2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Structural Control of Drift-free Bilinear Systems under Link Failures and Sparsity Constraints

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Abstract— **We study structural controllability properties of drift-free bilinear systems under link failures using a graph theoretic approach. We give an equivalent condition for structural controllability in the presence of link failures, which can be checked in polynomial time. We show that the problem of finding a sparsest structure for structural controllability under link failures is equivalent to a known NP-hard problem. We also consider the case of probabilistic link failures. The problem of deciding whether the underlying system is structurally controllable with a certain probability is equivalent to an NP-hard network reliability problem. Finally, we observe that the problem of designing a structurally controllable bilinear system from an uncontrollable one under nonuniform cost constraints is equivalent to an NP-complete strong connectivity augmentation problem.**

I. Introduction

With the continuous advancement in embedded systems, such as the Internet of Things and networked systems, there has been a notable surge in the literature on networked control and complex systems over the past two decades (see e.g. [1] and the references therein). In numerous instances, the exact numerical values within these large-scale systems are not precisely known; rather, only the zero/nonzero structure is discerned based on the interconnection pattern. Additionally, even when numerical values are known up to a certain precision, the computational cost remains high in large-scale complex systems. Consequently, the focus shifts towards exploring the structural properties of the underlying system. Moreover, there is a growing interest in investigating properties of systems that are independent of numerical values and hold for nearly any numerical realization [2], [3]. Structural controllability emerges as one such property of interest, offering insights into the controllability of systems based on their inherent structure rather than relying on precise numerical details.

Structural controllability, initially introduced for Linear Time-Invariant (LTI) systems in [4], has garnered significant attention in recent years within the realm of complex systems and networked control, as evidenced by studies such as [1], [5]–[7]. The progress made in this field is comprehensively surveyed in [2], [3], providing a valuable overview of the advancements in structural controllability. In the analysis of structural controllability for both linear and bilinear systems, graph theory plays a prominent role [2], [3], [5]. Recent works, such as [8], [9], have successfully provided graph-theoretic characterizations for the controllability of bilinear systems on Lie groups. Applications of bilinear systems span diverse fields, including quantum control [10], biology [11], socioeconomics [12], and so on. Bilinear models are also used as approximations to nonlinear models [12]. The early work on nonlinear control of bilinear systems and Lie groups is extensively covered in [13]–[15].

Article [5] considers drift-free bilinear systems on $\mathbb{R}^n$ of the form

$$\dot{\mathbf{x}}(t) = \left(\sum_{i=1}^{m} u_i(t) B_i\right)\mathbf{x}(t), \qquad (1)$$

where $B_1, \ldots, B_m$ are $(n \times n)$ structured matrices which belong to the Lie algebra $\mathcal{GL}(n)$, $\mathcal{SL}(n)$, $\widetilde{\mathcal{SO}}(n)$ of Lie groups $\mathbb{GL}(n)$, $\mathbb{SL}(n)$, $\widetilde{\mathbb{SO}}(n)$, respectively. The necessary and sufficient conditions for structural controllability were obtained using graph-theoretic methods. The main advantage of the graph-theoretic characterization is that it allows one to apply well-known graph-theoretic algorithms [16], [17] to solve control and optimization problems. Graph theory provides a versatile framework for modeling and analyzing dynamic systems independent of specific numerical values. This independence circumvents the challenges associated with numerical errors, offering a more robust and reliable approach for analyzing control and optimization problems.

In networked systems, link failures are quite common [18], [19], and handling them is a fundamental task for any communication scheme. Link failures can stem from various sources, including faults in interconnections or external attacks on cyber-physical systems, such as jamming or denial of service attacks [20] or packet dropouts [21]. Thus, analyzing a fundamental property like structural controllability of networked systems in the presence of link failures is crucial. Articles [22], [23] discuss the structural controllability of multiagent systems under communication link failures. Similarly, the structural controllability of power systems under transmission line failures is investigated in [24]. These studies contribute to the broader understanding of how structural properties can impact the controllability of systems experiencing link failures in various domains.

In this paper, building upon the results of [5], we give an equivalent condition for the structural controllability of bilinear systems (1) in the presence of link failures. Note that the nonzero entries of structured matrices can be considered as interconnection links in networked control systems. Subsequently, an equivalent condition for structural controllability in the presence of link failures gives a robust structural

The work of Chirayu D. Athalye was supported by BITS Pilani under the NFSG Grant.

