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L_∞/L_2 Hankel Norm Analysis and Characterization of Critical Instants for Continuous-Time Linear Periodically Time-Varying Systems

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***Abstract*— This paper is concerned with the Hankel norm analysis of linear periodically time-varying systems. An arbitrary $\Theta \in [0, h)$ is first taken as the instant separating past and future, where h denotes the period of such systems, and what is called the quasi L_∞/L_2 Hankel norm at Θ is defined. Then, a computation method of this norm for each Θ is derived. The supremum of the quasi L_∞/L_2 Hankel norms over $\Theta \in [0, h)$ is further defined as *the* L_∞/L_2 Hankel norm, and it is also shown that it can be computed directly without dealing with any quasi L_∞/L_2 Hankel norms. A relevant question of whether the supremum is attained as the maximum is also studied. In particular, it is discussed when and how the existence/absence of a critical instant attaining the maximum (and all the values of critical instants, if one exists) can be determined without computing all (or any of) the quasi L_∞/L_2 Hankel norms over $\Theta \in [0, h)$.**

I. Introduction

The Hankel operator is known to be an important notion for dynamical systems, as an operator describing the relation between the past input and the future output ([1]–[6]). In some studies, the input and output spaces are not limited to L_2 (e.g., [4],[7]–[9]). Among them, the case where the input space is L_2 and the output space is L_∞ is closely related to the well-known H_2 norm, and the present paper is also interested in this case, but for continuous-time linear periodically time-varying (LPTV) systems [10].

Let us consider h-periodic systems. Then, the periodicity obviously implies that the relation between the past input and the future output depends generally on the time instant separating past and future. This situation is exactly the same also in the relevant study on the Hankel operator of sampled-data systems [11]–[13]. However, the pioneering study on such a topic [14] merely considered the case where past and future is separated only at a sampling instant, and the subsequent studies in [15]–[17] have introduced a general instant $\Theta \in [0, h)$ to separate past and future. Following this key idea, the present paper also takes $\Theta \in [0, h)$ as the time instant separating past and future, and the associated operator mapping the past input belonging to the L_2 space to the future output viewed as an element in L_∞ is called the quasi L_∞/L_2 Hankel operator at Θ. Its norm is called the quasi L_∞/L_2 Hankel norm at Θ, and the supremum of the norms over $\Theta \in [0, h)$ is called *the* L_∞/L_2 Hankel norm.

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The present paper is interested in the quasi L_∞/L_2 Hankel norm at $\Theta \in [0, h)$ together with *the* L_∞/L_2 Hankel norm, and first aims at characterizing these norms in such a way that their numerical computations are readily feasible. It turns out that naively computing the L_∞/L_2 Hankel norm through the computation of all the quasi L_∞/L_2 Hankel norms over $\Theta \in [0, h)$ leads to the treatment of a double-supremum of a two-variate function and thus is rather demanding. This is because the quasi L_∞/L_2 Hankel norm at each $\Theta \in [0, h)$ is actually characterized as the supremum of a (Θ-dependent) function in $\theta \in [0, h)$. This paper shows that such a double-supremum can actually be avoided. This implies that if only the computation of the L_∞/L_2 Hankel norm is the target, then it can be computed directly without referring to any quasi L_∞/L_2 Hankel norm, despite the reference to it in the definition of the L_∞/L_2 Hankel norm.

This sophisticated computation method of the L_∞/L_2 Hankel norm, by the way, further gives rise to another important question studied in this paper. To describe it, we first introduce the notion of a critical instant in the L_∞/L_2 Hankel norm analysis of LPTV systems, as in the relevant studies for sampled-data systems [15]–[17]. That is, if the supremum of the quasi L_∞/L_2 Hankel norms over $\Theta \in [0, h)$ is actually attained as the maximum, then Θ is called a critical instant whenever the quasi L_∞/L_2 Hankel norm at Θ attains the maximum. By this definition of a critical instant, it may be reasonably considered to be hard to avoid the treatment of the quasi L_∞/L_2 Hankel norms to determine a critical instant. However, this is not always the case, and this paper aims at studying when and how the existence (or absence) of critical instants and their values could be determined without referring to the quasi L_∞/L_2 Hankel norm at all.

