{bmatrix}.$$

By Lemma 2, we know that there exists a local smooth solution of the Hamilton-Jacobi equation

$$\alpha V(x) - H(x, \frac{\partial V}{\partial x}(x)) = 0.$$

Using the solution one can construct an local optimal feedback control which make the system asymptotic stable.

### *B. Linear-quadratic approximation GBCS*

For the symbolic simplicity, we will only consider the following special situation

$$f(x, u_1, \cdots, u_L, u) = h(x) + \sum_{i=1}^{n} g_i(x)u_i + g(x)u,$$

$$l_i(x, u_1, \cdots, u_L, u) = x^T Q_i(x) x + \sum_{i=1}^{L} u_i^T R_{ij}(x) u_j,$$

where $R_i(0) > 0$ and $h(0) = 0$. More general nonlinear case can be treated using the similar method.

Define the following matrices

$$\begin{aligned} &A=\frac{\partial h}{\partial x}\Big|_{x=0},\ B=g(0),\ B_i=g_i(0),\\ &Q_i=\frac{\partial^2 Q_i(x)}{\partial x^2}\Big|_{x=0},\ R_{ij}=R_{ij}(0),\ i\in[L]\\ &Q_i=Q_i(0),\ R_i=R_i(0). \end{aligned} \tag{17}$$

Using these matrices, we can construct a linear quadratic GBCS as follows.

$$\begin{cases} \dot{x}(t)=Ax(t)+\sum_{i=1}^{L}B_iu_i(t)+Bu(t),\\ x(0)=x_0, \end{cases} \tag{18}$$

and the payoff function to be minimized by $u_i(\cdot)$ of the agent $i(i\in[L])$ is

$$\begin{aligned} &J_i(x_0,u_i,u_{-i},u)\\ =&\int_0^\infty e^{-\alpha t}\Big[x^T(t)Q_ix(t)+\sum_{i=1}^{L}u_i^T(t)R_{ij}u_j(t)\Big]dt, \end{aligned} \tag{19}$$

We call the GBCS above the linear-quadratic approximation of the GBCS (9)-(10) at 0.

The stabilizability of the linear-quadratic GBCS has been investigated in [10]. In the linear case, we mainly concern the linear-feedback control and linear-feedback Nash equilibrium. The reason for this can be found in Reference [10]. The following is the definition of the stabilizability in the reference which will be used in the next subsection.

Definition 4: The linear-quadratic GBCS (18)-(19) is stabilizable, if there exits a control $u=Fx$ of the regulator such that the noncooperative differential game formed by the agents has a linear feedback Nash equilibrium $(u_1=F_1x,\cdots,u_L=F_Lx)$ and the system is asymptotically stable (or equivalent to exponentially stable) when $u=Fx$ and $(u_1=F_1x,\cdots,u_L=F_Lx)$ for any initial state $x_0\in\mathbb{R}^n$.

### C. Stabilizability of GBCS

Before give the main result, we give some assumptions which are weak conditions for making the optimization problem well-posed.

Assumption 3: All the functions in GBCS (9)-(10) are smooth, $h(0)=0$ and $R_{ii}>0$.

Theorem 2: Suppose Assumption 3 holds and $L=1$. If the linear-quadratic GBCS (18)-(19) is stabilizable, then the nonlinear GBCS (9)-(10) is locally stabilizable.

Proof: By the condition of the theorem, there exists $u=Fx$ such that the linear-quadratic optimal control of the agent has a stable linear-feedback control $u_1=Kx$.

Now let the regulator's control of the nonlinear GBCS (9)-(10) be $u=Fx$. Then the optimal control (18)-(19) is just a linear-quadratic approximation of the nonlinear optimal control in (9)-(10). By Theorem 1, we get that the nonlinear optimal control problem admits a stable local optimal feedback control. By definition, the GBCS (9)-(10) is stabilizable.

Theorem 3: Suppose Assumption 3 holds, $L=2$ and $l_1+l_2=0$. If the linear-quadratic GBCS (18)-(19) is stabilizable, then the nonlinear GBCS (9)-(10) is locally stabilizable.

Proof: Under the assumptions of the theorem, the game formed by the agents is zero-sum and we can use a non-coupled nonlinear Hamilton-Jacobi equation to characterize the Nash equilibrium. In linear-quadratic GBCS case, the equation degenerates to a algebraic Riccati equation. So we can prove this result using a method similar to that of Theorem 2.

## V. CONCLUSIONS

In this paper, we have investigated the stabilizability of Nash equilibrium of the nonlinear game-based control systems (GBCSs). The motivation for studying it comes from the Lyaupunov indirect method, which uses the linearization of a system to determine the local stability of the original system. In this paper, we first study the relationship between the nonlinear and linear-quadratic optimal control problems using the method of contact geometry, then using the corresponding results gives some conditions for the stabilizability of nonlinear GBCS. It is obvious that this is just a few steps towards the stabilizability problem of general GBCS and there are many problems to be further studied.

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