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Generic Multi-class Cell Transmission Model for Traffic Control

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Abstract— **Recent years have witnessed renewed interest in multi-class traffic models, inspired in no small part by the impending arrival of Connected and Autonomous Vehicles, whose behaviour is likely to differ from that of Human-Driven Vehicles. Although numerous multi-class traffic models have been proposed, consistent overarching theory is lacking. In this paper, we propose a generic first-order multi-class traffic modelling framework, intended to be sufficiently versatile to represent most of the traffic phenomena relevant to freeway control applications. Based on this framework, we are able to instantiate different specific multi-class models by appropriate choice of a few design functions. We restrict the design space by introducing a set of assumptions that these functions should follow, helping guide the modelling process. Finally, we study a simple control example where a single class of vehicles is controlled in order to dissipate congestion on a highway, and test the control law using several multi-class model variants. The simulation results show that proposed control is able to dissipate the congestion and harmonize the traffic without relying on knowing the exact underlying traffic dynamics.**

I. Introduction

The fine details of traffic flow dynamics, emerging from diverse interactions between individual vehicles on the road, can be extremely complex and hard to model. The advent of Connected and Autonomous Vehicles (CAVs) is likely to make these interactions even more complicated, while offering an unprecedented opportunity to deploy new traffic control schemes over a road transportation system, as long as we have tools capable of capturing the intrinsic heterogeneity of its components – the individual vehicles.

The well-known Lighthill-Whitham-Richards (LWR) continuous macroscopic traffic model [1, 2], and its discrete counterpart, the Cell Transmission Model (CTM) [3], have long been successfully used to capture the aggregate traffic dynamics with homogeneous vehicles and stationary traffic flow, defined through the flow-density relationship, i.e., the fundamental diagram (FD) [4]. The real-world traffic data, however, exhibits a significant scattering around the predicted values, which may in part be explained by the intrinsic heterogeneity of traffic. While every vehicle and driver is unique, it is often reasonable to subdivide them into a limited number of classes, each representing an homogeneous group with respect to all the relevant features (e.g. vehicle type, driving style, etc.), leading to the multi-class models.

Looking only at first-order macroscopic models, we fist have multi-commodity models [5, 6], where different classes have the same driving behaviour and FD. Another viable approach is to represent the heterogeneity of the traffic stream through explicitly representing some slow vehicles [7, 8] or platoons [9, 10], surrounded by a continuous stream of faster vehicles, as a micro-macro model. While the most general approach would be to define class-specific FDs that depend on the traffic densities of all classes, it is often enough to express the influence of other classes through the aggregate traffic density, especially in congestion [11].

Even if different vehicles have different individual contributions to the congestion levels, due to their size or sluggish behaviour, it is possible to convert them into passenger car equivalents (PCE), potentially also depending on the traffic state [12]. In FASTLANE [13] the PCE conversion depends on both a class-specific gross stopping distance and the current speed, mediated by the minimum class-specific time headway value, leading to larger PCE values associated with the free-flow regime. Alternatively, it is possible to model the influence of other vehicle classes by scaling the class-specific nominal FDs according to the portion of the road allotted to them [14], assuming user-equilibrium between the different populations, or by defining perceived equivalent densities [15], effectively converting the other vehicles to an appropriate amount of vehicles of the same class. Similarly, in [16], the overall FD is reshaped to reflect the properties of class-specific driving behaviours, focusing in particular on mixed CAV-human-driven traffic.

The main contribution of this work is in proposing a generic cell-based first-order multi-class traffic modelling framework, versatile enough to capture complex interactions between different vehicle classes and represent most of the traffic phenomena relevant to freeway control. Instead of focusing on one specific model instance, we outline the procedure for defining the different design functions that determine the model dynamics, through stating a set of assumptions that can act as guidelines for choosing them. Furthermore, we are able to formulate a control law that efficiently dissipates traffic congestion, agnostic to the exact underlying traffic model.

The rest of this paper is structured as follows. First, in Section II, we propose our generic multi-class modelling framework, give guiding modelling assumptions for choice of specific model components, and exemplify three specific multi-class models that fit within the framework. Then, in Section III, we describe a simple VSL-like control scheme for congestion dissipation, which is put to the test in simulations in Section IV, applied on the presented multi-class traffic model instances. Finally, in Section V, we conclude and outline directions for future work.

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II. Multi-class Traffic Modelling

In this section, we introduce the proposed generic multi-class traffic modelling framework. First, we briefly outline the the well-known (single-class) Cell Transmission Model (CTM) [3], and then discuss extending it to a generic multi-class model, capturing the interactions between different classes of traffic. The functions that specify the model are given in as generic form as practical, and we propose a set of assumptions to help guide the process of designing them to capture a particular type of interactions of interest. Finally, we illustrate the model design process by instantiating the generic model to some of the more prominent multi-class models from literature.