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controllability condition. We also consider the problem of obtaining a sparsest control structure in the presence of link failures. In [25], a similar notion of resilience of structured matrices under k-edge erasure is studied and polynomial time solutions are provided using max-flow problems on bipartite graphs. However, the context is different from the one considered here. Besides, we consider the case of probabilistic link failures with the possibility of having a structurally controllable system with a given probability. Finally, we address the problem of designing a structurally controllable system from an uncontrollable one under the given cost constraints. While these problems, except for the first one, are computationally challenging [26], polynomial time approximation algorithms with approximation guarantees can be employed from the existing literature on graph theory.

Organization: This paper is organized as follows. The next section introduces essential preliminaries and outlines the problems under consideration. Section III presents our observations, delving into the computational complexity of the discussed problems. Finally, the paper is concluded in Section IV.

Notation: $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and positive integers, respectively. Abstract matrix Lie groups are denoted by $\mathbb{LG}$ and the corresponding Lie algebra by $\mathcal{LG}$. We use $\mathbb{GL}(n)$, $\mathbb{SL}(n)$, $\mathbb{SO}(n)$ to denote the general linear, the special linear, and the special orthogonal groups, respectively; whereas $\mathcal{GL}(n)$, $\mathcal{SL}(n)$, $\mathcal{SO}(n)$ are used to denote their respective Lie algebras. $\mathcal{G}(\mathcal{V}, \mathcal{E})$ denotes a graph with $\mathcal{V}$ as the vertex set and $\mathcal{E}$ as the edge set. The set of structured matrices with nonzero pattern Λ is denoted as $\mathcal{B}_\Lambda$. The entries of structured matrices taking arbitrary real values are denoted by $*$. The symbol $\oplus$ denotes the direct sum of vector spaces. The cardinality of a set Λ is denoted by $|\Lambda|$.

II. Preliminaries and Problem Formulation

A. Preliminaries and Background

We first define the matrix Lie groups $\mathbb{GL}(n), \mathbb{SL}(n)$, and $\widetilde{\mathbb{SO}}(n)$ as follows [5]

$$\begin{aligned}
&\mathbb{GL}(n) = \{A \in \mathbb{R}^{n\times n} \mid \det(A) \neq 0\},\\
&\mathbb{SL}(n) = \{A \in \mathbb{GL}(n) \mid \det(A) = 1\},\\
&\widetilde{\mathbb{SO}}(n) = \{A \in \mathbb{GL}(n) \mid A^\top A = \alpha I \text{ and } \det(A) > 0\}.
\end{aligned}$$

Lie groups also have the structure of a manifold [27]. The tangent space to the Lie group manifold $\mathbb{LG}$ at the identity element is called its Lie algebra $\mathcal{LG}$. The Lie algebra of a matrix Lie group is a vector space closed under the Lie bracket operation given by

$$[A_1, A_2] := A_1A_2 - A_2A_1,$$

where $A_1, A_2 \in \mathcal{LG}$. It turns out that the Lie algebras $\mathcal{GL}(n)$, $\mathcal{SL}(n)$, $\widetilde{\mathcal{SO}}(n)$ of the Lie groups $\mathbb{GL}(n)$, $\mathbb{SL}(n)$, $\widetilde{\mathbb{SO}}(n)$ are given by

$$\begin{aligned}
&\mathcal{GL}(n) = \{A \in \mathbb{R}^{n\times n}\},\\
&\mathcal{SL}(n) = \{A \in \mathbb{R}^{n\times n} \mid \text{trace}(A) = 0\},\\
&\widetilde{\mathcal{SO}}(n) = \{A \in \mathbb{R}^{n\times n} \mid A = -A^\top\} \oplus \text{span}\{I\}.
\end{aligned}$$

Note that for $\{A_1, \ldots, A_k\} \in \mathcal{LG}$, the vector space obtained by the repeated Lie bracket operation on the set $\{A_1, \ldots, A_k\}$ is called the Lie subalgebra generated by $\{A_1, \ldots, A_k\}$. For more details on Lie groups and Lie algebras, we refer the reader to [27].

