2023 62nd IEEE Conference on Decision and Control (CDC)
December 13-15, 2023. Marina Bay Sands, Singapore

Prescribed-Time Nonlinear Control with Multiplicative Noise*

Wuquan Li and Miroslav Krstic

Abstract— **We study the prescribed-time design problem for strict-feedback nonlinear systems with multiplicative measurement noise. With the assumption that the noise is small and linearly vanishing, we propose a new postulated feedback to solve the prescribed-time mean-square stabilization problem. In contrast to the existing stochastic prescribed-time designs, the merit of our design is that it can effectively deal with multiplicative measurement noise. The existence of measurement noise makes the design rather challenging since the resulting process noise intensity, in closed loop, depends on the feedback gains and even goes to infinity. Finally, a simulation example is given to illustrate the design.**

I. INTRODUCTION

The control design with sensor uncertainties has attracted much attention in the past two decades due to their wide engineering applications [1]-[2]. [3]-[4] focus on the output-feedback stabilization design for nonlinear systems with unknown measurement sensitivity, which are deterministic constants or bounded time-varying functions. As shown in [5], it is more reasonable to study the eye and arm movements based on the assumption that the neural control signals are corrupted by noise, which motivates the study of the stochastic sensor sensitivity. For linear systems whose white noise sources have intensities affinely related to the variance of the signal they corrupt, [6] provides necessary and sufficient conditions to guarantee the mean-square state-feedback stabilization; Recently, [7] proposes two designs to solve the mean-square stabilization problems for lower-triangular/upper-triangular nonlinear systems with multiplicative stochastic sensor uncertainty. It should be emphasized that, [6]-[7] only achieve mean-square stabilization in asymptotic sense. However, many real applications require that the mean-square stabilization be achieved in prescribed-time, rather than as time goes to infinity.

Prescribed-time control has been receiving increasing attention due to its wide applications in tactical missile guidance [8] and other applications in which there exists a short, finite amount of time remaining to achieve control objectives. The advantage of such control is that it allows the user to prescribe the convergence times a priori and irrespective of initial conditions. There are fruitful results for the prescribed-time control of deterministic systems [9]-[17]. When it turns to the stochastic prescribed-time control, [18] proposes a new nonscaling backstepping state-feedback design, which is the first result on the prescribed-time mean-square stabilization and inverse optimality control for stochastic strict-feedback nonlinear systems; [19] adopts scaled quartic Lyapunov functions to reduce the control effort in [18]; [20] solves the prescribed-time output-feedback control problems for stochastic nonlinear systems without/with sensor uncertainty; [21] proposes a prescribed-time mean-nonovershooting stabilizing feedback law for stochastic nonlinear systems with noise that vanishes in finite time. Although [18]-[21] concentrate on the prescribed-time control of stochastic nonlinear systems, they don't consider systems with multiplicative measurement noise. Noting that stochastic sensor uncertainty is ubiquitous in engineering, it is imperative to study the prescribed-time control for nonlinear systems with multiplicative measurement noise.

Motivated by the above observations, we study the prescribed-time stabilization problem for strict-feedback nonlinear systems with multiplicative measurement noise. The contributions of this paper are two-fold:

(1) We present a new design framework for nonlinear systems with multiplicative measurement noise. Unlike the design for linear systems in [6] where the control gain and the noise intensity are coupled in a linear matrix inequality, we develop a step by step gain design for nonlinear systems, which clearly shows what the control gains are. Different from the design for nonlinear systems in [7] where time-invariant controllers are designed to achieve asymptotic mean-square stability, our design can drive the system to be prescribed-time mean-square stable.

(2) The existence of multiplicative noise makes stochastic prescribed-time designs in [18]-[21] inapplicable. In order to handle the multiplicative noise, we propose a new postulated controller whose gains are designed step by step. Different from the designs in [18]-[21] where the controller is designed recursively, in our design, the feedback is inserted into the system, which leads to the process noise intensity actually being nonzero, depending on the feedback gains, and even going to infinity at the terminal time. How to select the control gains to prescribed-time stabilize the system in the presence of the nonlinearities is a hard problem.

The remainder of this paper is organized as follows. Section II is on problem formulation. Section III is focused on the prescribed-time design and stability analysis. Section Section IV gives a example to illustrate the theoretical results. Section V includes some concluding remarks.

*The work of the first author is supported by National Natural Science Foundation of China under Grant (No. 62373179), the Taishan Scholars Program of Shandong Province of China under Grant (No. tstp20221133), and Fundamental Research Projects of Science & Technology Innovation and Development Plan in Yantai City (No. 2023JCYJ051). The work of the second author was supported by US Air Force Office of Scientific Research under grant FA9550-22-1-0265. (Corresponding author: Wuquan Li.)