A. Single-class Cell Transmission Model

In this work, we focus on a multi-lane stretch of road with with homogeneous geometry an no on- and off-ramps. The traffic state at a discrete time t is given by the traffic densities $\rho_i(t)$ in cells i. The traffic state evolves according to

$$\rho_i(t+1) = \rho_i(t) + \frac{T}{L}\left(q_{i-1}(t) - q_i(t)\right) \tag{1}$$

where T is the time step, L the cell length, and $q_i(t)$ the traffic flow leaving cell i, given by

$$q_i(t) = \min\{d_i(t), s_i(t)\}. \tag{2}$$

Here $d_i(t)$ is the *demand* of $s_i(t)$ is the *supply* available to the vehicles in cell i, depending on the traffic state in the downstream cell $i+1$,

$$d_i(t) = D(\rho_i(t)), \tag{3}$$
$$s_i(t) = S(\rho_{i+1}(t)), \tag{4}$$

where $D(\rho)$ is some concave nondecreasing demand function with $D(0) = 0$, $S(\rho)$ some nonincreasing supply function with $S(P) = 0$, and P is the jam density. The concavity assumption on the demand function ensures that the wavefronts propagate at speed no higher than the speed of the traffic. The resulting FD can be defined as

$$Q(\rho) = \min\{D(\rho), S(\rho)\},$$

yielding critical density $\rho_{\mathrm{cr}} = \arg\max_{\rho\in[0,P]} Q(\rho)$, for which $D(\rho_{\mathrm{cr}}) = S(\rho_{\mathrm{cr}}) = Q(\rho_{\mathrm{cr}})$. Note that in order for the model to be numerically stable, there needs to exist some $V \in (0, \frac{L}{T}]$ such that $V\rho \geq D(\rho)$, corresponding to the Courant-Freidrichs-Lewy condition.

B. Generic Multi-class Cell Transmission Model

Let $\mathcal{K}$ be the set of vehicle classes with potentially different behaviours sharing the road. The traffic density of vehicles of class $\kappa \in \mathcal{K}$ in cell i at time t is denoted $\rho_i^\kappa(t)$, and the aggregate traffic density is given by

$$\rho_i(t) = \sum_{\kappa\in\mathcal{K}} \rho_i^\kappa(t).$$

All traffic densities are represented in terms of passenger car equivalents. The evolution of the traffic density of each class is of the same form as (1),

$$\rho_i^\kappa(t+1) = \rho_i^\kappa(t) + \frac{T}{L}\left(q_{i-1}^\kappa(t) - q_i^\kappa(t)\right), \quad \kappa\in\mathcal{K},$$

with similar expressions for the traffic flows of each class,

$$q_i^\kappa(t) = \min\{d_i^\kappa(t), s_i^\kappa(t)\}, \quad \kappa\in\mathcal{K},$$

resulting in the aggregate traffic flow

$$q_i(t) = \sum_{\kappa\in\mathcal{K}} q_i^\kappa(t).$$

Compared to the single-class case, the differences arise in how the demand and supply are defined. Whereas in the single-class case, these were naturally defined as a function of the aggregate traffic density, here the traffic flow of each class $\kappa \in \mathcal{K}$ depends not only on the traffic density of that class $\rho^\kappa(t)$, but also on the traffic densities of all other classes that share the road $\rho^c(t)$, $c \in \mathcal{K}\setminus\kappa$.

The generic form of class κ demand $d_i^\kappa(t)$ and supply $s_i^\kappa(t)$ that we adopt in this work is

$$d_i^\kappa(t) = \delta_i^\kappa(t) d_i(t)$$
$$s_i^\kappa(t) = \sigma_i^\kappa(t) s_i(t)$$

Here, $d_i(t)$ is the *aggregate demand*, $s_i(t)$ is the *aggregate supply* available to cell i, $\delta_i^\kappa(t)$ is the *demand allocation function*, and $\sigma_i^\kappa(t)$ the *supply allocation function*, with

$$\sum_{\kappa\in\mathcal{K}} \delta_i^\kappa(t) = \sum_{\kappa\in\mathcal{K}} \sigma_i^\kappa(t) = 1.$$

We define $d_i(t)$, $\delta_i^\kappa(t)$, and $\sigma_i^\kappa(t)$ as functions of the traffic state in cell i, $\rho_i^c(t)$, $c \in \mathcal{K}$, and $s_i(t)$ as a function of the downstream traffic state, in cell $i+1$, $\rho_{i+1}^c(t)$, $c \in \mathcal{K}$.

Remark. *The aggregate traffic flow* $q_i(t) = \sum_{\kappa\in\mathcal{K}} q_i^\kappa(t)$ *differs from* (2) *unless* $d_i^\kappa(t) \leq s_i^\kappa(t)$, $\forall\kappa\in\mathcal{K}$, *when we have* $q_i(t) = d_i(t)$, *or* $d_i^\kappa(t) \geq s_i^\kappa(t)$, $\forall\kappa\in\mathcal{K}$, *when we have* $q_i(t) = s_i(t)$, *i.e., thus defined* $d_i(t)$ *and* $s_i(t)$ *do not correspond to the supply and demand of the whole traffic in the sense of* (3) *and* (4).

