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# Multidimensional signed Friedkin-Johnsen model

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***Abstract*— The multidimensional signed Friedkin-Johnsen (SFJ) model introduced in this paper describes opinion dynamics on a signed network in which the agents hold opinions on multiple interconnected topics and are allowed to be stubborn. In the paper, we establish sufficient conditions for the stability of the multidimensional SFJ model, and analyze convergence to consensus in concatenated instances of this model.**

## I. INTRODUCTION

Opinion dynamics models are essential for examining the dissemination of information or opinions on different topics among individuals within a community or group [1]. The models have found extensive practical applications in diverse fields, such as parliamentary systems [2], finance [3], social behavior [4], and psychology [5]. These models help investigate the dynamics of social attitudes and their evolution over time. A graph-based approach is normally adopted, where individuals (i.e., agents) serve as nodes, the interactions between different individuals represent the edges, and the opinions of each individual are represented by the states of the agent.

The fundamental model employed is the DeGroot model [6]. This model operates through an averaging rule based on the interactions between agents, which drives the dynamics towards consensus. This model was expanded into a Friedkin-Johnsen (FJ) model in [7], incorporating the stubbornness of the agents into the dynamics. In the FJ model, the opinions of the agents tend to approach each other, yet they do not normally reach a consensus. If an FJ model represents a discussion event, a series of discussion events on the same topics can be represented by a concatenated FJ model [8]–[10]. This model can be examined using a two-time-scale framework, similar to the one introduced in [11]. The motivation behind employing a concatenation of FJ models lies in the observation that if each single instance of the FJ model leads to opinions that get closer to each other, then concatenating FJ models (i.e., using the endpoint of the preceding discussion as the starting point for the next discussion) may gradually lead to a consensus opinion, in spite of the persistent stubbornness.

In [12] and [13] the DeGroot and concatenated FJ models were investigated within the framework of signed networks, in which the antagonism among agents is introduced via negative edge weights. These models leverage the algebraic property of matrices that are Eventually Stochastic (ES) for the discrete time (DT) case, and Eventually Exponentially Positive (EEP) for the continuous time (CT) case [14]. In particular, in [13] we refer to the FJ model over a signed network as signed FJ (SFJ) model, and to its concatenated version as concatenated SFJ.

Assume now that our opinion dynamics models are used to investigate multiple topics that are up for discussion simultaneously, and that these multiple topics are interdependent. An example could be students' opinions on different subjects, where preferences for one subject may negatively correlate with preferences for another (e.g., a preference for mathematics may be negatively correlated with a preference for humanities), and positively correlate with yet another subject (e.g., a preference for mathematics may positively correlate with a preference for physics). Such interdependences among multiple topics are typically captured by the so-called multidimensional models [9], [15]–[19]. The paper [16], for instance, proposes a multidimensional extension of the DT FJ model in which, alongside the usual interaction graph among the agents, a second interaction graph for topics is considered. For CT dynamics, [19] proposes two models that introduce interdependency among topics, and provides sufficient conditions for convergence of the resulting model. The models of [16], [19] both necessitate to have nonnegative interaction weights among agents, and so do [9], [15], [17], [18].

In this paper, we explore the behaviour of both CT and DT multidimensional opinion models incorporating both stubbornness and antagonistic interactions. In particular, our SFJ models describe the agent-to-agent interaction graph using an ES matrix for DT and an EEP Laplacian for CT. Consequently, our models exhibit the flexibility to be either stable or unstable, contingent on the selection of the interaction graph, stubbornness values, and topic dependencies. Several cases of topic interdependence matrices are investigated in the paper, and sufficient conditions for convergence are established. The only other model of multidimensional opinion dynamics models with antagonistic inter-agent interactions we are aware of is [20], which however is limited to a much simpler agent-to-agent signed interaction matrix, that is always Schur stable unless it is structurally balanced (the rows of the interaction matrix in absolute value sum to 1, CT equivalent of what in [21] is called an "opposing" Laplacian). The stability analysis of this model is rather straightforward, and its dynamical behavior much more limited, even in the multidimensional case.

In the paper we also provide necessary conditions for consensus in the concatenation of such multidimensional SFJ

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model. To the best of our knowledge, this has not been treated before in the literature.