We use the following notation in this paper. The sets of positive integers and nonnegative integers are denoted respectively by $\mathbb{N}$ and $\mathbb{N}_0$. The set of n-dimensional real vectors is denoted by $\mathbb{R}^n$ while the set of $m \times n$ real matrices is denoted by $\mathbb{R}^{m\times n}$. For $v \in \mathbb{R}^n$, their 2-norm and ∞-norm are denoted respectively by $|v|_2$ $(:= (v^T v)^{1/2})$ and $|v|_\infty$ $(:= \max_{i=1,\ldots,n} |v_i|)$. For a real symmetric matrix, its maximum diagonal entry and maximum eigenvalue are denoted respectively by $d_{\max}(\cdot) =: \mu_\infty(\cdot)$ and $\lambda_{\max}(\cdot) =: \mu_2(\cdot)$. For a real matrix X, the symbol $\mathrm{sq}(\mathrm{X})$ denotes XX^T. We say that $w \in L_2(-\infty, \Theta)$ to mean that w is defined on $(-\infty, \infty)$ but $w(t) = 0$ for $t \geq \Theta$ while its $L_2(-\infty, \Theta)$

norm defined by

$$\|w(\cdot)\|_{2-}^{[\Theta]} := \left(\int_{-\infty}^{\Theta} w^T(t) w(t) dt \right)^{1/2} \tag{1}$$

is finite. For a vector-valued function z defined on $[\Theta, \infty)$, its $L_{\infty,p}[\Theta, \infty)$ norm is defined by

$$\|z(\cdot)\|_{\infty,p}^{[\Theta]} := \operatorname*{ess\,sup}_{t \in [\Theta,\infty)} |z(t)|_p \qquad (p = 2, \infty) \tag{2}$$

and the set of z for which the right hand side is finite is denoted by $L_{\infty,p}[\Theta, \infty)$. We sometimes use the notation $L_\infty[\Theta, \infty)$ for simplicity when the underlying p is obvious from the context or when both $p = 2$ and $p = \infty$ are meant. For $h > 0$ and $t \geq 0$, we mean by $t \bmod h$ the smallest nonnegative τ such that $t - \tau$ is an integer multiple of h, and $(t_1 + t_2) \bmod h$ is meant by $t_1 + t_2 \bmod h$.

II. Linear Periodically Time-Varying Systems, Their quasi L_∞/L_2 Hankel Norms and the L_∞/L_2 Hankel Norm

This paper deals with the stable h-periodic linear periodically time-varying (LPTV) system $\mathcal{P}$ described by

$$\frac{dx(t)}{dt} = A(t)x(t) + B(t)w(t) \tag{3}$$

$$z(t) = C(t)x(t) \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $w(t) \in \mathbb{R}^{n_w}$ and $z(t) \in \mathbb{R}^{n_z}$. Even though a much weaker assumption could also be adopted [18], we assume for simplicity that $A(t) = A(t + h)$ is piecewise continuous and right continuous at every $t \in \mathbb{R}$, and similarly for $B(t)$ and $C(t)$. The state transition matrix of $\mathcal{P}$ with respect to the interval $[t_1, t_2]$ is denoted by $\Phi(t_2; t_1)$.

This paper is concerned with the (quasi) L_∞/L_2 Hankel norm analysis of $\mathcal{P}$, where we take an arbitrary $\Theta \in [0, h)$ as the time instant separating past and future. More precisely, we first consider the operator, denoted by $\mathbf{H}_p^{[\Theta]}$ and called the quasi $L_{\infty,p}/L_2$ Hankel operator at Θ, representing the input-output relation of $\mathcal{P}$ with respect to the past input $w \in L_2(-\infty, \Theta)$ and the future output $z \in L_{\infty,p}[\Theta, \infty)$. Its norm defined by

$$\|\mathbf{H}_p^{[\Theta]}\|_{\infty,p/2-} := \sup_{w \in L_2(-\infty,\Theta)} \frac{\|z\|_{\infty,p}^{[\Theta]}}{\|w\|_{2-}^{[\Theta]}} \qquad (p = 2, \infty) \tag{5}$$

(simply denoted by $\|\mathbf{H}_p^{[\Theta]}\|$ in the following) is called the quasi $L_{\infty,p}/L_2$ Hankel norm at Θ. Furthermore, *the* $L_{\infty,p}/L_2$ Hankel norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|_{\mathrm{H},p}$, is defined by

$$\|\mathcal{P}\|_{\mathrm{H},p} := \sup_{\Theta \in [0,h)} \|\mathbf{H}_p^{[\Theta]}\| \qquad (p = 2, \infty) \tag{6}$$

If the right-hand side is attained as the maximum over $\Theta \in [0, h)$, each maximum-attaining Θ is called a critical instant. We sometimes use the notation $\Theta^\star$ to mean that $\Theta = \Theta^\star$ is a critical instant. The terms quasi L_∞/L_2 Hankel norm/operator and L_∞/L_2 Hankel norm/operator are sometimes used when the underlying p is obvious from the context or when both $p = 2$ and $p = \infty$ are meant.

III. Characterizations of the Quasi L_∞/L_2 Hankel Norms and the L_∞/L_2 Hankel Norm

This section is devoted to characterizing the quasi L_∞/L_2 Hankel norm for each Θ and *the* L_∞/L_2 Hankel norm in such a way that their numerical computation is readily feasible.