C. Modelling Assumptions

In this work, we aim to provide an as generic as possible form of a multi-class cell-based discrete-time traffic model. Therefore, we only impose those assertions about the form of $\delta_i^\kappa(t)$, $d_i(t)$, $\sigma_i^\kappa(t)$, and $s_i(t)$ that are necessary for numerical stability,

$$\delta_i^\kappa(t) \geq 0, \quad d_i(t) \geq 0, \quad \sigma_i^\kappa(t) \geq 0, \quad s_i(t) \geq 0,$$
$$\left(\exists V \in \left(0, \frac{L}{T}\right]\right) V\rho_i^\kappa(t) \geq \delta_i^\kappa(t) d_i(t), \quad \kappa\in\mathcal{K}.$$

Apart from this condition, we allow these functions to take a general form, and instead focus on providing a catalog of assumptions to guide the process of choosing them. These assumptions are designed to restrict the design space of $\delta_i^\kappa(t)$, $d_i(t)$, $\sigma_i^\kappa(t)$, and $s_i(t)$, in order to impart some desirable properties to the model. The catalog of assumptions is loosely ordered from the most general ones, that most versions of multi-class models satisfy, to more specific ones.

(A1) ***Common jam density.*** Densities of each class of vehicles are expressed in passenger cars equivalents, with scaling factors chosen such that there exists a unique jam density P for all classes, and intrinsic supply functions of all classes have the same support,

$$\rho < P \Longrightarrow S^{\kappa}(\rho) > 0, \; \rho \geq P \Longrightarrow S^{\kappa}(\rho) = 0, \; \kappa \in \mathcal{K}.$$

(A2) ***Intrinsic class behaviour.*** For each class κ, we define its intrinsic behaviour describing the traffic dynamics in case vehicles of that class were alone on the road. This behaviour is represented by its intrinsic FD $Q^{\kappa}(\rho)$, defined through its intrinsic demand function $D^{\kappa}(\rho)$ and supply function $S^{\kappa}(\rho)$,

$$Q^{\kappa}(\rho) = \min\{D^{\kappa}(\rho), S^{\kappa}(\rho)\}.$$

Furthermore, there exist some maximum free flow speed $V > 0$ such that all class demand functions $D^{\kappa}(\rho)$ are V^{κ}–Lipschitz continuous, $V^{\kappa} \leq V$, and some maximum congestion wave speed $W > 0$ such that all class supply functions $S^{\kappa}(\rho)$ are W^{κ}–Lipschitz continuous, $W^{\kappa} \leq W$,

$$0 \leq \frac{\partial D^{\kappa}(\rho)}{\partial \rho} \leq V, \quad -W \leq \frac{\partial S^{\kappa}(\rho)}{\partial \rho} \leq 0.$$

Finally, if vehicles of a single class $\kappa \in \mathcal{K}$ *dominate* the traffic at some location, $\rho_i^{\kappa}(t) \approx \rho_i(t)$, $\rho_{i+1}^{\kappa}(t) \approx \rho_{i+1}(t)$, the dynamics of both the class κ traffic and the aggregate traffic are dominantly determined by its intrinsic behaviour,

$$d_i^{\kappa}(t) \approx d_i(t) \approx D_i^{\kappa}(\rho_i(t)),$$
$$s_{i+1}^{\kappa}(t) \approx s_{i+1}(t) \approx S_{i+1}^{\kappa}(\rho_{i+1}(t)),$$

yielding $\delta_i^{\kappa}(t) \approx 1$ and $\sigma_i^{\kappa}(t) \approx 1$. Note that we allow $D_i^{\kappa}(\rho)$ and $S_i^{\kappa}(\rho)$ to be defined differently for different cells in order to capture varying road geometry, traffic control actions, etc.

(A3) ***Near-vacuum and near-jam conditions.*** In free flow, with sufficiently low total traffic density, $\rho_i(t) \approx 0$, the demand of all classes is dominated by their intrinsic demand,

$$\rho_i(t) \approx 0 \implies d_i^{\kappa}(t) \approx D_i^{\kappa}(\rho_i^{\kappa}(t)), \quad \kappa \in \mathcal{K}.$$

Consequently, we have

$$\rho_i(t) \approx 0 \implies \delta_i^{\kappa}(t) \approx \frac{D_i^{\kappa}(\rho_i^{\kappa}(t))}{d_i(t)}, \quad \kappa \in \mathcal{K},$$

and the aggregate demand is given by

$$\rho_i(t) \approx 0 \implies d_i(t) \approx \sum_{\kappa \in \mathcal{K}} D_i^{\kappa}(\rho_i^{\kappa}(t)).$$

Therefore, one good default option for $\delta_i^{\kappa}(t)$ is

$$\delta_i^{\kappa}(t) = \frac{D_i^{\kappa}(\rho_i^{\kappa}(t))}{\sum\limits_{c \in \mathcal{K}} D_i^{c}(\rho_i^{c}(t))}. \tag{5}$$