W.Q. Li is with the School of Mathematics and Statistics Science, Ludong University, Yantai 264025, China sea81@126.com

Miroslav Krstic is with the Department of Mechanical and Aerospace Engineering, University of California, San Diego, La Jolla, CA 92093 USA krstic@ucsd.edu

II. PROBLEM FORMULATION

Consider a class of nonlinear systems described by

$$\dot{x}_i = x_{i+1} + f_i(t,x),\ i = 1,\dots,n-1, \tag{1}$$
$$\dot{x}_n = u + f_n(t,x), \tag{2}$$

where $x = (x_1,\dots,x_n)^T \in R^n$ and $u \in R$ are the system state and control input. The function $f_i : R^+ \times R^n \to R$ is piecewise continuous in t, locally bounded and locally Lipschitz continuous in x uniformly in $t \in R^+$, $f_i(t,0) = 0$, $i = 1,\dots,n$.

We observe the state x_i as y_i, which is described by

$$y_i = x_i(1 + g_i(t)\dot{\omega}_i),\ i = 1,\dots,n, \tag{3}$$

or

$$y_i dt = x_i dt + g_i(t)x_i d\omega_i,\ i = 1,\dots,n, \tag{4}$$

where $g_1(t),\dots,g_n(t)$ are continuous functions and $\omega_1,\dots,\omega_n$ are scalar independent standard Wiener processes defined on the complete filtered probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$ with a filtration $\mathcal{F}_t$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all P-null sets).

We introduce the following scaling function:

$$\mu(t) = \left(\frac{T}{t_0 + T - t}\right)^2, \forall\ t \in [t_0, t_0+T), \tag{5}$$

where $T > 0$ is the freely prescribed time.

Obviously, $\mu(t)$ is a monotonically increasing function on $[t_0, t_0+T)$ with $\mu(t_0) = 1$ and $\lim_{t\to t_0+T}\mu(t) = +\infty$ (In this paper, $\lim_{t\to t_0+T}$ means t approaches t_0+T "from the left" or "from below").

We design a new controller as

$$u = -k_1\mu^n y_1 - k_2\mu^{n-1}y_2 - \dots - k_n\mu y_n, \tag{6}$$

where $k_1,\dots,k_n$ are positive control gains to be designed later.

By (3)-(4) and (6), system (1)-(2) can be written as

$$dx_i = (x_{i+1} + f_i(t,x))dt,\ i = 1,\dots,n-1, \tag{7}$$
$$dx_n = \Big(-\sum_{i=1}^{n} k_i\mu^{n+1-i}x_i + f_n(t,x)\Big)dt + G(x)d\omega, \tag{8}$$

where $\omega = (\omega_1,\dots,\omega_n)^T$ and

$$G(x) = (-k_1 g_1\mu^n x_1,\dots,-k_n g_n \mu x_n). \tag{9}$$

For system (1)-(2), we need the following assumption.

Assumption 1. For $i = 1,\dots,n$, there exists a nonnegative constant c_i such that

$$|f_i(t,x)| \le c_i(|x_1| + \dots + |x_i|). \tag{10}$$

The noise intensity $g_i(t)$ satisfies the following linearly vanishing condition.

Assumption 2. For $i = 1,\dots,n$, there exists a positive constant δ_i such that

$$|g_i(t)| \le \delta_i\left(1 - \frac{t-t_0}{T}\right), \quad \forall t \in [t_0, t_0+T). \tag{11}$$

In this paper, with Assumptions 1-2, when the noise power δ_i is sufficient small, we aim to design the control gains $k_1,\dots,k_n$ to make system (7)-(9) achieve prescribed-time mean-square stable with $\lim_{t\to t_0+T} E|x|^2 = 0$.

III. PRESCRIBED-TIME DESIGN AND STABILITY ANALYSIS

In this section, we first develop a new design scheme for the control gains $k_1,\dots,k_n$, then analyze the stability of the closed-loop system.

Step 1. Define $V_1 = \frac{1}{2}\xi_1^2$, $\xi_1 = x_1$, then from (7) we have

$$\mathcal{L}V_1 = \xi_1(x_2 - x_2^*) + \xi_1 x_2^* + \xi_1 f_1. \tag{12}$$

From (10) we get

$$\xi_1 f_1 \le c_1\xi_1^2. \tag{13}$$

We choose

$$\alpha_1 = n + c_1, \tag{14}$$
$$x_2^* = -\alpha_1\mu\xi_1. \tag{15}$$

Substituting (13)-(15) into (12) yields

$$\mathcal{L}V_1 \le -n\mu\xi_1^2 + \xi_1(x_2 - x_2^*). \tag{16}$$

Step 2. Define $\xi_2 = x_2 - x_2^*$, from (15) we get

$$\xi_2 = x_2 + \alpha_1\mu\xi_1. \tag{17}$$

It follows from (7) and (17) that

$$d\xi_2 = \Big(x_3 + f_2 + \frac{2}{T}\mu^{3/2}\alpha_1\xi_1 + \mu\alpha_1(x_2 + f_1)\Big)dt. \tag{18}$$