A. Characterization of the quasi L_∞/L_2 Hankel norm

We begin by characterizing the quasi L_∞/L_2 Hankel norm at $\Theta \in [0, h)$. Since z is right continuous at every $t \in \mathbb{R}$ by the assumption on $C(\cdot)$, it follows from (5) that

$$\|\mathbf{H}_p^{[\Theta]}\| = \sup_{\|w\|_{2-}^{[\Theta]} \leq 1} \sup_{\theta \in [\Theta,\infty)} |z(\theta)|_p \tag{7}$$

Regarding the interval $[\Theta, \infty)$ on the right-hand side for θ, it is not hard to see that it may be replaced by $[\Theta, \Theta + h)$. Hence, we have

$$\begin{aligned} \|\mathbf{H}_p^{[\Theta]}\| &= \sup_{\|w\|_{2-}^{[\Theta]} \leq 1} \sup_{\theta \in [\Theta,\Theta+h)} |z(\theta)|_p \\ &= \sup_{\theta \in [0,h)} \sup_{\|w\|_{2-}^{[\Theta]} \leq 1} |z(\Theta + \theta)|_p \end{aligned} \tag{8}$$

Assuming that $x(-\infty) = 0$ and noting that $w(t) = 0$ for $t \geq \Theta$, it follows from (3) and (4) that

$$\begin{aligned} z(\Theta + \theta) &= \int_{-\infty}^{\Theta} C(\Theta + \theta) \Phi(\Theta + \theta; \tau) B(\tau) w(\tau) d\tau \\ &=: (\mathcal{F}_\Theta w)(\theta) \end{aligned} \tag{9}$$

We proceed with the following arguments separately for $p = \infty$ and $p = 2$. For $p = \infty$, we denote the ith row of $C(\cdot)$ by $C_i(\cdot)$ and the ith entry of $\mathcal{F}_\Theta w$ by $(\mathcal{F}_\Theta w)_i$. The Cauchy-Schwarz inequality applied to

$$(\mathcal{F}_\Theta w)_i(\theta) = \int_{-\infty}^{\Theta} C_i(\Theta + \theta) \Phi(\Theta + \theta; \tau) B(\tau) w(\tau) d\tau \tag{10}$$

leads to

$$\begin{aligned} &|(\mathcal{F}_\Theta w)_i(\theta)| \\ &\leq \left(\int_{-\infty}^{\Theta} \mathrm{sq}(C_i(\Theta + \theta) \Phi(\Theta + \theta; \tau) B(\tau)) d\tau \right)^{1/2} \\ &\qquad \times \left(\int_{-\infty}^{\Theta} w^T(\tau) w(\tau) d\tau \right)^{1/2} \end{aligned} \tag{11}$$

where the equality holds if and only if

$$w(\tau) \equiv \lambda B^T(\tau) \Phi^T(\Theta + \theta; \tau) C_i^T(\Theta + \theta) \tag{12}$$

Since the constant λ can be taken so that $\|w\|_{2-}^{[\Theta]} \leq 1$, we readily see that

$$\begin{aligned} &\sup_{\|w\|_{2-}^{[\Theta]} \leq 1} |(\mathcal{F}_\Theta w)(\theta)|_\infty \\ &= \max_i \left(\int_{-\infty}^{\Theta} \mathrm{sq}(C_i(\Theta + \theta) \Phi(\Theta + \theta; \tau) B(\tau)) d\tau \right)^{1/2} \\ &= d_{\max}^{1/2}(F_\Theta(\theta)) \end{aligned} \tag{13}$$

where

$$F_\Theta(\theta) := \int_{-\infty}^{\Theta} \mathrm{sq}(C(\Theta+\theta)\Phi(\Theta+\theta;\tau)B(\tau))d\tau \quad (14)$$

Combining (8), (9) and (13) leads to

$$\|\mathbf{H}_\infty^{[\Theta]}\| = \sup_{\theta\in[0,h)} d_{\max}^{1/2}(F_\Theta(\theta)) \quad (15)$$

Next, for $p=2$, we define the operator $\mathcal{F}_{\Theta\theta}$ by

$$\mathcal{F}_{\Theta\theta} w := \int_{-\infty}^{\Theta} C(\Theta+\theta)\Phi(\Theta+\theta;\tau)B(\tau)w(\tau)d\tau \quad (16)$$

so that

$$\sup_{\|w\|_{2-}^{[\Theta]}\le 1} |z(\Theta+\theta)|_2 = \sup_{\|w\|_{2-}^{[\Theta]}\le 1} |\mathcal{F}_{\Theta\theta} w|_2 \quad (17)$$