Conversely, in congestion, with sufficiently high total traffic density, $\rho_{i+1}(t) \approx P$, vehicles of all classes follow the same behaviour and move at the same speed,

$$\frac{s_i^{\kappa}(t)}{\rho_i^{\kappa}(t)} \approx \frac{s_i(t)}{\rho_i(t)},$$

since in this case the traffic flows depend only on the supply. Consequently, we have

$$\rho_{i+1}(t) \approx P \implies \sigma_i^{\kappa}(t) \approx \frac{\rho_i^{\kappa}(t)}{\rho_i(t)}, \quad \kappa \in \mathcal{K},$$

and a good default option for $\sigma_i^{\kappa}(t)$ is similarly

$$\sigma_i^{\kappa}(t) = \frac{\rho_i^{\kappa}(t)}{\rho_i(t)}. \tag{6}$$

(A4) ***Road space allocation.*** We may define the aggregate demand $d_i(t)$ and supply $s_i(t)$ as

$$d_i(t) = \sum_{\kappa \in \mathcal{K}} p_i^{\kappa}(t) D_i^{\kappa}\left(\frac{\rho_i^{\kappa}(t)}{p_i^{\kappa}(t)}\right), \tag{7}$$

$$s_i(t) = \sum_{\kappa \in \mathcal{K}} r_{i+1}^{\kappa}(t) S_{i+1}^{\kappa}\left(\frac{\rho_{i+1}^{\kappa}(t)}{r_{i+1}^{\kappa}(t)}\right), \tag{8}$$

with $p_i^{\kappa}(t)$ representing road space allocation to class κ in free-flow-like conditions, and $r_i^{\kappa}(t)$ representing road space allocation to class κ in congestion-like conditions. Similar approaches are commonly used to model multi-class traffic [14, 15]. Note that these road space allocation functions may, but need not correspond to the demand and supply allocation functions $\delta_i^{\kappa}(t)$ and $\sigma_i^{\kappa}(t)$.

x'Furthermore, the total allocated road space follows

$$\sum_{\kappa \in \mathcal{K}} p_i^{\kappa}(t) \leq 1, \quad \sum_{\kappa \in \mathcal{K}} r_i^{\kappa}(t) = 1,$$

where we assume that the entire road is always used in congestion. For both road space allocation functions, denoting by f either p or r, we have

$$\frac{\rho_i^{\kappa}(t)}{P} \leq f_i^{\kappa}(t) \leq 1 - \frac{\rho_i(t) - \rho_i^{\kappa}(t)}{P}.$$

In this case, one natural choice for function $\delta_i^{\kappa}(t)$ is

$$\delta_i^{\kappa}(t) = \frac{p_i^{\kappa}(t) D_i^{\kappa}\left(\frac{\rho_i^{\kappa}(t)}{p_i^{\kappa}(t)}\right)}{\sum_{c \in \mathcal{K}} p_i^{c}(t) D_i^{c}\left(\frac{\rho_i^{c}(t)}{p_i^{c}(t)}\right)}, \tag{9}$$

yielding a simple expression for class demands,

$$d_i^{\kappa}(t) = p_i^{\kappa}(t) D_i^{\kappa}\left(\frac{\rho_i^{\kappa}(t)}{p_i^{\kappa}(t)}\right). \tag{10}$$

(A5) ***Bounds on demand and supply.*** The aggregate demand $d_i(t)$ is lower-bounded by imposing the behaviour of the slowest class on all traffic,

$$d_i(t) \geq \min_{\kappa \in \mathcal{K}, \rho_i^{\kappa}(t) > 0} \{D_i^{\kappa}(\rho_i(t))\},$$

corresponding to the case when it is impossible to overtake the slow vehicles. The upper bound of $d_i(t)$ is achieved when the entire road space is allocated, $\sum_{\kappa \in \mathcal{K}} p_i^{\kappa}(t) = 1$, and road space allocation is such that it maximizes (7), depending on $\rho_i^{\kappa}(t)$ and $D_i^{\kappa}(\rho)$, $\kappa \in \mathcal{K}$.

Conversely, the aggregate supply is upper-bounded by imposing the behaviour of the most reactive class on all traffic,

$$s_i(t) \leq \max_{\kappa \in \mathcal{K}, \rho_{i+1}^{\kappa}(t) > 0} \left\{S_{i+1}^{\kappa}(\rho_{i+1}(t))\right\},$$

corresponding to the case when the traffic is permeated with vehicles of the highly reactive class.

Finally, the demand and supply of class κ do not exceed their intrinsic values, calculated for the case if vehicles of other classes would be ignored,

$$d_i^{\kappa}(t) \leq D_i^{\kappa}(\rho_i^{\kappa}(t)),$$
$$s_i^{\kappa}(t) \leq S_{i+1}^{\kappa}(\rho_{i+1}^{\kappa}(t)).$$

D. Multi-class Cell Transmission Model Instances

In order to demonstrate how the proposed generic multi-class traffic modelling framework can be used to represent various models, both known in the literature and novel, we present three examples.

1) Extended multi-class CTM: First, we give an extended version of the multi-class CTM from [10], reformulated to allow for general demand and supply functions. Given intrinsic demand $D_i^\kappa(\rho)$ and supply functions $S_i^\kappa(\rho)$, $\kappa \in \mathcal{K}$, the model is defined by

$$d_i(t) = \min\left\{\sum_{\kappa\in\mathcal{K}} D_i^\kappa(\rho_i^\kappa(t)), \bar{q}_i(t)\right\}, \tag{11}$$

$$s_i(t) = \min\left\{S^\kappa(\rho_{i+1}^\kappa(t)), \bar{q}_{i+1}(t)\right\}, \tag{12}$$

$$\bar{q}_i(t) = \frac{\sum_{\kappa\in\mathcal{K}} D_i^\kappa(\rho_i^\kappa(t))\bar{q}_i^\kappa}{\sum_{\kappa\in\mathcal{K}} D_i^\kappa(\rho_i^\kappa(t))},$$

$$\bar{q}_i^\kappa = \max_{\rho\in[0,P]} \min\{D_i^\kappa(\rho), S_i^\kappa(\rho)\},$$

with default forms of $\delta_i^\kappa(t)$ and $\sigma_i^\kappa(t)$, (5) and (6).