The structure of the paper is as follows: Section II presents basic concepts related to signed graphs and matrix theory. Section III introduces our DT and CT multidimensional SFJ models, along with sufficient stability conditions for the SFJ models. Furthermore, Section IV establishes sufficient conditions for the asymptotic stability of the multidimensional SFJ models. Lastly, Section V addresses consensus for concatenations of multidimensional SFJ models.

## II. Preliminary material

*Notations.* Lower case letters $y, z, \alpha, \dots$ denote real numbers, and bold letters $\mathbf{v}, \mathbf{w}, \mathbf{x}, \dots$ represent vectors. The vectors of 0 and 1 are written as $\mathbf{0}_n, \mathbb{1}_n \in \mathbb{R}^n$. The one-norm, Euclidean-norm and infinity-norm of a vector $\mathbf{x}$ is represented as $\|\mathbf{x}\|_1$, $\|\mathbf{x}\|_2$, and $\|\mathbf{x}\|_\infty$ respectively. Capital Latin or Greek letters $W, X, \Theta, \dots$ denote matrices. For a matrix $H = [H_{ij}] \in \mathbb{R}^{n\times n}$, its transpose and kernel are represented as $H^T$ and $\ker(H)$, respectively. For a matrix $H \in \mathbb{C}^{n\times n}$, $H^*$ represents the complex conjugate transpose. The diagonal matrix $\mathrm{diag}\,(\mathbf{x}) \in \mathbb{R}^{n\times n}$ has the entries of vector $\mathbf{x} \in \mathbb{R}^n$ on its diagonal. $\Lambda\,(H) = \{\lambda_1\,(H), \dots, \lambda_n\,(H)\}$ gives the spectrum of the matrix $H$. The maximum absolute eigenvalue $\rho\,(H) = \max_{i=1,\dots,n} |\lambda_i\,(H)|$ is called the spectral radius of the matrix $H$ and if $\rho(H) > |\lambda(H)|\ \forall\lambda \in \Lambda(H)$, $\lambda \neq \rho(H)$, then it is strictly dominant. For two sets $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{C}$, the product of sets is defined as $\mathcal{H}_1\mathcal{H}_2 \equiv \{h_1h_2 : h_1 \in \mathcal{H}_1,\ h_2 \in \mathcal{H}_2\}$. Furthermore, $Co(\mathcal{H})$ gives the convex hull of the set $\mathcal{H}$. Let $\mathcal{LHP}$ ($\mathcal{LHP}_c$) and $\mathcal{RHP}$ ($\mathcal{RHP}_c$) denote the open (closed) left and open (closed) right half plane of the complex plane, respectively.

### A. Signed Graphs

A directed graph (digraph) is a triplet $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ with $n$ nodes $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$, a link set $\mathcal{E}$ in which a pair $(i, j) \in \mathcal{E}$ signifies a link from node $v_i$ to $v_j$, and a weighted adjacency matrix $A$ where $A_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. In signed graphs, the entries of $A_{ij}$ may assume positive or negative values, reflecting a friendly or antagonistic relationship between the nodes. In an undirected graph, $A$ satisfies $A^T = A$, while in a weight balanced digraph, we have $A\mathbb{1} = A^T\mathbb{1}$. For a signed graph, the weighted in-degree vector is defined as $\sigma_{\mathrm{in}} = \begin{bmatrix} \sum_{j=1, i\neq j}^{n} A_{ij} \end{bmatrix}_{n\times 1} \in \mathbb{R}^n$. The "repelling Laplacian" [21] is defined as $L = \Sigma_{\mathrm{in}} - A$ where $\Sigma_{\mathrm{in}} = \mathrm{diag}\,(\sigma_{\mathrm{in}})$. By construction $L\mathbb{1} = \mathbf{0}$. A path from node $i$ to $k$ in the graph is given by the pairs $(i, 1), (1, 2), \dots, (j-1, j), (j, k) \in \mathcal{E}$ which is equivalent to $A_{kj}A_{j(j-1)}\dots A_{21}A_{1i} \neq 0$. In the graph, a directed spanning tree exists with root node $i$ if there is a directed path from node $i$ to all other nodes in the graph. A strongly connected graph has a directed path between each pair of nodes.