Next, we briefly discuss the notion of controllability for bilinear systems. The bilinear control system (1) is said to be controllable if for a given pair $\mathbf{x}_0, \mathbf{x}_f \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, there exist piecewise continuous control inputs $u_1(\cdot), \ldots, u_m(\cdot)$ which drive the state from $\mathbf{x}(0) = \mathbf{x}_0$ to $\mathbf{x}(T) = \mathbf{x}_f$. The bilinear system (1) is controllable if and only if the Lie subalgebra formed by $B_1, \ldots, B_m$ is equal to the Lie algebra $\mathcal{LG}$ [5], [14].

We consider drift-free bilinear systems [15] of the form (1). We define the set of structured matrices on the matrix Lie algebra $\mathcal{LG}(n) \subseteq \mathbb{R}^{n\times n}$ as follows

$$\begin{aligned}
\mathcal{B}_\Lambda := \{B \in \mathcal{LG}(n) \mid B(i,j) = * \text{ if } (i,j) \in \Lambda,\\
B(i,j) = 0 \text{ if } (i,j) \notin \Lambda\}, \quad (2)
\end{aligned}$$

where $\Lambda \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\}$. Note that in (1), $B_1, \ldots, B_m \in \mathcal{B}_\Lambda$ and $\mathcal{LG}$ could be $\mathcal{GL}(n), \mathcal{SL}(n)$ or $\widetilde{\mathcal{SO}}(n)$. Furthermore, entries $*$ in $\mathcal{B}_\Lambda$ specified by Λ could be arbitrary real numbers (that satisfy the properties of $\mathcal{LG}$) and the remaining entries are set to zero. In other words, Λ determines the zero pattern of $\mathcal{B}_\Lambda$. Note that the set of matrices with the structure $\mathcal{B}_\Lambda$ forms a vector space. The structure Λ corresponds to the interaction/non-interaction pattern among the state variables in networked systems.

Example 1: Consider 3×3 pattern matrices belonging to $\mathcal{GL}(3)$ as follows. Let $\Lambda_1 := \{(1,2), (1,3), (2,3)\}$ and $\Lambda_2 := \{(1,1), (3,1), (2,3), (3,2)\}$. Then,

$$\mathcal{B}_{\Lambda_1} := \begin{bmatrix} 0 & * & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{B}_{\Lambda_2} := \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ * & * & 0 \end{bmatrix}.$$

We now define the notion of structural controllability [5].

Definition 1: The pattern $\mathcal{B}_\Lambda$ is said to be structurally controllable (or m-structurally controllable) if there exists $m \in \mathbb{N}$ and $B_1, \ldots, B_m \in \mathcal{B}_\Lambda$ such that (1) is controllable.

Next, we succinctly cover some basics from graph theory required to analyze structural controllability. A directed graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, also known as digraph, consists of a vertex set $\mathcal{V}$ and an edge set $\mathcal{E}$ whose elements are ordered pair of vertices. A directed edge from vertex i to j is denoted as $(i,j) \in \mathcal{E}$. A directed edge from a vertex to itself is called a self-loop. A directed path between two vertices i_1, i_k is the sequence of vertices $\{i_1, i_2, \ldots, i_k\}$ such that $(i_j, i_{j+1}) \in \mathcal{E}$ for $j = 1, \ldots, k-1$. A digraph is said to be strongly connected if there exists a directed path between any two vertices. A digraph $\mathcal{G}$ is said to be k-edge connected if $|\mathcal{V}| > k+1$ and deletion of an arbitrary subset of $\mathcal{E}$ with cardinality strictly less than k leaves a strongly connected graph [17], [28]. A digraph is said to be complete if every vertex is connected to every other vertex by a directed edge. For ease of reference, we give below definitions from [5] of a digraph associated with the pattern matrix $\mathcal{B}_\Lambda$ and the transitive closure of digraphs .