2) Road-space-allocation-based model: Next we show how the commonly used multi-class traffic modelling framework relying on road space allocation and perceived density, proposed in [14] and discussed in [15], maps onto the proposed framework. Focusing on the model from [15], which is a specific form of the one from [14], we need to define functions $d_i(t)$, $s_i(t)$, $\delta_i^\kappa(t)$, and $\sigma_i^\kappa(t)$ that lead to the same outcomes. A corresponding model may be constructed based on assumption **(A4)**, with $d_i(t)$ given by (7) and $s_i(t)$ by (8). Functions $\delta_i^\kappa(t)$ and $p_i^\kappa(t)$ are defined such that they ensure that in free flow, with sufficiently low traffic density, the class traffic flow simplifies to

$$q_i^\kappa(t) \approx D_i^\kappa(\rho_i^\kappa(t)),$$

which may be achieved by using $\delta_i^\kappa(t)$ given by (9), resulting in the class demands of the appropriate form (10). Functions $\sigma_i^\kappa(t)$ and $r_i^\kappa(t)$ are defined so that the model would exhibit the same behaviour in congested regime, with all traffic moving at the same speed. It can be shown that this is achieved using $\sigma_i^\kappa(t)$ given by (6). Note that an equivalent for $r_i^\kappa(t)$ is not defined in [15], so in this case we may simply use $r_i^\kappa(t) = \sigma_i^\kappa(t)$.

3) Simple multi-lane-emulating model: Finally, we illustrate the versatility of the proposed modelling framework by specifying another model instance inspired by multi-lane models. In this specific case, we want to explicitly capture the phenomenon of a slow class of vehicles forcing the other vehicles to slow down by blocking the traffic in one or multiple lanes. Consider two vehicle classes $\mathcal{K} = \{\mathfrak{a}, \mathfrak{b}\}$, with the demand function of the slow vehicles $\mathfrak{a}$ being $D_i^\mathfrak{a}(\rho; t) = \min\{U_i^\mathfrak{a}(t)\rho, \bar{q}\}$, where $U_i^\mathfrak{a}(t)$ is the (potentially time-varying) free flow speed of class $\mathfrak{a}$ vehicles, and $\bar{q}$ is the road capacity. Additionally, let the demand function of class $\mathfrak{b}$ vehicles satisfy $D_i^\mathfrak{b}(\rho) = \bar{q}$ for $\rho \geq \rho_{\mathrm{cr}}$, $v_{\mathrm{cr}}\rho_{\mathrm{cr}} = \bar{q}$, and the supply of both classes be

$$S_i^\kappa(\rho) = \mathcal{S}(\rho) = \min\{W(P-\rho), \bar{q}\}, \quad s_i(t) = \mathcal{S}(\rho_{i+1}(t)).$$

Then, assuming the road has two lanes, the aggregate demand can be written

$$d_i(t) = \begin{cases} d_{I,i}(t), & U_i^\mathfrak{a}(t) \geq v_{\mathrm{cr}} \\ d_{II,i}(t), & U_i^\mathfrak{a}(t) < v_{\mathrm{cr}}, \frac{W+U_i^\mathfrak{a}(t)}{WP}\rho_i^\mathfrak{a}(t) < \frac{1}{2}, \\ d_{III,i}(t), & U_i^\mathfrak{a}(t) < v_{\mathrm{cr}}, \frac{W+U_i^\mathfrak{a}(t)}{WP}\rho_i^\mathfrak{a}(t) \geq \frac{1}{2}, \end{cases} \tag{13}$$

$$d_{I,i}(t) = p_i^\mathfrak{a}(t) D_i^\mathfrak{a}\left(\frac{\rho_i^\mathfrak{a}(t)}{p_i^\mathfrak{a}(t)}\right) + p_i^\mathfrak{b}(t) D_i^\mathfrak{b}\left(\frac{\rho_i^\mathfrak{b}(t)}{p_i^\mathfrak{b}(t)}\right),$$

$$d_{II,i}(t) = \frac{1}{2} D_i^\mathfrak{b}(\min\{2\rho_i^\mathfrak{b}(t), \rho_{\mathrm{cr}}\}) + \dots$$
$$\dots + U_i^\mathfrak{a}(t)(\rho_i^\mathfrak{a}(t) + \max\{0, \rho_i^\mathfrak{b}(t) - \frac{\rho_{\mathrm{cr}}}{2}\},$$

$$d_{III,i}(t) = U_i^\mathfrak{a}(t)\rho_i(t).$$

We may adopt the default forms of $\delta_i^\kappa(t)$ and $\sigma_i^\kappa(t)$, (5) and (6), respectively, and also let road space allocation functions $p_i^\kappa(t)$ for the case when class $\mathfrak{a}$ vehicles are not slower than class $\mathfrak{b}$ vehicles be $p_i^\kappa(t) = \delta_i^\kappa(t)$.