Choose the new scaled Lyapunov function

$$V_2 = V_1 + \frac{1}{2\mu^2}\xi_2^2. \tag{19}$$

By (16), (18) and (19) we get

$$\begin{aligned}\mathcal{L}V_2 \le{}& -n\mu\xi_1^2 + \xi_1\xi_2 + \frac{1}{\mu^2}\xi_2 x_3 - \frac{2}{T}\mu^{-3/2}\xi_2^2\\ &+ \frac{1}{\mu^2}\xi_2\Big(f_2 + \frac{2}{T}\mu^{3/2}\alpha_1\xi_1 + \mu\alpha_1(x_2 + f_1)\Big).\end{aligned} \tag{20}$$

By Young's inequality in [22] we obtain

$$\xi_1\xi_2 \le \frac{1}{2}\mu\xi_1^2 + \frac{1}{2\mu}\xi_2^2. \tag{21}$$

From (10) and (17) we have

$$\begin{aligned}&\left|f_2 + \frac{2}{T}\mu^{3/2}\alpha_1\xi_1 + \mu\alpha_1(x_2 + f_1)\right|\\ &\le c_2(|x_1| + |x_2|) + \frac{2}{T}\alpha_1\mu^{3/2}|\xi_1| + \alpha_1\mu(|x_2| + c_1|x_1|)\\ &\le \Big(c_1\alpha_1\mu + c_2 + \frac{2}{T}\alpha_1\mu^{3/2}\Big)|\xi_1| + (c_2 + \alpha_1\mu)|\xi_2\\ &\quad - \alpha_1\mu\xi_1|\\ &= \Big(c_1\alpha_1\mu + c_2 + \frac{2}{T}\alpha_1\mu^{3/2} + c_2\alpha_1\mu + \alpha_1^2\mu^2\Big)|\xi_1|\\ &\quad + (c_2 + \alpha_1\mu)|\xi_2|\\ &\le \Big(c_1\alpha_1 + c_2 + \frac{2}{T}\alpha_1 + c_2\alpha_1 + \alpha_1^2\Big)\mu^2|\xi_1|\\ &\quad + (c_2 + \alpha_1)\mu|\xi_2|.\end{aligned} \tag{22}$$

By (22) and Young's inequality we get

$$\begin{aligned}
&\frac{1}{\mu^2}\xi_2\Big(f_2+\frac{2}{T}\mu^{3/2}\alpha_1\xi_1+\mu\alpha_1(x_2+f_1)\Big)\\
&\le \frac{1}{\mu}(c_2+\alpha_1)\xi_2^2+\Big(c_1\alpha_1+c_2+\frac{2}{T}\alpha_1+c_2\alpha_1\\
&\quad +\alpha_1^2\Big)|\xi_1||\xi_2|\\
&\le \frac{1}{2}\mu\xi_1^2+\frac{1}{2\mu}\Big(2c_2+2\alpha_1+\Big(c_1\alpha_1+c_2+\frac{2}{T}\alpha_1\\
&\quad +c_2\alpha_1+\alpha_1^2\Big)^2\Big)\xi_2^2. \qquad (23)
\end{aligned}$$

Substituting (21) and (23) into (20) yields

$$\begin{aligned}
\mathcal{L}V_2 \;\le\; & -(n-1)\mu\xi_1^2+\frac{1}{\mu^2}\xi_2(x_3-x_3^*)+\frac{1}{\mu^2}\xi_2x_3^*\\
&+\frac{1}{2\mu}\Big(1+2c_2+2\alpha_1+\Big(c_1\alpha_1+c_2+\frac{2}{T}\alpha_1\\
&+c_2\alpha_1+\alpha_1^2\Big)^2\Big)\xi_2^2. \qquad (24)
\end{aligned}$$

If we choose

$$\begin{aligned}
\alpha_2 \;=\;& n-1+\frac{1}{2}\Big(1+2c_2+2\alpha_1+\Big(c_1\alpha_1+c_2\\
&+\frac{2}{T}\alpha_1+c_2\alpha_1+\alpha_1^2\Big)^2\Big), \qquad (25)\\
x_3^* \;=\;& -\alpha_2\mu\xi_2, \qquad (26)
\end{aligned}$$

then we have

$$\mathcal{L}V_2 \le -(n-1)\mu\xi_1^2-(n-1)\frac{1}{\mu}\xi_2^2+\frac{1}{\mu^2}\xi_2(x_3-x_3^*). \qquad (27)$$

Deductive Step. Assume that at step k, there are a set of virtual controllers $x_2^*,\dots,x_{k+1}^*$ defined by

$$\begin{aligned}
x_2^* &= -\alpha_1\mu\xi_1, & \xi_1 &= x_1, & (28)\\
x_3^* &= -\alpha_2\mu\xi_2, & \xi_2 &= x_2-x_2^*, & (29)\\
&\vdots & &\vdots &\\
x_{k+1}^* &= -\alpha_k\mu\xi_k, & \xi_k &= x_k-x_k^*, & (30)
\end{aligned}$$

such that

$$\begin{aligned}
\mathcal{L}V_k \;\le\;& -\sum_{i=1}^{k}(n-k+1)\frac{1}{\mu^{2i-3}}\xi_i^2\\
&+\frac{1}{\mu^{2k-2}}\xi_k(x_{k+1}-x_{k+1}^*), \qquad (31)
\end{aligned}$$

where $\alpha_1,\dots,\alpha_k$ are positive constants and

$$V_k=\sum_{i=1}^{k}\frac{1}{2\mu^{2i-2}}\xi_i^2. \qquad (32)$$

To complete the induction, at the $k+1$th step, we consider the ξ_{k+1}-system.