Definition 2: Given a pattern matrix $\mathcal{B}_\Lambda$, the digraph associated with it is denoted by $\mathcal{G}(\mathcal{B}_\Lambda)$ with the vertex set $\mathcal{V} = \{1, \ldots, n\}$ and the edge set

$$\mathcal{E} = \{(i, j) \in V \times V \mid \mathcal{B}_\Lambda(i, j) \neq 0\}.$$

Definition 3 (Transitive closure): Let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be a digraph with $|\mathcal{V}| = n$. Let $\mathcal{E}^{(r)} \subseteq \mathcal{V} \times \mathcal{V}$ be the ordered pair of vertices (i, j) such that there exists a path of length at most r between i and j $(r \geq 1)$. Then, the digraph $\mathcal{G}^{(r)} := \mathcal{G}(\mathcal{V}, \mathcal{E} \cup \mathcal{E}^{(r)})$ is called the r^{th} transitive closure of $\mathcal{G}$. The n^{th} transitive closure is the transitive closure of $\mathcal{G}(\mathcal{V}, \mathcal{E})$.

The following result from [5] gives a graph theoretic characterization of the structural controllability of (1).

Theorem 1: The following are equivalent:

(i) $\mathcal{B}_\Lambda$ is structurally controllable.
(ii) The transitive closure of $\mathcal{G}(\mathcal{B}_\Lambda)$ is a complete graph.
(iii) $\mathcal{G}(\mathcal{B}_\Lambda)$ is strongly connected.

Theorem 1 holds when the pattern $\mathcal{B}_\Lambda$ is associated with $\mathcal{GL}(n)$, $\mathcal{SL}(n)$, and $\widetilde{\mathcal{SO}}(n)$. The strong connectivity of $\mathcal{G}(\mathcal{B}_\Lambda)$ can be checked in polynomial time using algorithms in the literature [16], [17]. For $\mathcal{B}_\Lambda$ to be m-structurally controllable, the conditions given by Theorem 1 must be satisfied for at least one numerical realization of $B_1, \ldots, B_m$. Note that if the system (1) is m-structurally controllable, then it is controllable for almost all numerical realizations of $B_1, \ldots, B_m$ [5, Th. III.1].

We briefly cover below the background of the complexity theory; we refer the reader to [26] for more details. If there is an algorithm that solves a decision problem in time bounded by a polynomial function on the size of the inputs, then the problem is said to belong to class P. A decision problem is in class NP whenever a solution to the problem can be verified in polynomial time. Class P forms a subset of class NP and it is not known if the two classes are equal. A problem $\mathcal{P}$ is said to be NP-hard if every problem in the class NP can be reduced to $\mathcal{P}$ in polynomial time, and it is said to be NP-complete if it is NP-hard and also belongs to the class NP. Every optimization problem can be converted into a decision problem and if a decision problem is NP-complete, then the optimization problem is NP-hard.

B. Problem Formulation

We consider the structured bilinear system (1) where, due to a number of link failures, some nonzero entries of the pattern $\mathcal{B}_\Lambda$ become zero. The first problem is to obtain a necessary and sufficient condition for structural controllability in the presence of these link failures.

Problem 1: Find a graph theoretic equivalent condition for the structural controllability of $\mathcal{B}_\Lambda$ under k link failures.

The answer to Problem 1 (Theorem 2) gives a robust characterization of structural controllability. Subsequently, using Theorem 2, we discuss the computational complexity of the following two problems.

Problem 2: Minimize $|\Lambda|$ subjected to the structural controllability of $\mathcal{B}_\Lambda$ under k link failures.

Problem 3: Let p be a probability of link failure for any edge $e \in \mathcal{G}(\mathcal{B}_\Lambda)$ and suppose edge failures are independent. Is $\mathcal{B}_\Lambda$ structurally controllable with probability $q > 0$ or higher?

Problem 2 deals with designing a sparsest pattern Λ subjected to structural controllability under k link failures. Suppose there is a uniform cost associated with each nonzero entry in the pattern matrix $\mathcal{B}_\Lambda$; each of these nonzero entry corresponds to an interconnection between the state variables. In that case, a solution of Problem 2 gives a pattern Λ, which is structurally controllable under k link failures, with minimum cost. Problem 2 is naturally applicable in the design of large-scale systems, where the aim is to achieve structural controllability with minimum interconnections (i.e. avoiding superfluous links).

In Problem 3, we consider the case of probabilistic link failures. This becomes significant in real-world scenarios where uncertainties in the environment can cause link failures in networked systems. We also consider the problem of designing a structurally controllable pattern from an uncontrollable structure B_Λ by adding more interconnections under the cost constraints; this is addressed in Theorem 5.