Since $\mathcal{F}_{\Theta\theta}$ is a linear bounded operator between Hilbert spaces, we have

$$\sup_{\|w\|_{2-}^{[\Theta]}\le 1} |\mathcal{F}_{\Theta\theta} w|_2 = \|\mathcal{F}_{\Theta\theta}\mathcal{F}_{\Theta\theta}^*\|_2^{1/2} = \lambda_{\max}^{1/2}(\mathcal{F}_{\Theta\theta}\mathcal{F}_{\Theta\theta}^*) \quad (18)$$

where $\mathcal{F}_{\Theta\theta}^*$ denotes the adjoint operator of $\mathcal{F}_{\Theta\theta}$ given by

$$(\mathcal{F}_{\Theta\theta}^* v)(\tau) = B^T(\tau)\Phi^T(\Theta+\theta;\tau)C^T(\Theta+\theta)v \quad (19)$$

and $\|\mathcal{F}_{\Theta\theta}\mathcal{F}_{\Theta\theta}^\star\|_2$ denotes the (induced) 2-norm of the matrix

$$\mathcal{F}_{\Theta\theta}\mathcal{F}_{\Theta\theta}^\star = \int_{-\infty}^{\Theta} \mathrm{sq}(C(\Theta+\theta)\Phi(\Theta+\theta;\tau)B(\tau))d\tau \quad (20)$$

which is nothing but $F_\Theta(\theta)$ given in (14). It follows from (17) and (18) that

$$\sup_{\|w\|_{2-}^{[\Theta]}\le 1} |z(\Theta+\theta)|_2 = \lambda_{\max}^{1/2}(F_\Theta(\theta)) \quad (21)$$

This together with (8) leads to

$$\|\mathbf{H}_2^{[\Theta]}\| = \sup_{\theta\in[0,h)} \lambda_{\max}^{1/2}(F_\Theta(\theta)) \quad (22)$$

These arguments can be summarized as follows.

Theorem 1: The quasi L_∞/L_2 Hankel norm of the stable LPTV system $\mathcal{P}$ is given by

$$\|\mathbf{H}_p^{[\Theta]}\| = \sup_{\theta\in[0,h)} \mu_p^{1/2}(F_\Theta(\theta)) \qquad (p=2,\infty) \quad (23)$$

where $F_\Theta(\theta)$ is defined by (14), and μ_p denotes $\lambda_{\max}$ for $p=2$ and $d_{\max}$ for $p=\infty$.

B. Characterization of the L_∞/L_2 Hankel norm

This subsection is devoted to the treatment of *the* L_∞/L_2 Hankel norm. Since we have characterized in the preceding subsection the quasi L_∞/L_2 Hankel norm at each $\Theta \in [0,h)$, the definition (6) immediately characterizes the L_∞/L_2 Hankel norm. However, such a characterization leads to a double-supremum in θ as well as Θ, and thus does not provide a clear perspective. We show that such a double-supremum can be avoided, and only a single-supremum actually suffices to characterize the L_∞/L_2 Hankel norm of $\mathcal{P}$.

First, recall that

$$\mu_p^{1/2}(F_\Theta(\theta)) = \sup_{w\in L_2(-\infty,\Theta)} \frac{|z_+(\Theta+\theta)|_p}{\|w\|_{2-}^{[\Theta]}} \qquad (p=2,\infty) \quad (24)$$

by (13) and (21). First replacing θ with 0 and then substituting $\Theta+\theta \bmod h$ into Θ leads to

$$\begin{aligned}\mu_p^{1/2}&(F_{\Theta+\theta \bmod h}(0)) \\ &= \sup_{w\in L_2(-\infty,\Theta+\theta \bmod h)} \frac{|z_+(\Theta+\theta \bmod h)|_p}{\|w\|_{2-}^{[\Theta+\theta \bmod h]}} \\ &= \sup_{w\in L_2(-\infty,\Theta+\theta)} \frac{|z_+(\Theta+\theta)|_p}{\|w\|_{2-}^{[\Theta+\theta]}}\end{aligned} \quad (25)$$

where the last equality follows from the h-periodicity of $\mathcal{P}$. Since $(-\infty,\Theta) \subset (\infty,\Theta+\theta)$ for $\Theta\in[0,h)$ and $\theta\in[0,h)$, we see by the comparison of (24) and (25) that

$$\mu_p^{1/2}(F_\Theta(\theta)) \le \mu_p^{1/2}(F_{\Theta+\theta \bmod h}(0)) \quad (26)$$

Taking the supremums over $\Theta\in[0,h)$ and $\theta\in[0,h)$ leads to

$$\begin{aligned}&\sup_{\Theta\in[0,h)}\sup_{\theta\in[0,h)} \mu_p^{1/2}(F_\Theta(\theta)) \\ &\quad\le \sup_{\Theta\in[0,h)}\sup_{\theta\in[0,h)} \mu_p^{1/2}(F_{\Theta+\theta \bmod h}(0)) \\ &\quad= \sup_{\theta\in[0,h)} \mu_p^{1/2}(F_\theta(0))\end{aligned} \quad (27)$$

where the last equality is an immediate consequence of the fact that $\Theta+\theta \bmod h$ ranges over $[0,h)$.