Let $\xi_{k+1}=x_{k+1}-x_{k+1}^*$, from (28)-(30) we have

$$\xi_{k+1}=x_{k+1}+\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}x_s. \qquad (33)$$

By (7) and (33) we get

$$\begin{aligned}
d\xi_{k+1} \;=\;& \Big(x_{k+2}+f_{k+1}+\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}(x_{s+1}+f_s)\\
&+\frac{2}{T}\sum_{s=1}^{k}(k+1-s)\alpha_s\dots\alpha_k\mu^{k-s+3/2}x_s\Big)dt. \qquad (34)
\end{aligned}$$

We choose the new Lyapunov function

$$V_{k+1} \;=\; V_k+\frac{1}{2\mu^{2k}}\xi_{k+1}^2. \qquad (35)$$

It follows from (31), (34)-(35) and Itô's formula that

$$\begin{aligned}
\mathcal{L}V_{k+1}\le\;& -\sum_{i=1}^{k}(n-k+1)\frac{1}{\mu^{2i-3}}\xi_i^2+\frac{1}{\mu^{2k-2}}\xi_k\xi_{k+1}\\
&-\frac{2}{T}k\mu^{-2k+1/2}\xi_{k+1}^2+\frac{1}{\mu^{2k}}\xi_{k+1}x_{k+2}+\frac{1}{\mu^{2k}}\xi_{k+1}\\
&\cdot\Big(f_{k+1}+\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}(x_{s+1}+f_s)\\
&+\frac{2}{T}\sum_{s=1}^{k}(k+1-s)\alpha_s\dots\alpha_k\mu^{k-s+3/2}x_s\Big). \qquad (36)
\end{aligned}$$

From Young's inequality we get

$$\frac{1}{\mu^{2k-2}}\xi_k\xi_{k+1} \;\le\; \frac{1}{2\mu^{2k-3}}\xi_k^2+\frac{1}{2\mu^{2k-1}}\xi_{k+1}^2. \qquad (37)$$

By (10) we obtain

$$\begin{aligned}
&\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}(x_{s+1}+f_s)\\
&\le\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}(c_s(|x_1|+\dots+|x_s|)+|x_{s+1}|)\\
&=\sum_{s=1}^{k}\sum_{j=1}^{s}c_s\alpha_s\dots\alpha_k\mu^{k+1-s}|x_j|\\
&\quad+\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}|x_{s+1}|\\
&=\Big(\sum_{s=1}^{k}c_s\alpha_s\dots\alpha_k\mu^{k+1-s}\Big)|x_1|+\alpha_k\mu|x_{k+1}|\\
&\quad+\sum_{j=2}^{k}\Big(\alpha_{j-1}\dots\alpha_k\mu^{k+2-j}+\sum_{s=j}^{k}c_s\alpha_s\dots\alpha_k\\
&\quad\cdot\mu^{k+1-s}\Big)|x_j|. \qquad (38)
\end{aligned}$$

Denoting

$$\triangle_{k+1,1}=c_{k+1}+\sum_{s=1}^{k}c_s\alpha_s\dots\alpha_k+\frac{2}{T}k\alpha_1\dots\alpha_k, \qquad (39)$$

$$\begin{aligned}
\triangle_{k+1,j}=\;& c_{k+1}+\alpha_{j-1}\dots\alpha_k+\sum_{s=j}^{k}c_s\alpha_s\dots\alpha_k\\
&+\frac{2}{T}(k+1-j)\alpha_j\dots\alpha_k,\;\; 2\le j\le k, \qquad (40)\\
\triangle_{k+1,k+1}=\;& c_{k+1}+\alpha_k, \qquad (41)
\end{aligned}$$

we have

$$\begin{aligned}
& f_{k+1}+\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}(x_{s+1}+f_s) \\
& \quad +\frac{2}{T}\sum_{s=1}^{k}(k+1-s)\alpha_s\dots\alpha_k\mu^{k-s+3/2}x_s \\
& \le \triangle_{k+1,1}\mu^{k+1}|x_1|+\sum_{j=2}^{k}\triangle_{k+1,j}\mu^{k+2-j}|x_j| \\
& \quad +\triangle_{k+1,k+1}\mu|x_{k+1}| \\
& \le \triangle_{k+1,1}\mu^{k+1}|\xi_1|+\sum_{j=2}^{k}\triangle_{k+1,j}\mu^{k+2-j}|\xi_j| \\
& \quad +\sum_{j=2}^{k}\triangle_{k+1,j}\mu^{k+3-j}\alpha_{j-1}|\xi_{j-1}| \\
& \quad +\triangle_{k+1,k+1}\mu|\xi_{k+1}|+\triangle_{k+1,k+1}\mu^2\alpha_k|\xi_k| \\
& =\sum_{j=1}^{k}(\triangle_{k+1,j}+\triangle_{k+1,j+1}\alpha_j)\mu^{k+2-j}|\xi_j| \\
& \quad +\triangle_{k+1,k+1}\mu|\xi_{k+1}|. \qquad (42)
\end{aligned}$$