III. Structural Controllability under Link Failures

In this section, we discuss our approach towards addressing the problems mentioned in the previous section. We leverage the graph-theoretic characterization of the structural controllability of drift-free bilinear systems given by Theorem 1 to answer Problem 1. It turns out that Problems 2 and 3 are equivalent to well-known NP-hard problems on graphs. We give references to the available approximation algorithms in the literature and also mention further classification (on a finer scale) of these problems from the literature. The following theorem follows from Theorem 1 and the definition of $(k+1)$-edge connected graphs [16], [17].

Theorem 2: $\mathcal{B}_\Lambda$ is structurally controllable under k link failures $(k \geq 0)$ if and only if $\mathcal{G}(\mathcal{B}_\Lambda)$ is $(k+1)$-edge connected.

Proof: $(\Rightarrow)$: By Theorem 1, structural controllability of $\mathcal{B}_\Lambda$ under k link failures implies that after removing k edges in $\mathcal{G}(\mathcal{B}_\Lambda)$, the digraph is strongly connected. This shows that $\mathcal{G}(\mathcal{B}_\Lambda)$ is $(k+1)$-edge connected.

$(\Leftarrow)$: $\mathcal{G}(\mathcal{B}_\Lambda)$ is $(k+1)$-edge connected implies that after removing k edges, it is still strongly connected. Therefore, by Theorem 1, $\mathcal{B}_\Lambda$ is structurally controllable after k link failures. ■

The k-edge connectivity of a digraph can be checked by polynomial time algorithms [17].

Example 2: Consider a bilinear system (1) with $\mathcal{B}_\Lambda \subseteq \mathcal{GL}(3)$ with $\Lambda_1 = \{(1,2), (2,1), (2,3), (3,2), (1,3), (3,1)\}$

(a) $\mathcal{G}^{(1)}(\mathcal{B}_{\Lambda_1})$ (b) $\mathcal{G}^{(1)}(\mathcal{B}_{\Lambda_2})$

Fig. 1: $\mathcal{G}(\mathcal{B}_{\Lambda_1})$ and $\mathcal{G}(\mathcal{B}_{\Lambda_2})$ for the system in Example 2.

and $\Lambda_2 = \{(1,2),(2,3),(3,1),(3,2)\}$. Their respective graphs $\mathcal{G}(\mathcal{B}_{\Lambda_1})$ and $\mathcal{G}(\mathcal{B}_{\Lambda_2})$ are as shown in Fig. 1. Notice that $\mathcal{G}(\mathcal{B}_{\Lambda_1})$ is 2-edge connected, whereas $\mathcal{G}(\mathcal{B}_{\Lambda_2})$ is strongly connected (i.e., 1-edge connected) but not 2-edge connected. Therefore, unlike $\mathcal{B}_{\Lambda_2}$, the pattern matrix $\mathcal{B}_{\Lambda_1}$ is structurally controllable under 1 link failure.

Theorem 3: Problem 2 is NP-hard.

Proof: Problem 2 is equivalent to the minimum $(k+1)$-edge connected spanning subgraph problem (known as the minimum $(k+1)$-ECSS problem [28]) which is a known NP-hard problem [26], [28] for $k \geq 0$. ■

In [28], a $(1+4/\sqrt{k})$-approximation algorithm is given with time complexity $O(k^3|\mathcal{V}|^2 + [\mathcal{E}]^{1.5}(\log|\mathcal{V}|)^2)$. The approximation factor was improved in [29] to $1+\sqrt{2/k}$ for $k \geq 15$, and to $1+5/k$ for $5 < k < 15$. These algorithms can be used to give approximate sparse structures with the mentioned approximation guarantees to solve Problem 2. In [30], it is shown that for $c > 0$ and any integer $k \geq 1$, a polynomial time algorithm for the minimum k-ECSS problem on digraphs within an approximation ratio $1 + c/k$ would imply $P = NP$. Linear-time approximation algorithms with approximation guarantees are given in [31].

Theorem 4: Problem 3 is NP-hard.

Proof: Consider an instance where both p and q are rational numbers. By Theorem 1, the problem is equivalent to $\mathcal{G}(\mathcal{B}_\Lambda)$ being strongly connected with probability q or higher. This is the well-known network reliability problem on digraphs known to be NP-hard [26]. ■

Network reliability problems are known to be $\#P$-complete [32], i.e., NP-hard problems which may require exponential time even if $P = NP$. In [33], a polynomial time approximation algorithm is given for the case where there is a fixed node (called the root) and the problem is to find a path to every other node in the network.