### B. Matrix Theory

The matrix $H$ is called Hurwitz (resp. Schur) stable if $\Re\,(\lambda\,(H)) < 0$ (resp. $|\lambda\,(H)| < 1$). It is marginally stable if $\Re\,(\lambda\,(H)) \leq 0$ (resp. $|\lambda\,(H)| \leq 1$), and the eigenvalues such that $\Re\,(\lambda\,(H)) = 0$ (resp. $|\lambda\,(H)| = 1$) are simple eigenvalues. The matrix $H$ is said reducible if $\exists$ a permutation matrix $P$ such that $P^THP = \begin{bmatrix} H_1 & H_2 \\ 0 & H_3 \end{bmatrix}$ where $H_1$ and $H_3$ are non-trivial square matrices. It is irreducible if it is not reducible. A positive matrix, $H > 0$, has positive entries, $H_{kl} > 0$, and a nonnegative matrix, $H \geq 0$, has non-negative entries, $H_{kl} \geq 0$. The dimension of $\ker\,(H)$ is known as corank of a matrix $H$. For a normal matrix $H$, we have $HH^T = H^TH$. A positive definite (resp. semi-definite) matrix $H$, represented as $H \succ 0$ (resp. $H \succeq 0$), is defined as $\mathbf{x}^TH_{sym}\mathbf{x} > 0$ (resp. $\mathbf{x}^TH_{sym}\mathbf{x} \geq 0$) $\forall\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$ where $H_{sym} = (H + H^T)/2$. A negative definite (resp. semi-definite) matrix is denoted $H \prec 0$ (resp. $H \preceq 0$) and obviously corresponds to $-H \succ 0$ (resp. $-H \succeq 0$). The Kronecker product between two matrices $H_1$ and $H_2$ is given by $H_1 \otimes H_2$ and $\lambda(H_1 \otimes H_2) = \lambda(H_1)\lambda(H_2)$.

### C. Perron-Frobenious property and Eventual Positivity

The following definitions and properties can be found in [12], [13].

- The matrix $H \in \mathbb{R}^{n\times n}$ satisfies the Perron-Frobenius (PF) property (denoted $H \in \mathcal{PF}$) if $\exists\lambda \in \Lambda(H)$ real and positive which is a simple and strictly dominant eigenvalue, and the corresponding right eigenvector is positive.
- The matrix $H \in \mathbb{R}^{n\times n}$ is said Eventually Positive (EP) if $H^k > 0$ for $k \geq k_o$, $k$, $k_o \in \mathbb{N}$ and it is represented as $H \overset{\vee}{>} 0$.
- For a EP matrix $H \in \mathbb{R}^{n\times n}$, we have $H^T \overset{\vee}{>} 0$ and both $H$, $H^T \in \mathcal{PF}$.
- The matrix $H \in \mathbb{R}^{n\times n}$ is said Eventually Exponentially Positive (EEP), if $H + dI \overset{\vee}{>} 0$ for some $d \in \mathbb{R}_{\geq 0}$.

### D. Eventually Stochastic Matrices

The following definition and property can be found in [14].

- The matrix $H \in \mathbb{R}^{n\times n}$ is said Eventually Stochastic (ES) if $H \overset{\vee}{>} 0$ and $H\mathbb{1} = \mathbb{1}$.
- For a ES matrix $H$, $1 \in \Lambda\,(H)$ is a simple and strictly dominant eigenvalue, with right and left eigenvectors $\mathbf{v} > 0$ and $\mathbf{w} > 0$.

### E. Numerical Range

The following definitions and properties can be found in [13], [22].

- The numerical range of a matrix $H \in \mathbb{C}^{n\times n}$ is defined as the compact and convex set in $\mathbb{C}$ given by $\omega(H) = \{\mathbf{x}^*H\mathbf{x} : \mathbf{x} \in \mathbb{C}^n, \|\mathbf{x}\| = 1\}$.
- The numerical range of a matrix $H$ encapsulates all eigenvalues of $H$, i.e., $\Lambda(H) \subset \omega(H)$.
- The numerical radius of a matrix $H \in \mathbb{C}^{n\times n}$ is defined as $r(H) = \max\{|z| : z \in \omega(H)\}$.
- For a matrix $H$, we have the following relation between spectral radius $\rho(H)$, numerical radius $r(H)$ and norm $\|H\|$.

$$\rho(H) \leq r(H) \leq \|H\| \leq 2r(H)$$

- A matrix $H \in \mathbb{C}^{n\times n}$ is called radial if $\rho(H) = \|H\|_2 (= r(H))$.
- The matrix $H$ is radial *iff* $\rho(H)I_n - H^T H \succeq 0$. By construction, symmetric (Hermitian), skew-symmetric (skew-Hermitian), unitary, and normal matrices are examples of radial matrices.