By Young's inequality we get

$$\begin{aligned}
& \frac{1}{\mu^{2k}}|\xi_{k+1}|\sum_{j=1}^{k-1}(\triangle_{k+1,j}+\triangle_{k+1,j+1}\alpha_j)\mu^{k+2-j}|\xi_j| \\
& =\sum_{j=1}^{k-1}(\triangle_{k+1,j}+\triangle_{k+1,j+1}\alpha_j)\mu^{2-j-k}|\xi_j||\xi_{k+1}| \\
& \le \frac{1}{4\mu^{2k-1}}\sum_{j=1}^{k-1}(\triangle_{k+1,j}+\triangle_{k+1,j+1}\alpha_j)^2\xi_{k+1}^2 \\
& \quad +\sum_{j=1}^{k-1}\frac{1}{\mu^{2j-3}}\xi_j^2 \qquad (43)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{1}{\mu^{2k}}|\xi_{k+1}|(\triangle_{k+1,k}+\triangle_{k+1,k+1}\alpha_k)\mu^2|\xi_k| \\
& =\frac{1}{\mu^{2k-2}}(\triangle_{k+1,k}+\triangle_{k+1,k+1}\alpha_k)|\xi_k||\xi_{k+1}| \\
& \le \frac{1}{2\mu^{2k-1}}(\triangle_{k+1,k}+\triangle_{k+1,k+1}\alpha_k)^2\xi_{k+1}^2 \\
& \quad +\frac{1}{2\mu^{2k-3}}\xi_k^2. \qquad (44)
\end{aligned}$$

It can be inferred from (42)-(44) that

$$\begin{aligned}
& \frac{1}{\mu^{2k}}|\xi_{k+1}|\Big|f_{k+1}+\sum_{s=1}^{k}\alpha_s\dots\alpha_k\mu^{k+1-s}(x_{s+1}+f_s)| \\
& \quad +\frac{2}{T}\sum_{s=1}^{k}(k+1-s)\alpha_s\dots\alpha_k\mu^{k-s+3/2}x_s\Big| \\
& \le \sum_{j=1}^{k-1}\frac{1}{\mu^{2j-3}}\xi_j^2+\frac{1}{2\mu^{2k-3}}\xi_k^2+\frac{1}{4\mu^{2k-1}}\Big(4\triangle_{k+1,k+1} \\
& \quad +2(\triangle_{k+1,k}+\triangle_{k+1,k+1}\alpha_k)^2 \\
& \quad +\sum_{j=1}^{k-1}(\triangle_{k+1,j}+\triangle_{k+1,j+1}\alpha_j)^2\Big)\xi_{k+1}^2. \qquad (45)
\end{aligned}$$

Substituting (37) and (45) into (36) yields

$$\begin{aligned}
\mathcal{L}V_{k+1} \le & -\sum_{i=1}^{k}(n-k)\frac{1}{\mu^{2i-3}}\xi_i^2+\frac{1}{\mu^{2k}}\xi_{k+1}(x_{k+2}-x_{k+2}^*) \\
& +\frac{1}{4\mu^{2k-1}}\Big(2+2(\triangle_{k+1,k}+\triangle_{k+1,k+1}\alpha_k)^2 \\
& +\sum_{j=1}^{k-1}(\triangle_{k+1,j}+\triangle_{k+1,j+1}\alpha_j)^2 \\
& +4\triangle_{k+1,k+1}\Big)\xi_{k+1}^2+\frac{1}{\mu^{2k}}\xi_{k+1}x_{k+2}^*. \qquad (46)
\end{aligned}$$

Choosing the virtual controller

$$\begin{aligned}
\alpha_{k+1}= & \, n-k+\frac{1}{4}\Big(2+2(\triangle_{k+1,k}+\triangle_{k+1,k+1}\alpha_k)^2 \\
& +\sum_{j=1}^{k-1}(\triangle_{k+1,j}+\triangle_{k+1,j+1}\alpha_j)^2+4\triangle_{k+1,k+1}\Big), \qquad (47) \\
x_{k+2}^*= & -\alpha_{k+1}\mu\xi_{k+1}, \qquad (48)
\end{aligned}$$

then we have

$$\begin{aligned}
\mathcal{L}V_{k+1} \quad \le \quad & -\sum_{i=1}^{k+1}(n-k)\frac{1}{\mu^{2i-3}}\xi_i^2 \\
& +\frac{1}{\mu^{2k}}\xi_{k+1}(x_{k+2}-x_{k+2}^*). \qquad (49)
\end{aligned}$$