We now consider the problem of designing a structurally controllable system from a structurally uncontrollable system under cost constraints.

Theorem 5: Suppose $\mathcal{B}_\Lambda$ is structurally uncontrollable. Let $w(i,j) \in \mathbb{N}$ be the cost associated with making the entry $\mathcal{B}(i,j)$ nonzero. Suppose the costs $w(i,j)$ are not uniform. Then, the problem of deciding if there exists a set $\Lambda_1 \subset \{1,\ldots,n\} \times \{1,\ldots,n\}$ such that $\sum_{(i,j)\in\Lambda_1} w(i,j) \leq b \in \mathbb{N}$ and $\mathcal{B}_{\Lambda\cup\Lambda_1}$ is structurally controllable is NP-complete.

Proof: Since $\mathcal{B}_\Lambda$ is structurally uncontrollable, it follows from Theorem 1 that $\mathcal{G}(\mathcal{B}_\Lambda)$ is not strongly connected. The costs $w(i,j)$ can be identified with edge weights and the set Λ_1 can be identified with the edge augmentation of $\mathcal{G}(\mathcal{B}_\Lambda)$ to $\mathcal{G}(\mathcal{B}_{\Lambda\cup\Lambda_1})$. Furthermore, by Theorem 1, the structural controllability of $\mathcal{B}_{\Lambda\cup\Lambda_1}$ is equivalent to $\mathcal{G}(\mathcal{B}_{\Lambda\cup\Lambda_1})$ being strongly connected. This is the strong connectivity augmentation problem which is NP-complete for non uniform costs [26], [34]. ■

It turns out that if the edges in the graph all have equal weight, then the various augmentation problems also have efficient algorithms [34], [35]. The following corollary follows from the $O(|\mathcal{V}|+|\mathcal{E}|)$ time algorithm of [34], [36] for the strong connectivity augmentation problem with uniform costs.

Corollary 5.1: If the costs $w(i,j)$ are identical for all entries in Theorem 5, then the decision problem is polynomial time solvable.

In [35], a 2-approximation algorithm is given which runs in $O(|\mathcal{V}|^2)$ time for the strong connectivity augmentation problem. This can be used to find an approximate solution to the cost-constrained structural controllability problem discussed above. It is shown in [37] that the problem is fixed time tractable with running time $2^{O(k\log k)}n^{O(1)}$. (Fixed time tractability is one of the types of parameterized complexity that allows the classification of NP-hard problems on a finer scale).

Problems similar to the ones mentioned here are considered in [38] for the structural controllability of LTI systems. One of them is the minimal cost link-addition problem to make LTI systems controllable, which is similar to the case of Theorem 5. The other two are the minimum cost link-deletion problem and the minimum cost input-deletion problem such that the LTI system is structurally uncontrollable.

In [39], a problem similar to Theorem 5 is considered for the structural controllability of multiagent systems subject to cost constraints. Both input and output measurement costs are accounted for, with the objective of minimizing the cost while adhering to structural controllability constraints. The structural controllability is characterized using the notion of dynamic graphs. A polynomial time algorithm is given to solve the minimum cost control configuration problem for single input discrete-time bilinear systems subject to the structural controllability constraints.

Remark 1: We have considered a single pattern $\mathcal{B}_\Lambda$ for structured systems. In general, one can have different pattern matrices $\mathcal{B}_{\Lambda_1},\ldots,\mathcal{B}_{\Lambda_l}$ in (1). In [5, Th. IV.2], equivalent structural controllability conditions for such cases are obtained using colored graphs where the edges of $\mathcal{G}$ are colored according to the pattern $\mathcal{B}_{\Lambda_i}$ they are associated with. The equivalent conditions require strong connectivity of $\mathcal{G}$. The observations mentioned in this paper extend to these general cases as well in the presence of link failures. Furthermore, these observations also hold for structural accessibility of bilinear systems with drift in the presence of link failures.

IV. Conclusion

We extended graph theoretic equivalent conditions of [5] for the structural controllability of driftless bilinear systems to the case where there are link failures. We used results from graph theory to study complexity issues of a few more problems such as finding the sparsest patterns in the presence of link failures and deciding structural controllability under probabilistic link failures. Finally, we showed that the problem of designing a structurally controllable bilinear system from an uncontrollable one under cost constraints is NP-hard.