III. Control

Using the proposed generic model, we formulate a simple VSL-like congestion dissipation control law. We focus on a simple two-class traffic case, $\mathcal{K} = \{\mathfrak{a}, \mathfrak{b}\}$, where class $\mathfrak{a}$ consists of connected vehicles that can be controlled from the infrastructure, and class $\mathfrak{b}$ of human-driven background traffic, which cannot be directly controlled. In particular, we assume class $\mathfrak{a}$ vehicles in each cell within some range $i = \underline{I}_U, \underline{I}_U + 1, \dots, \bar{I}_U$ can be issued a reference speed $U_i^\mathfrak{a}(t) \in [\underline{U}, V]$ that they follow, unless they are forced to slow down due to congestion. Here $\underline{U} > 0$ is the minimum assignable reference speed, V is the overall maximum allowed speed on the road, and $U_i^\mathfrak{a}(t) = V$ for $i \notin [\underline{I}_U, \bar{I}_U]$. The effect of this control is modelled through the intrinsic demand function of class $\mathfrak{a}$, defined as

$$D_i^\mathfrak{a}(\rho; t) = \min\left\{U_i^\mathfrak{a}(t)\rho, \mathcal{D}^\mathfrak{a}(\rho)\right\}, \tag{14}$$

whereas the intrinsic demand function of class $\mathfrak{b}$ and the intrinsic supply functions of both classes take the same form in all cells,

$$D_i^\mathfrak{b}(\rho) = \mathcal{D}^\mathfrak{b}(\rho), \quad S_i^\kappa(\rho) = \mathcal{S}^\kappa(\rho), \quad \kappa \in \{\mathfrak{a}, \mathfrak{b}\}. \tag{15}$$

Here we define by $\mathcal{D}^\mathfrak{a}(\rho)$ and $\mathcal{D}^\mathfrak{b}(\rho)$ the default (uncontrolled) intrinsic demand functions of the two classes, by $\mathcal{S}^\kappa(\rho)$ the default intrinsic supply function, and we have $\mathcal{D}^\mathfrak{a}(\rho) \leq V\rho$ and $\mathcal{D}^\mathfrak{b}(\rho) \leq V\rho$. Note that the intrinsic demand function of class $\mathfrak{a}$ now varies in time, together with $U_i^\mathfrak{a}(t)$.

The influence of $U_i^\mathfrak{a}(t)$ on the aggregate traffic flow $q_i(t)$ can be approximated as

$$q_i(t) \approx \hat{q}_i(t) + \hat{A}_i(t)(U_i^\mathfrak{a}(t) - V),$$

where $\hat{A}_i(t)$ is the sensitivity of $q_i(t)$ to $U_i^\mathfrak{a}(t)$. We denote the uncontrolled aggregate traffic flow by $\hat{q}_i(t)$, corresponding to the case when $U_i^\mathfrak{a}(t) = V$ and $D_i^\mathfrak{a}(\rho) = \mathcal{D}^\mathfrak{a}(\rho)$, and define

$$u_i^\mathfrak{a}(t) = U_i^\mathfrak{a}(t) - V.$$

The evolution of aggregate traffic density can now be written

$$\rho_i(t{+}1) \approx \hat{\rho}_i(t{+}1) + \frac{T}{L}\left(\hat{A}_{i-1}(t)u^{\mathfrak{a}}_{i-1}(t) - \hat{A}_i(t)u^{\mathfrak{a}}_i(t)\right),$$

where $\hat{\rho}_i(t+1)$ is the uncontrolled updated aggregate traffic density, with $u^{\mathfrak{a}}_{i-1}(t) = u^{\mathfrak{a}}_i(t) = 0$,

$$\hat{\rho}_i(t+1) = \rho_i(t) + \frac{T}{L}\left(\hat{q}_{i-1}(t) - \hat{q}_i(t)\right).$$

A straightforward way to achieve congestion dissipation is to restrict the inflow to congested cells, thus keeping $\rho_i(t)$ close to some traffic density $\rho^*_i(t)$ that maximises the traffic flow. This admits a recursively formulated control law

$$\overleftarrow{u}^{\mathfrak{a}}_i(t) = \min\left\{0, \frac{\frac{L}{T}(\rho^*_{i+1}(t) - \hat{\rho}_{i+1}(t)) + \hat{A}_{i+1}(t)\,\overleftarrow{u}^{\mathfrak{a}}_{i+1}(t)}{\hat{A}_i(t)}\right\},$$

starting from $\overleftarrow{u}^{\mathfrak{a}}_{\bar{I}_U}(t)$, with $\overleftarrow{u}^{\mathfrak{a}}_{\bar{I}_U+1}(t) = 0$, corresponding to $U^{\mathfrak{a}}_{\bar{I}_U+1}(t) = V$, and then propagating the control computation upstream until $\overleftarrow{u}^{\mathfrak{a}}_{\underline{I}_U}(t)$. Note that $\overleftarrow{u}^{\mathfrak{a}}_i(t)$ is deliberately not defined lower-bounded in the process of recursively computing the control action, allowing the information about congestion multiple cells downstream of cell i to affect $\overleftarrow{u}^{\mathfrak{a}}_i(t)$.