On the other hand, it is obvious that

$$\sup_{\Theta\in[0,h)}\sup_{\theta\in[0,h)} \mu_p^{1/2}(F_\Theta(\theta)) \ge \sup_{\Theta\in[0,h)} \mu_p^{1/2}(F_\Theta(0)) \quad (28)$$

It then immediately follows from (27) and (28) that

$$\sup_{\Theta\in[0,h)}\sup_{\theta\in[0,h)} \mu_p^{1/2}(F_\Theta(\theta)) = \sup_{\theta\in[0,h)} \mu_p^{1/2}(F_\theta(0)) \quad (29)$$

Combining this relation, (6) and (23) and defining

$$F(\theta) := F_\theta(0) \quad (30)$$

leads to the following result, as claimed, in terms of a single-supremum.

Theorem 2: The L_∞/L_2 Hankel norm of the stable LPTV system $\mathcal{P}$ is given by

$$\|\mathcal{P}\|_{\mathrm{H},p} = \sup_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta)) \qquad (p=2,\infty) \quad (31)$$

where

$$F(\theta) := \int_{-\infty}^{\theta} \mathrm{sq}(C(\theta)\Phi(\theta;\tau)B(\tau))d\tau \quad (32)$$

IV. Characterization of a Critical Instant

This section is concerned with characterizing a critical instant, if one exists for $\mathcal{P}$ (or more strongly, all the critical instants, or alternatively, determining the absence of a critical instant). By definition, a critical instant is a maximum attaining point at which the quasi L_∞/L_2 Hankel norm takes the maximum over $\Theta \in [0, h)$. A direct and naive interpretation of this definition might suggest that characterizing a critical instant requires us to compute all the quasi L_∞/L_2 Hankel norms at $\Theta \in [0, h)$. Regarding *the* L_∞/L_2 Hankel norm, however, which is by definition the supremum over $\Theta \in [0, h)$ of all the quasi L_∞/L_2 Hankel norms, we have given its simplified characterization in Theorem 2. What is interesting about this theorem is that it involves no Θ. That is, it does not refer to anything that is directly related to the treatment of the quasi L_∞/L_2 Hankel norms at $\Theta \in [0, h)$. The topic to be studied in this section is motivated by this aspect, and we study whether similar results could be derived about the characterization of a critical instant. More precisely, we are interested in the question whether it suffices, in some situations, to deal only with $F(\theta)$ defined in (32) (without involving Θ) rather than $F_\Theta(\theta)$ defined in (14) to characterize a critical instant $\Theta = \Theta^\star$ (if one exists), or more strongly, all the critical instants. Another relevant question is related to the case where no critical instant actually exists, in which case we are interested in whether dealing only with $F(\theta)$ could somehow conclude the absence of a critical instant Θ.

A. A basic result

In the following arguments, we say that $\mu_p^{1/2}(F(\theta))$ is maximum-attaining if $\sup_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$ is attained as the maximum over $\theta \in [0, h)$, in which case we say that $\theta = \theta^\star$ is a maximum-attaining point of $\mu_p^{1/2}(F(\theta))$ if $\mu_p^{1/2}(F(\theta^\star)) = \max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$.

We first give the following basic result, by which we are readily led to a positive answer to one of the questions raised above; when $\mu_p^{1/2}(F(\theta))$ is maximum-attaining, one can conclude that a critical instant exists (and for each maximum-attaining point $\theta = \theta^\star$, the instant $\Theta = \theta^\star$ is indeed a critical instant).

Theorem 3: Let p be $p = 2$ or $p = \infty$. If

$$\sup_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta)) = \mu_p^{1/2}(F(\theta^\star)) \tag{33}$$

then $\Theta = \theta^\star$ is a critical instant (i.e., $\|\mathcal{P}\|_{\mathrm{H},p} = \|\mathbf{H}_p^{[\theta^\star]}\|$).

To study further those aspects that are not yet answered by the above theorem, the following arguments are divided into two cases, where we first consider the simplest case with continuous $C(\cdot)$ and then consider the case with discontinuous $C(\cdot)$.

B. The case where $C(\cdot)$ is continuous

Regarding the assumption in Theorem 3 that $\mu_p^{1/2}(F(\theta))$ is maximum-attaining, we actually have the following result in this case.

Proposition 1: If $C(\cdot)$ is continuous, then $F(\theta)$ is continuous on $[0, h)$, and $\lim_{\theta\to h-0} F(\theta)$ exists and coincides with $F(0)$.

The following result is an immediate consequence of Theorem 3 and Proposition 1.

Corollary 1: If $C(\cdot)$ is continuous, then $\mu_p^{1/2}(F(\theta))$ is maximum-attaining, and for each maximum attaining point $\theta = \theta^\star$, the instant $\Theta = \theta^\star$ is a critical instant. In particular, a critical instant always exists if $C(\cdot)$ is continuous.