REFERENCES

[1] Y. Liu, J. Slotine, and A. Barabasi, "Controllability of complex networks," *Nature*, vol. 473, pp. 167–173, 2011.

[2] J.-M. Dion, C. Commault, and J. van der Woude, "Generic properties and control of linear structured systems: a survey," *Automatica*, vol. 39, no. 7, pp. 1125–1144, 2003.

[3] G. Ramos, A. P. Aguiar, and S. Pequito, "An overview of structural systems theory," *Automatica*, vol. 140, p. 110229, 2022.

[4] C.-T. Lin, "Structural Controllability," *IEEE Transactions on Automatic Control*, vol. 19, no. 3, pp. 201–208, 1974.

[5] A. Tsopelakos, M.-A. Belabbas, and B. Gharesifard, "Classification of the Structurally Controllable Zero-Patterns for Driftless Bilinear Control Systems," *IEEE Transactions on Control of Network Systems*, vol. 6, no. 1, pp. 429–439, 2019.

[6] S. Moothedath, P. Chaporkar, and M. N. Belur, "A Flow-Network-Based Polynomial-Time Approximation Algorithm for the Minimum Constrained Input Structural Controllability Problem," *IEEE Transactions on Automatic Control*, vol. 63, no. 9, pp. 3151–3158, 2018.

[7] S. Pequito, S. Kar, and A. Aguiar, "On the complexity of the constrained input selection problem for structural linear systems," *Automatica*, vol. 62, pp. 193–199, 2015.

[8] X. Wang, B. Li, J.-S. Li, I. R. Petersen, and G. Shi, "Controllability and Accessibility on Graphs for Bilinear Systems Over Lie Groups," *IEEE Transactions on Automatic Control*, vol. 68, no. 4, pp. 2277–2292, 2023.

[9] G. Cheng, W. Zhang, and J.-S. Li, "Combinatorics-Based Approaches to Controllability Characterization for Bilinear Systems," *SIAM Journal on Control and Optimization*, vol. 59, no. 5, pp. 3574–3599, 2021.

[10] N. Boussaid, M. Caponigro, and T. Chambrion, "Weakly Coupled Systems in Quantum Control," *IEEE transactions on automatic control*, vol. 58, no. 9, pp. 2205–2216, 2013.

[11] D. Williamson, "Observation of bilinear systems with application to biological control," *Automatica*, vol. 13, no. 3, pp. 243–254, 1977.

[12] R. R. Mohler and W. J. Kolodziej, "An Overview of Bilinear System Theory and Applications," *IEEE Transactions on Systems, Man, and Cybernetics*, vol. 10, no. 10, pp. 683–688, 1980.

[13] V. Jurdjevic and H. J. Sussmann, "Control systems on Lie groups," *Journal of Differential Equations*, vol. 12, no. 2, pp. 313–329, 1972.

[14] W. M. Boothby and E. N. Wilson, "Determination of the Transitivity of Bilinear Systems," *SIAM Journal on Control and Optimization*, vol. 17, no. 2, pp. 212–221, 1979.

[15] D. Elliot, *Bilinear Control Systems: Matrices in Action*, vol. 169. Springer, 1 ed., 2009.

[16] D. West, *Introduction to Graph Theory*. Prentice-Hall, 2 ed., 2000.

[17] J. Bang-Jensen and G. Gutin, *Digraphs: Theory, Algorithms and Applications*. Springer London, 2 ed., 2009.

[18] A. Clauset, C. Moore, and M. E. Newman, "Hierarchical structure and the prediction of missing links in networks," *Nature*, vol. 453, pp. 98–101, 2008.

[19] B. Yang, J. Liu, S. Shenker, J. Li, and K. Zheng, "Keep Forwarding: Towards $K-$link Failure Resilient Routing," in *IEEE INFOCOM 2014 - IEEE Conference on Computer Communications*, pp. 1617–1625, 2014.

[20] D. Senejohnny, P. Tesi, and C. De Persis, "A Jamming-Resilient Algorithm for Self-Triggered Network Coordination," *IEEE Transactions on Control of Network Systems*, vol. 5, no. 3, pp. 981–990, 2018.