While control defined by $U^{\mathfrak{a}}_i(t) = \max\{\underline{U}, V + \overleftarrow{u}^{\mathfrak{a}}_i(t)\}$ is effective in dissipating congestion, it may also cause congestion at the upstream end of the control zone, in cell $i = \underline{I}_U$, which has the potential to aggravate the traffic situation instead of improving it. Therefore, it is necessary to lower-bound the control action to avoid this, which can also be defined recursively by

$$\overrightarrow{u}^{\mathfrak{a}}_i(t) = \max\left\{\underline{U} - V, \min\left\{0, \frac{\frac{L}{T}(\hat{\rho}_i(t) - \rho^*_i(t)) + \hat{A}_{i-1}(t)\,\overrightarrow{u}^{\mathfrak{a}}_{i-1}(t)}{\hat{A}_i(t)}\right\}\right\},$$

starting with from $\overrightarrow{u}^{\mathfrak{a}}_{\underline{I}_U}(t)$, with $\overrightarrow{u}^{\mathfrak{a}}_{\underline{I}_U-1}(t) = 0$ and $U^{\mathfrak{a}}_{\underline{I}_U-1}(t) = V$, and then propagating the control computation downstream until $\overrightarrow{u}^{\mathfrak{a}}_{\bar{I}_U}(t)$. Finally, the control action to be applied is defined by

$$U^{\mathfrak{a}}_i(t) = V + \max\left\{\overleftarrow{u}^{\mathfrak{a}}_i(t), \overrightarrow{u}^{\mathfrak{a}}_i(t)\right\}. \qquad (16)$$

IV. Simulation Results

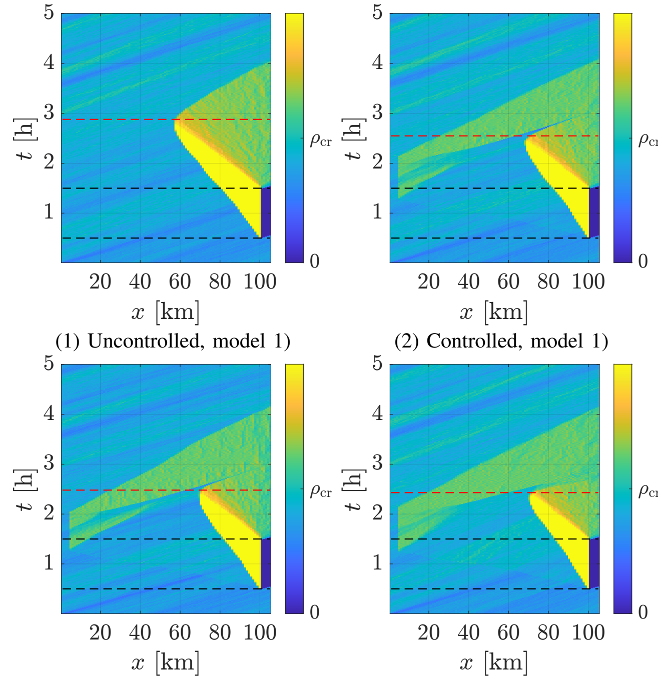
We demonstrate the effectiveness of the proposed control law in simulations, on a two-class congestion dissipation example, with class $\mathfrak{a}$ being vehicles we can control, and class $\mathfrak{b}$ vehicles we cannot control. We studied a $N_x L = 105$ km long road with no on- or off-ramps, with $N_t T = 5$ h long simulation runs. The traffic flow around position $x = 100$ km is blocked for one hour, during $tT = [0.5, 1.5]$ h, causing the congestion that the control law is tasked with dissipating.

We compare the control performance using the three model instances described in Section II-D:

1) Extended multi-class CTM from [10], (11), (12),
2) Road-space-allocation-based model similar to the ones in, e,g., [14] and [15],
3) Simple multi-lane-emulating model (13).

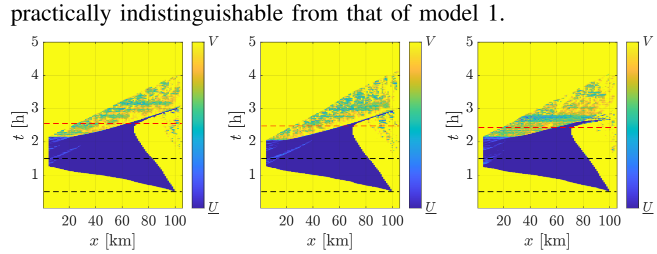
For all three models, we adopt the same default intrinsic demand and supply functions for both classes $\kappa \in \{\mathfrak{a}, \mathfrak{b}\}$,

$$\mathcal{D}^{\mathfrak{a}}(\rho) = \mathcal{D}^{\mathfrak{b}}(\rho) = V\left(\min\{\rho, \rho_{\text{cr}}\} - \alpha(\min\{\rho, \rho_{\text{cr}}\})^2\right),$$
$$\mathcal{S}^{\mathfrak{a}}(\rho) = \mathcal{S}^{\mathfrak{b}}(\rho) = W(P - \max\{\rho, \rho_{\text{cr}}\}),$$



(1) Uncontrolled, model 1) (2) Controlled, model 1) (3) Controlled, model 2) (4) Controlled, model 3)

Fig. 1: Aggregate traffic density $\rho_i(t)$. Horizontal dashed red line indicate the time when congestion is dissipated, and horizontal dashed black line outline the time when the traffic flow is blocked. The uncontrolled case for models 2 and 3 are omitted due to being practically indistinguishable from that of model 1.