The following lemma gives the numerical range of Kronecker products of matrices.

**Lemma 1** *[22] Let $H_1 \in \mathbb{R}^{n\times n}$, and $H_2 \in \mathbb{R}^{m\times m}$, then*

(a) $\omega(H_1)\omega(H_2) \subseteq Co(\omega(H_1)\omega(H_2)) \subseteq \omega(H_1 \otimes H_2)$.

(b) *If $H_1$ is normal, then $\omega(H_1 \otimes H_2) = Co(\omega(H_1)\omega(H_2))$.*

(c) *If $e^{i\vartheta}H_1 \succeq 0$ for some $\vartheta \in [0, 2\pi)$, then $\omega(H_1 \otimes H_2) = \omega(H_1)\omega(H_2)$.*

## III. Problem Formulation

In this section, we explore the multidimensional SFJ model for both discrete time (DT) and continuous time (CT). Multidimensional implies that individuals hold opinions on multiple topics, and the opinions on these topics are interconnected. Let there be $m$ topics and $n$ agents.

### A. Discrete-time multidimensional SFJ

The multidimensional model is similar to the one provided in [16] but with signed weighted adjacency matrix. The dynamics for agent $i$ is given by

$$x_i(t+1) = (1-\theta_i)C\sum_{j=1}^{n} w_{ij}x_j(t) + \theta_i x_i(0),$$

where $x_i(t) \in \mathbb{R}^m$ denote the opinions of agent $i$ on the $m$ topics, $\theta_i \in (0, 1)$ gives the stubbornness value, $w_{ij} \in \mathbb{R}$ represents the signed weighted interaction of the agent $j$ towards $i$ satisfying $\sum_{j=1}^{n} w_{ij} = 1$, and $C \in \mathbb{R}^{m\times m}$ describes the inter-dependency of topics common to all agents and is referred to as multi-issues dependence structure (MiDS) [16]. In matrix form using Kronecker product, we can write

$$\mathbf{x}(t+1) = ((I_n - \Theta)W \otimes C)\,\mathbf{x}(t) + (\Theta \otimes I_m)\,\mathbf{x}(0), \quad (1)$$

where $\mathbf{x}(t) = [x_1^T(t) \dots x_n^T(t)]^T \in \mathbb{R}^{nm}$, $W = [w_{ij}]_{n\times n}$ is the signed weighted adjacency pattern describing the communication graph $\mathcal{G}$, and $\Theta = \mathrm{diag}([\theta_1, \dots, \theta_n])$. When $C = I_m$, we get $m$ independent SFJ models in which the topics converge to equilibrium points independently. The DT SFJ model (1) with $C = I_m$ converge to an equilibrium point according to Lemma 2, where the results of [13] are enhanced to include radial matrices and the following assumption that all agents are partially stubborn is used.

**Assumption 1** *All agents in the network are partially stubborn, i.e., $\theta_i \in (0, 1)\ \forall\, i = 1, \dots, n$.*

**Lemma 2 ( [13] Theorem 1, 2)** *Under Assumption 1, if $C = I_m$ and $W$ is a ES and radial matrix, then $\rho((I_n - \Theta)W) < 1$, and the independent models in* (1) *converge to $\mathbf{x}^* = (P_W \otimes I_m)\mathbf{x}(0)$ where $P_W = (I_n - (I_n - \Theta)W)^{-1}\Theta$ is an ES matrix with left eigenvector $\mathbb{1}_n^T(I_n - \Theta)^{-1}\Theta$ corresponding to the eigenvalue* 1.

The affine model (1) has an equilibrium point if $\rho((I_n - \Theta)W \otimes C) < 1$, and the equilibrium point is given by $\mathbf{x}^* = Q_W\mathbf{x}(0)$ where

$$Q_W = (I_{nm} - (I_n - \Theta)W \otimes C)^{-1}(\Theta \otimes I_m). \quad (2)$$

**Lemma 3** *Under Assumption 1, if $W$ is ES and radial, then $r((I_n - \Theta)W) < 1$.*

**Proof.** Using the Cauchy–Schwarz inequality $\|(I_n - \Theta)W\|_2 \le \|(I_n - \Theta)\|_2\|W\|_2 < 1$, as $\|(I_n - \Theta)\|_2 < 1$ and $\|W\|_2 = 1$. With $r((I_n - \Theta)W) \le \|(I_n - \Theta)W\|_2$, the Lemma is proved. ■