[21] A. S. A. Dilip, "Probabilistic State transfer, Estimation and Measures for Optimal Actuator/Sensor Placement for Linear Systems With Packet Dropouts," *IEEE Control Systems Letters*, vol. 3, no. 4, pp. 865–870, 2019.

[22] S. Jafari, A. Ajorlou, and A. G. Aghdam, "Leader localization in multi-agent systems subject to failure: A graph-theoretic approach," *Automatica*, vol. 47, no. 8, pp. 1744–1750, 2011.

[23] M. A. Rahimian and A. G. Aghdam, "Structural controllability of multi-agent networks: Robustness against simultaneous failures," *Automatica*, vol. 49, no. 11, pp. 3149–3157, 2013.

[24] G. Ramos, S. Pequito, A. P. Aguiar, and S. Kar, "Analysis and design of electric power grids with p-robustness guarantees using a structural hybrid system approach," in *2015 European Control Conference (ECC)*, pp. 3542–3547, 2015.

[25] M.-A. Belabbas, X. Chen, and D. Zelazo, "On Structural Rank and Resilience of Sparsity Patterns," *IEEE Transactions on Automatic Control*, vol. 68, no. 8, pp. 4783–4795, 2023.

[26] M. Garey and D. Johnson, *Computers And Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman and Company, 1979.

[27] B. Hall, *Lie Groups. Lie Algebras and Representations: An Elementary Introduction*. Springer Cham, 2 ed., 2015.

[28] J. Cheriyan and R. Thurimella, "Approximating minimum-size k-connected spanning subgraphs via matching," *SIAM Journal on Computing*, vol. 30, no. 2, pp. 528–560, 2000.

[29] H. N. Gabow, "Special edges, and approximating the smallest directed k-edge connected spanning subgraph," SODA '04, (USA), pp. 234–243, Society for Industrial and Applied Mathematics, 2004.

[30] H. N. Gabow, M. X. Goemans, É. Tardos, and D. P. Williamson, "Approximating the smallest k-edge connected spanning subgraph by lp-rounding," *Networks*, vol. 53, pp. 345–357, 2009.

[31] L. Georgiadis, G. F. Italiano, A. Karanasiou, C. Papadopoulos, and N. Parotsidis, "Sparse certificates for $2-$connectivity in directed graphs," *Theoretical Computer Science*, vol. 698, pp. 40–66, 2017. Algorithms, Strings and Theoretical Approaches in the Big Data Era (In Honor of the 60th Birthday of Professor Raffaele Giancarlo).

[32] C. J. Colbourn, *The Combinatorics of Network Reliability*. USA: Oxford University Press, Inc., 1987.

[33] H. Guo and M. Jerrum, "A Polynomial-Time Approximation Algorithm for All-Terminal Network Reliability," *SIAM Journal on Computing*, vol. 48, no. 3, pp. 964–978, 2019.

[34] K. P. Eswaran and R. E. Tarjan, "Augmentation problems," *SIAM Journal on Computing*, vol. 5, no. 4, pp. 653–665, 1976.

[35] G. N. Frederickson and J. Ja'Ja', "Approximation algorithms for several graph augmentation problems," *SIAM Journal on Computing*, vol. 10, no. 2, pp. 270–283, 1981.

[36] S. Raghavan, "A Note on Eswaran and Tarjan's Algorithm for the Strong Connectivity Augmentation Problem," in *The Next Wave in Computing, Optimization, and Decision Technologies* (B. Golden, S. Raghavan, and E. Wasil, eds.), (Boston, MA), pp. 19–26, Springer US, 2005.

[37] K. V. Klinkby, P. Misra, and S. Saurabh, "Strong connectivity augmentation is FPT," in *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pp. 219–234, 2021.

[38] Y. Zhang and T. Zhou, "Minimal structural perturbations for controllability of a networked system: Complexities and approximations," *International Journal of Robust and Nonlinear Control*, vol. 29, no. 12, pp. 4191–4208, 2019.

[39] P. Liu, Y. Zhang, and Y.-P. Tian, "Scheduling Algorithm of Observation and Controlling for Multi-Agent Systems to Guarantee Structural Controllability," in *2017 13th IEEE International Conference on Control & Automation (ICCA)*, pp. 672–677, 2017.