The above arguments clarify a situation where the existence of a critical instant can be ensured (and the value of a critical instant can be determined) only through the treatment of $F(\theta)$. However, the arguments do not necessarily clarify whether all the critical instants can be determined through the treatment of $F(\theta)$ in such a situation. Furthermore, it is not still clear whether one can conclude that no critical instant exists if $\mu_p^{1/2}(F(\theta))$ is not maximum-attaining (which can be the case when $C(\cdot)$ is not continuous). The following arguments aim at giving some further results relevant to these issues. Before proceeding to such a direction, we first consider classifying a critical instant into a few different categories in the following subsection.

C. Classification of critical instants

We next consider the general case where the intersection of the set of the discontinuities of $C(\cdot)$ and the interval $[0, h)$ consists of $t_1, \ldots, t_K$. For notational simplicity, we define $\mathbb{T} := \{t_1, \ldots, t_K\}$ as well as $\mathbb{K} := \{1, \ldots, K\}$ and

$$t_{k\Theta} := \begin{cases} t_k & \text{if } t_k \in (\Theta, h) \quad (34a) \\ h + t_k & \text{if } t_k \in [0, \Theta] \quad (34b) \end{cases}$$

for the underlying $\Theta \in [0, h)$. Note that $t_{k\Theta} \in (\Theta, \Theta + h]$ and $t_{k\Theta} = t_k \bmod h$ for $k = 1, \ldots, K$.

The arguments in the following subsection aim at studying whether some issues relevant to critical instants could eventually be discussed only through the treatment of $F(\theta)$. As a key idea as in [16] dealing with a relevant problem of (quasi) L_∞/L_2 Hankel norm of sampled-data systems, it is quite helpful to recall (23) and introduce for each $\Theta \in [0, h)$ the following 'quantity' (where $t_{k\Theta} - \Theta - 0$ is regarded as being different from $t_{k\Theta} - \Theta$, whose precise meaning will become clear in the following arguments):

$$\theta_p(\Theta) := \begin{cases} \arg\max\limits_{\theta\in[0,h)} \mu_p^{1/2}(F_\Theta(\theta)) \\ \quad (\text{if } \|\mathbf{H}_p^{[\Theta]}\| = \max_{\theta\in[0,h)} \mu_p^{1/2}(F_\Theta(\theta))) & (35a) \\ t_{k\Theta} - \Theta - 0 \\ \quad (\text{otherwise; with } k \in \mathbb{K} \text{ such that} \\ \qquad \|\mathbf{H}_p^{[\Theta]}\| = \lim\limits_{\theta\to t_{k\Theta}-\Theta-0} \mu_p^{1/2}(F_\Theta(\theta))) & (35b) \end{cases}$$

For simplicity, $\theta_p(\Theta)$ is often denoted by $\theta(\Theta)$ in the following especially when the underlying p is clear or does not matter crucially. Furthermore, when Θ is a critical instant, we sometimes use the notation $\theta^\star(\Theta)$ to stress that we are talking about $\theta(\Theta)$ *for a critical* Θ.

Remark 1: We take the standpoint that $\theta_p(\Theta)$ is determined uniquely for each $\Theta \in [0, h)$ by the following rule:

1) If $\mu_p^{1/2}(F_\Theta(\theta))$ is maximum-attaining with respect to θ, the $\arg\max$ in (35a) is regarded to imply the smallest value among all the possibilities.
2) Otherwise, the smallest $t_{k\Theta}$ is taken in applying (35b).

We are now in a position to classify $\theta^\star(\Theta)$ for a critical instant Θ as follows.

(I) $\theta^\star(\Theta) = 0$
(II) $0 < \theta^\star(\Theta) < h - \Theta$
(III) $h - \Theta \leq \theta^\star(\Theta) < h$
(IV) $\theta^\star(\Theta) = t_{k\Theta} - \Theta - 0$ for some $k \in \mathbb{K}$

In the arguments in the following subsection, we deal with each case of this classification separately and study the condition for the existence of a critical instant $\Theta \in [0, h)$ whose associated $\theta^\star(\Theta)$ is classified as each of the four cases.

D. The case with general $C(\cdot)$

The arguments in this subsections are organized as follows. We first consider each of the four cases with a critical instant classified as (I)–(IV) and derive some results on critical instants relevant to each case. Existence of a critical instant classified as either of (II), (III) and (IV) might be considered a somewhat queer situation, and we are interested in a condition that can negate the possibility. To proceed to such a direction, we then study a condition that ensures the state transition matrix of $\mathcal{P}$ to be analytic. Finally, we combine these results to derive some integrated results about when and how the existence (or absence) of critical instants and their values could be determined only through the treatment of a univariate function $F(\theta)$.