Step n. Defining

$$\xi_n \quad = \quad x_n+\alpha_{n-1}\mu\xi_{n-1}, \qquad (50)$$

by (7), (8), (33) and (50) we have

$$\begin{aligned}
d\xi_n \quad = \quad & \Big(-\sum_{i=1}^{n}k_i\mu^{n+1-i}x_i+f_n(t,x) \\
& +\sum_{s=1}^{n-1}\alpha_s\dots\alpha_{n-1}\mu^{n-s}(x_{s+1}+f_s) \\
& +\frac{2}{T}\sum_{s=1}^{n-1}(n-s)\alpha_s\dots\alpha_{n-1}\mu^{n-s+1/2}x_s\Big)dt \\
& +G(x)d\omega. \qquad (51)
\end{aligned}$$

Choosing

$$V_n=\sum_{i=1}^{n}\frac{1}{2\mu^{2i-2}}\xi_i^2, \qquad (52)$$

by (9), (51) and (52), similar to (46) we have

$$\begin{aligned}
\mathcal{L}V_n \quad \le \quad & -\sum_{i=1}^{n-1}\frac{1}{\mu^{2i-3}}\xi_i^2+\frac{1}{\mu^{2(n-1)}}\xi_n x_{n+1}^* \\
& +\frac{1}{\mu^{2(n-1)}}\xi_n\Big(-\sum_{i=1}^{n}k_i\mu^{n+1-i}x_i-x_{n+1}^*\Big) \\
& +\frac{1}{2\mu^{2n-2}}\sum_{i=1}^{n}k_i^2g_i^2\mu^{2n+2-2i}x_i^2 \\
& +\frac{1}{4\mu^{2n-3}}\triangle_n\xi_n^2, \qquad (53)
\end{aligned}$$

where $\triangle_n > 0$ is a constant.

Choosing

$$\alpha_n = -1 + \frac{1}{4}\triangle_n, \quad (54)$$
$$x_{n+1}^* = -\alpha_n \mu \xi_n, \quad (55)$$

then we get

$$\mathcal{L}V_n \leq -\sum_{i=1}^{n} \frac{1}{\mu^{2i-3}}\xi_i^2 + \frac{1}{2}\sum_{i=1}^{n} k_i^2 g_i^2 \mu^{4-2i} x_i^2 + \frac{1}{\mu^{2(n-1)}}\xi_n\Big(-\sum_{i=1}^{n} k_i \mu^{n+1-i} x_i - x_{n+1}^*\Big). \quad (56)$$

It follows from (28)-(30), (48) (50) and (55) that

$$x_{n+1}^* = -\sum_{i=1}^{n} \alpha_i \dots \alpha_n \mu^{n+1-i} x_i. \quad (57)$$

If we choose

$$k_i = \prod_{s=i}^{n} \alpha_s, \quad (58)$$

from (56)-(58) we get

$$\mathcal{L}V_n \leq -\sum_{i=1}^{n} \frac{1}{\mu^{2i-3}}\xi_i^2 + \frac{1}{2}\sum_{i=1}^{n} k_i^2 g_i^2 \mu^{4-2i} x_i^2. \quad (59)$$

We are now ready to state the main stability results on system (1)-(2).

Theorem 1. Consider the plant consisting of (1)-(2), (3) and (6). If Assumptions 1-2 hold and the noise power δ_i satisfies

$$0 < \delta_n < \frac{1}{\alpha_{n-1}\alpha_n}, \quad (60)$$
$$0 < \delta_i < \min\left\{ \frac{\sqrt{1 - (\prod_{s=i+1}^{n} \alpha_s)^2 \delta_{i+1}^2 \alpha_i^2}}{\prod_{s=i}^{n} \alpha_s}, \frac{1}{\prod_{s=i-1}^{n} \alpha_s} \right\}, 1 \leq i \leq n-1, \quad (61)$$

where $\alpha_0 = 1$, then the following conclusions hold:

1) The plant has an almost surely unique solution on $[t_0, t_0 + T)$;

2) The plant is prescribed-time mean-square stabilized with $\lim\limits_{t \to t_0+T} E|x|^2 = 0$. Specifically, $\forall\ t \in [t_0, t_0 + T)$, we have

$$E|x|^2 \leq 2\mu^{2n}(1+\alpha)e^{-c_0 T^2(\frac{1}{t_0+T-t} - \frac{1}{T})} \cdot\Big(x_1^2(t_0) + \sum_{k=2}^{n}\Big(x_k(t_0) + \sum_{s=1}^{k-1}\prod_{j=s}^{k-1} \alpha_j x_s(t_0)\Big)^2\Big). \quad (62)$$

Proof. Due to the page limit, the proof of this theorem is omitted here.

Remark 1. From (60) and (61), the noise powers are determined in the following sequence: $\delta_n, \delta_{n-1}, \dots, \delta_1$. These powers not only ensures the stability, but also guarantees that the inequality (61) is well defined. Specifically, they make the radicand in the square root of (61) nonnegative.