(1) Model 1) (2) Model 2) (3) Model 3)

Fig. 2: Control action $U^{\mathfrak{a}}_i(t)$ for the three controlled cases.

with $V = 100$ km/h, critical density $\rho_{\text{cr}} = 38$ veh/km, parameter α such that $\bar{q} = V(\rho_{\text{cr}} - \alpha{\rho_{\text{cr}}}^2) = 3000$ veh/h, jam density $P = 120$ veh/km, and congestion wave speed W such that $W(P - \rho_{\text{cr}}) = \bar{q}$. Specific intrinsic demands and supplies are given by (14) and (15), with $D^{\mathfrak{a}}_i(\rho; t)$ depending on class $\mathfrak{a}$ reference speed $U^{\mathfrak{a}}_i(t)$, acting as the control input. The initial traffic density, as well as the density of the flow arriving from upstream, take values in $\rho^{\mathfrak{a}}_i(0) \in [0, 12]$ veh/km, $\rho^{\mathfrak{b}}_i(0) \in [3, 27]$ veh/km. Finally, a random normally distributed multiplicative noise, with a mean of 1 and standard deviation of 0.05, is applied to all traffic speeds, limited to ensure that CFL conditions hold.

The applied control law is described in Section III and given by (16). The controlled case for all three models is compared by a corresponding uncontrolled case, for which we set $U^{\mathfrak{a}}_i(t) = V$. We approximate the uncontrolled aggregate traffic flow as

$$\hat{q}_i(t) = \min\{v_{\text{cr}}\rho_i(t), \bar{q}\},$$

and the sensitivity to control as

$$\hat{A}_i(t) = \min\{\rho^{\mathfrak{a}}_i(t), \rho_{\text{cr}}\}.$$

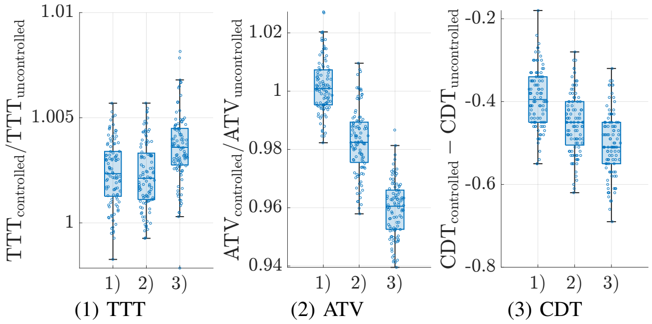


Fig. 3: Box plots of relative TTT and ATV, and CDT difference.

The reference traffic density is set to $\rho_i^*(t) = \rho_{\mathrm{cr}}$, and the minimum allowed reference speed is taken to be $\underline{U} = V/3$.

We executed 100 simulation runs for all three modes, both uncontrolled and controlled. The results of one representative simulation run are shown in Figure 1, depicting the space-time profile of aggregate traffic density $\rho_i(t)$, and Figure 2, depicting the control action $U_i^{\mathfrak{a}}(t)$. We can see that the applied control manages to accelerate the congestion dissipation in all three cases, with very similar behaviour of the traffic. Class $\mathfrak{a}$ vehicles are controlled to slow down upstream of the congestion, thus reducing the inflow of traffic to it and helping dissipate it faster, essentially spreading the congestion more evenly along the road. After the initial congestion is dissipated, the control discharges the remaining vehicles at road capacity, keeping them at critical density.

We consider three indices of control performance: Total Travel Time (TTT), Average Total Variation (ATV),

$$\mathrm{TTT} = \sum_{t=1}^{N_t} \sum_{i=1}^{N_x} LT\rho_i(t), \;\; \mathrm{ATV} = \sum_{t=1}^{N_t} \sum_{i=1}^{N_x - 1} \frac{|\rho_{i+1}(t) - \rho_i(t)|}{N_t(N_x - 1)},$$

and Congestion Dissipation Time (CDT), defined here as the length of the period from when the traffic flow blockage is removed to when all $\rho_i(t)$ is lower than some limit (in this case $\rho_{\mathrm{cr}} + 10$). As can be seen from the statistics of these indices are shown as box plots in Figure 3, the control manages to reduce the CDT and slightly improve the ATV of $\rho_i(t)$, with a negligible increase in TTT.

V. Conclusion

In this work we propose a generic multi-class traffic modelling framework, capable of capturing complex interactions between vehicles with different behaviours. This framework covers the majority of hitherto proposed multi-class models, including those based on the popular road-space-allocation approach. Instead of defining the exact mechanism of interaction between different vehicle classes, we present a set of assumptions which can be used to design functions that capture the phenomena of interest. The simple control law, based on this modelling framework, is shown in simulations to be able to accelerate congestion dissipation without knowing the exact underlying traffic model. This suggests that it might be possible to apply similar control schemes to more realistic microscopic, or even real-world traffic, without the need for an overly complex description of the interactions between the individual vehicles.

Among the plethora of potential directions for continuing this work we particularly emphasize the potential for this modelling framework to work with design functions that include uncertainty. In this case, robust control could explicitly be designed, and different model components could be learned from data. Finally, a detailed analysis of the properties of various models within this framework is forthcoming.

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