1) Necessary and sufficient condition for the existence of a critical instant Θ classified as case (I):

If there exists a critical instant $\Theta \in [0, h)$ classified as (I), it follows from the definition of $\theta^\star(\Theta)$ in (35) that $\|\mathcal{P}\|_{\mathrm{H},p} = \|\mathbf{H}_p^{[\Theta]}\| = \mu_p^{1/2}(F_\Theta(0))$, and thus $\|\mathcal{P}\|_{\mathrm{H},p} = \mu_p^{1/2}(F(\Theta))$ by (30). This together with (31) implies that $\mu_p^{1/2}(F(\theta))$ is maximum-attaining and $\Theta = \arg\max_{\theta \in [0,h)} \mu_p^{1/2}(F(\theta))$.

On the other hand, if $\mu_p^{1/2}(F(\theta))$ is maximum-attaining, then for each of its maximum-attaining point $\theta \in [0, h)$, which we denote by Θ in this paragraph, it follows from (31) that $\|\mathcal{P}\|_{\mathrm{H},p} = \mu_p^{1/2}(F(\Theta))$ and thus $\|\mathcal{P}\|_{\mathrm{H},p} = \mu_p^{1/2}(F_\Theta(0))$ by (30). This together with (6) and (23) immediately implies that Θ is a critical instant. Furthermore, by the definition of $\theta^\star(\Theta)$ in (35a), we can readily see that $\theta^\star(\Theta) = 0$, i.e., the critical instant Θ is classified as (I).

Combining the above arguments, we are now in a position to state the following result; it is valid even for the case where $K = 0$ (i.e, $C(\cdot)$ is continuous), and provides in that case some additional information over the one given by Theorem 3 with respect to the classification of a critical instant.

Theorem 4: (i) There exists a critical instant Θ classified as (I) if and only if $\mu_p^{1/2}(F(\theta))$ is maximum-attaining. (ii) If $\mu_p^{1/2}(F(\theta))$ is maximum-attaining, then for each of its maximum-attaining points $\theta = \theta^\star$, the instant $\Theta = \theta^\star$ is a critical instant classified as (I). (iii) Conversely, if there exists a critical instant $\Theta^\star \in [0, h)$ classified as (I), then $\mu_p^{1/2}(F(\theta))$ is maximum-attaining and $\theta = \Theta^\star$ is a maximum-attaining point.

2) Necessary condition for the existence of a critical instant Θ classified as case (II) or (III):

Next, let us consider the case where a critical instant $\Theta \in [0, h)$ exists and is classified as either (II) or (III), i.e., $0 < \theta^\star(\Theta) < h$.

Let $\Theta_1 := \Theta$ and $\theta_1^\star := \theta^\star(\Theta_1)$, and consider $F(\Theta^\bullet) = F_{\Theta^\bullet}(0)$ where $\Theta^\bullet := \Theta_1 + \theta_1^\star \bmod h\ (\in [0, h))$. Here, $F(\theta)$ given in (32) is h-periodic and thus $F_{\Theta^\bullet}(0) = F_{\Theta_1 + \theta_1^\star}(0)$ by (30). Hence, we have

$$\begin{aligned} &F_{\Theta^\bullet}(0) - F_{\Theta_1}(\theta_1^\star) = F_{\Theta_1+\theta_1^\star}(0) - F_{\Theta_1}(\theta_1^\star) \\ &= \int_{\Theta_1}^{\Theta_1+\theta_1^\star} \mathrm{sq}(C(\Theta_1+\theta_1^\star)\Phi(\Theta_1+\theta_1^\star;\tau)B(\tau))d\tau \end{aligned} \quad (36)$$

If $p = \infty$, take i such that the ith diagonal entry is the maximum among the diagonal entries of $F_{\Theta_1}(\theta_1^\star)$, and let e_i be the ith column of the identity matrix in $\mathbb{R}^{n_z}$. If $p = 2$, take $v \in \mathbb{R}^{n_z}$ such that $|v|_2 = 1$ and $\lambda_{\max}(F_{\Theta_1}(\theta_1^\star)) = v^T F_{\Theta_1}(\theta_1^\star) v$. Then, v_p is defined as follows.

$$v_p := \begin{cases} e_i & (p = \infty) \quad (37a) \\ v & (p = 2) \quad (37b) \end{cases}$$

Then, we can derive the following result, where the first assertions is non-surprising, while the second assertion is the nontrivial and crucial one for the subsequent more significant arguments at the end of this subsection.