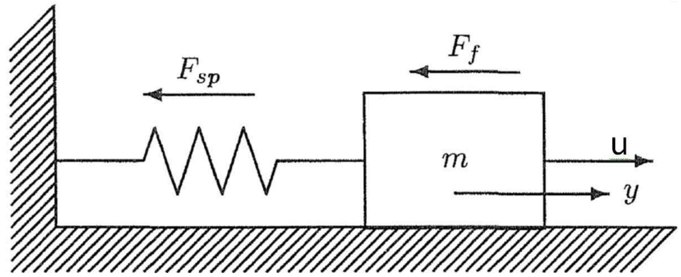

Fig. 1. Mass-spring mechanical system.

Remark 2. In this section, we propose a new postulated feedback (6) to stabilize system (1)-(2) in prescribed-time. The existence of multiplicative noise makes all the existing stochastic prescribed-time designs in [18]-[21] inapplicable. In order to handle the multiplicative noise, the feedback is already inserted into (8). As shown in [9]-[17], even $G = 0$, how to design the feedback gains $k_1, \dots, k_n$ is nontrivial. More importantly, note that, the perturbation G is actually nonzero, contains the feedback gains $k_1, \dots, k_n$ and even goes to infinity as $t \to t_0 + T$. How to design $k_1, \dots, k_n$ to prescribed-time stabilize the system in the presence of the nonlinearities f_i is a hard problem.

Remark 3. Different from the existing stochastic prescribed-time designs where scaling-free quartic Lyapunov functions [18], scaled quartic Lyapunov functions [19] and [21], or scaled quadratic Lyapunov function [20] are used, the design in this section is based on a new scaled quadratic Lyapunov function $\sum_{i=1}^{n} \frac{1}{2\mu^{2i-2}}\xi_i^2$ where the power of μ is lower than that in [20]. The advantage of this Lyapunov function is that it simplifies the design process, which yields a relative simpler controller.

IV. A SIMULATION EXAMPLE

In this section, we give a simulation example to show the effectiveness of control scheme developed in the last section.

Example 1. Consider the mass-spring mechanical system shown in Fig. 1, where a mass m is attached to a wall through a spring and sliding on a horizontal surface. The mass is driven by an external force which serves as a control variable. Let y be the displacement from a reference position. By Newton's law of motion, the system is described as [23]

$$m\ddot{y} + F_f + F_{sp} = u, \quad (63)$$

where F_f is a resistive force due to friction and F_{sp} is the restoring force of the spring. We assume that the displacement is relative small and thus F_{sp} can be written as $F_{sp} = ky$, where k is a spring parameter. Meantime, we assume the resistive force is linear viscous friction and write F_f as $F_f = c\dot{y}$, where c is a friction parameter.

To obtain a state model for the mass-spring mechanical system, take the state variables as $x_1 = y$ and $x_2 = \dot{y}$. Then, from (63) we get the state-space form as

$$\dot{x}_1 = x_2, \quad (64)$$
$$\dot{x}_2 = \frac{u}{m} - \frac{k}{m}x_1 - \frac{c}{m}x_2. \quad (65)$$

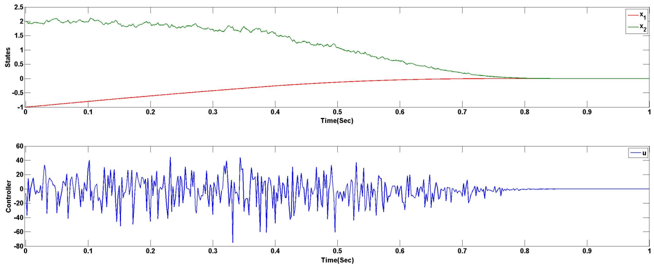

Fig. 2. The response of the closed-loop system (64)-(68).

Choosing $m=1, k=0.2, c=0.1$, Assumption 1 is satisfied with $c_1=0$ and $c_2=0.2$.

Let $t_0=0$ and $T=1$. (5) can be rewritten as

$$\mu(t) = \left(\frac{1}{1-t}\right)^2, \forall\, t \in [0,1). \tag{66}$$

We observe the state x_i as y_i, which is described by

$$y_1 = x_1(1+0.007(1-t)\dot{\omega}_1), \tag{67}$$
$$y_2 = x_2(1+0.01(1-t)\dot{\omega}_2). \tag{68}$$

From the design in Section III, we get $\alpha_1 = 2$ and $\alpha_2 = 40.68$. From (60) and (61), we have

$$0 < \delta_1 < 0.0072, \tag{69}$$
$$0 < \delta_2 < 0.012. \tag{70}$$

From (67)-(70), Assumption 2 holds.

By following the design procedure developed in Section III, we obtain the controller as

$$u = -81.36\mu^2 y_1 - 40.68\mu y_2. \tag{71}$$

For simulation, we randomly set the initial conditions as $x_1(0) = -1$, $x_2(0) = 2$. Fig. 2 gives the responses of the controller and states, which shows that $\lim_{t\to 1} E|x|^2 = 0$. Therefore, the effectiveness of the control scheme developed in Section Section III is demonstrated.