Proposition 2: Suppose that $\Theta_1 \in [0, h)$ is a critical instant classified as (II) or (III) (and thus $\theta_1^\star = \theta^\star(\Theta_1) \in (0, h)$). Then, $\Theta^\bullet := \Theta_1 + \theta_1^\star \bmod h\ (\in [0, h))$ is also a critical instant and classified as (I). Furthermore, we have

$$\int_{\Theta_1}^{\Theta_1+\theta_1^\star} \mathrm{sq}(v_p^T C(\Theta_1+\theta_1^\star)\Phi(\Theta_1+\theta_1^\star;\tau)B(\tau))d\tau = 0 \quad (38)$$

3) Necessary condition for the existence of a critical instant Θ classified as case (IV):

In addition, we are led to the following result, where v_p is given by (37) with i and v determined as follows (under a given $k \in \mathbb{K}$). If $p = \infty$, take i such that the ith diagonal entry is the maximum among the diagonal entries of $\lim_{\theta \to t_{k\Theta_1} - \Theta_1 - 0} F_{\Theta_1}(\theta)\ (=: F^-)$. If $p = 2$, take $v \in \mathbb{R}^{n_z}$ such that $|v|_2 = 1$ and $\lambda_{\max}(F^-) = v^T F^- v$.

Proposition 3: Suppose that $\Theta_1 \in [0, h)$ is a critical instant classified as (IV) (and thus $\theta^\star(\Theta_1) = t_{k\Theta_1} - \Theta_1 - 0$ for some $k \in \mathbb{K}$). Then, we have

$$\int_{\Theta_1}^{t_{k\Theta_1}} \mathrm{sq}(v_p^T C^-(t_{k\Theta_1})\Phi(t_{k\Theta_1};\tau)B(\tau))d\tau = 0 \quad (39)$$

where $C^{-}(t_{k\Theta_1}) := \lim_{t\to t_{k\Theta_1}-0} C(t)$.

The above result is also a nontrivial and crucial one for the subsequent more integrated arguments at the end of this subsection.

By the way, our numerical studies with several examples of $\mathcal{P}$ have confirmed the existence of the critical instants classified as (IV) (as well as (I), (II) and (III)), in general. However, we can show the following very specific result, which gives a sufficient condition for the absence of a critical instant classified as (IV).

Theorem 5: If $C(\cdot)$ is scalar-valued and upper semicontinuous at each discontinuity $t_k \in \mathbb{T}$ $(k \in \mathbb{K})$, then no critical instant classified as (IV) exists.

4) Sufficient condition for analyticity of the transition matrix:

The following result, which is shown through arguments similar to those developed in [19], gives a sufficient condition for the state transition matrix $\Phi(\theta;\tau)$ of $\mathcal{P}$ to be analytic in τ over $\mathbb{R}$ regardless of θ. It plays an important role to apply the preceding results, relevant to the classification of a critical instant into four types, to the derivation of some more sophisticated result about when and how dealing only with $F(\theta)$ could determine existence/absence of a critical instant and all the values of critical instants.

Lemma 1: If $A(t)$ is (entry-wise) analytic in t over $\mathbb{R}$, then $\Phi(\theta;\tau)$ also is in τ over $\mathbb{R}$ regardless of θ.

5) Insight into critical instants and when and how all the critical instants can be detected only through the treatment of $F(\theta)$:

We are now in a position to combine the preceding results to have a deeper insight into the critical instants of $\mathcal{P}$ as well as when and how their existence/absence and values could be determined only through the treatment of $F(\theta)$.

First, Proposition 2 immediately implies the following (non-surprising) result.

Proposition 4: If there exists a critical instant classified as (II) or (III), then there always exists a critical instant classified as (I).

This result together with Theorem 4 further leads to the following result.

Proposition 5: There exists a critical instant classified as either of (I), (II) and (III) if and only if $\mu_p^{1/2}(F(\theta))$ is maximum-attaining.

Regarding the issue of whether all the critical instants could be detected only through the treatment of $F(\theta)$, we have the following result from Propositions 2 and 3, Lemma 1 and Theorem 4.

Theorem 6: Suppose that $C(t)$ is continuous at every $t \in [0,h)\setminus\mathbb{T}$ and $A(t)$ and $B(t)$ are analytic in t over $\mathbb{R}$. Then, (i) each critical instant Θ is classified as (I) and satisfies $\Theta = \arg\max_{\theta\in[0,h)} \mu_p^{1/2}(F(\theta))$; (ii) conversely, if $\mu_p^{1/2}(F(\theta))$ is maximum-attaining, then for each maximum-attaining point $\theta=\theta^\star$ of $\mu_p^{1/2}(F(\theta))$, the instant $\Theta=\theta^\star$ is a critical instant classified as (I).

In particular, the assertion (i) of Theorem 6 immediately leads to the following result.

Corollary 2: Under the assumptions of Theorem 6, there exists no critical instant Θ classified as (II), (III) or (IV). Furthermore, if $\mu_p^{1/2}(F(\theta))$ is not maximum-attaining, then no critical instant exists.

Theorem 6 and Corollary 2 obviously imply that, under the assumptions of Theorem 6, the set of all the critical instants coincides with the set of all the maximum-attaining points of $\mu_p^{1/2}(F(\theta))$, which is true even when either of the two sets is empty.

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