V. CONCLUDING REMARKS

In this paper we have addressed the prescribed-time designs for strict-feedback nonlinear systems with multiplicative measurement noise. When the noise is small and linearly vanishing, we propose a new postulated feedback to solve the prescribed-time mean-square stabilization problem. In order to handle the multiplicative noise, in our new designs, the feedback is inserted into the system, which leads to that the noise intensity is actually nonzero, contains the feedback gains and even goes to infinity in the terminal time, how to design the control gains to prescribed-time stabilize the system in the presence of the nonlinearities is a hard problem.

For the prescribed-time designs with multiplicative measurement noise, many important issues are still open and worth investigating, such as output-feedback control, prescribed-time control for more general systems, etc.

REFERENCES

[1] J.J. Carr, *Sensors and Circuits*, Prentice-Hall: Englewood Cliffs, 1993.

[2] E. Lantto, *Robust control of magnetic bearings in subcritical machines*, Ph.D. dissertation, Department of Electrical Engineering, Helsinki University of Technology, Espoo, Finland, 1999.

[3] C.C. Chen, C.J. Qian, Z.Y. Sun, Y.W. Liang, Global output feedback stabilization of a class of nonlinear systems with unknown measurement sensitivity, *IEEE Transactions on Automatic Control*, 63(7): 2212–2217, 2018.

[4] W.Q. Li, X.X. Yao, M. Krstic, Adaptive-gain observer-based stabilization of stochastic strict-feedback systems with sensor uncertainty, *Automatica*, 120, 109112,

[5] C.M. Harris, D.W. Wolpert, Signal-dependent noise determines motor planning, *Nature*, 394(6695): 780–784, 1998.

[6] M.C. De Oliveira, R.E. Skelton, State feedback control of linear systems in the presence of devices with finite signal-to-noise ratio, *International Journal of Control*, 74(15): 1501–1509, 2001.

[7] W.Q. Li, M. Krstic, Stabilization of triangular nonlinear systems with multiplicative stochastic state sensing noise, *IEEE Transactions on Automatic Control*, DOI: 10.1109/TAC.2022.3201032, 2022.

[8] P. Zarchan, *Tactical and strategic missile guidance*, Sixth Edition, Progress in Astronautics and Aeronautics, 2012.

[9] Y.D. Song, Y.J. Wang, J.C. Holloway, M. Krstic, Time-varying feedback for robust regulation of normal-form nonlinear systems in prescribed finite time, *Automatica*, 83: 243–251, 2017.

[10] Y.D. Song, Y.J. Wang, M. Krstic, Time-varying feedback for stabilization in prescribed finite time, *International Journal of Robust and Nonlinear Control*, 29(3): 618–633, 2019.

[11] J. Holloway, M. Krstic, Prescribed-time observers for linear systems in observer canonical form, *IEEE Transactions on Automatic Control*, 64(9): 3905–3912, 2019.

[12] J. Holloway, M. Krstic, Prescribed-time output feedback for linear systems in controllable canonical form, *Automatica*, 107: 77–85, 2019.

[13] P. Krishnamurthy, F. Khorrami, M. Krstic, A dynamic high-gain design for prescribed-time regulation of nonlinear systems, *Automatica*, 115: 108860, 2020.

[14] P. Krishnamurthy, F. Khorrami, M. Krstic, Robust adaptive prescribed-time stabilization via output feedback for uncertain nonlinear strict-feedback-like systems, *European Journal of Control*, 55: 14–23, 2020.

[15] P. Krishnamurthy, F. Khorrami, M. Krstic, Adaptive output-feedback stabilization in prescribed time for nonlinear systems with unknown parameters coupled with unmeasured states, *International Journal of Adaptive Control and Signal Processing*, 35(2): 184–202, 2021.

[16] D. Steeves, M. Krstic, R. Vazquez, Prescribed-time estimation and output regulation of the linearized Schrödinger equation by backstepping, *European Journal of Control*, 55: 3–13, 2020.

[17] D. Steeves, M. Krstic, R. Vazquez, Prescribed-time H^1-stabilization of reaction-diffusion equations by means of output feedback, *Proceedings of 2019 European Control Conference*, 1932–1937, 2019.

[18] W.Q. Li, M. Krstic, Stochastic nonlinear prescribed-time stabilization and inverse optimality, *IEEE Transactions on Automatic Control*, 67(3): 1179–1193, 2022.

[19] W.Q. Li, M. Krstic, Prescribed-time control of stochastic nonlinear systems with reduced control effort, *Journal of Systems Science & Complexity*, 34(5): 1782–1800, 2021.

[20] W.Q. Li, M. Krstic, Prescribed-time output-feedback control of stochastic nonlinear systems, *IEEE Transactions on Automatic Control*, 68(3): 1431–1446, 2023.

[21] W.Q. Li, M. Krstic, Prescribed-time mean-nonovershooting control under finite-time vanishing noise, *SIAM Journal of Control and Optimization*, 61(3): 1187–1212, 2023.

[22] H. Deng, M. Krstic, Stochastic nonlinear stabilization, part I: a backstepping design, *Systems and Control Letters*, 32: 143–150, 1997.

[23] H.K. Khalil, *Nonlinear Systems*, 3rd edition, Prentice Hall, 2002.