### B. Continuous-time multidimensional SFJ

The dynamics for the agent $i$ in CT can be written as

$$\begin{aligned}\dot{x}_i(t) = (1-\theta_i)C\sum_{j\in\mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)) \\ - (I_m - C)x_i(t) - \theta_i(Cx_i(t) - x_i(0)),\end{aligned}$$

where $x_i(t) \in \mathbb{R}^m$ defines the opinions on the $m$ topics, $\theta_i \in (0, 1)$ represents the stubbornness value, $A = [a_{ij}]_{n\times n}$ is the adjacency matrix of the signed graph $\mathcal{G}$, and $C \in \mathbb{R}^{m\times m}$ is the MiDS matrix. A compact version in matrix form with Kronecker products is given by

$$\begin{aligned}\dot{\mathbf{x}}(t) = -\Big(I_{nm} - ((I_n - \Theta)(I_n - L)) \otimes C\Big)\mathbf{x}(t) \\ + (\Theta \otimes I_m)\,\mathbf{x}(0),\end{aligned} \quad (3)$$

where $\mathbf{x}(t) \in \mathbb{R}^{nm}$, $\Theta = \mathrm{diag}([\theta_1, \dots, \theta_n])$, and $L = \Sigma_{in} - A$ is the signed Laplacian matrix. When $C = I_m$, we get the $m$ independent SFJ models. The CT SFJ model (3) with $C = I_m$ converges to an equilibrium point as per Lemma 4 where the results of [13] now include radial matrices.

**Lemma 4 ( [13] Theorem 3, 4)** *Under Assumption 1, if $C = I_m$ and $-L$ is an EEP and a radial matrix, then $-((I_n - \Theta)L + \Theta)$ is Hurwitz, and the independent models in* (3) *converge to $\mathbf{x}^* = (P_L \otimes I_m)\mathbf{x}(0)$ where $P_L = (\Theta + (I_n - \Theta)L)^{-1}\Theta$ is an ES matrix with left eigenvector $\mathbb{1}_n^T(I_n - \Theta)^{-1}\Theta$ corresponding to the eigenvalue* 1.

The model (3) has an equilibrium point if $-(I_{nm} - ((I_n - \Theta)(I_n - L)) \otimes C)$ is Hurwitz, and the equilibrium point is given by $\mathbf{x}^* = Q_L\mathbf{x}(0)$ where

$$Q_L = (I_{nm} - (I_n - \Theta)(I_n - L) \otimes C)^{-1}(\Theta \otimes I_m). \quad (4)$$

## IV. Convergence in multidimensional SFJ

In this section, we provide sufficient condition for the multidimensional SFJ model to converge to an equilibrium point for both CT and DT in the more interesting case of $C \neq I_m$. In the following theorem, we use the Assumption 1 that all agents are partially stubborn.

**Theorem 1** *Under Assumption 1, if*

(i) *[DT case:] $W$ is ES and radial matrix and $\rho(C) < 1/\rho((I_n - \Theta)W)$, then $\rho((I_n - \Theta)W \otimes C) < 1$.*
(ii) *[CT case:] $I_n - L$ is ES and radial matrix and $\rho(C) < 1/\rho((I_n - \Theta)(I_n - L))$, then $-(I_{nm} - ((I_n - \Theta)(I_n - L)) \otimes C)$ is Hurwitz.*

*The multidimensional DT and CT SFJ models* (1) *and* (3) *converge to $\mathbf{x}^* = Q\mathbf{x}(0)$ where $Q \in \{Q_W, Q_L\}$ (with $Q_W$ and $Q_L$ defined in* (2) *and* (4)*). Furthermore, if $\lambda(C) = 1$ with the corresponding right and left eigenvectors as $\mathbf{v_C}$ and $\mathbf{w_C}$, respectively, then we have $Q\mathbf{v} = \mathbf{v}$ and $\mathbf{w}^T Q = \mathbf{w}^T$ where $\mathbf{v} = \mathbb{1}_n \otimes \mathbf{v_C}$ and $\mathbf{w} = (I_n - \Theta)^{-1}\Theta\mathbb{1}_n \otimes \mathbf{w_C}$.*

**Proof.** The analysis of stability for both the affine systems (1), (3) and the linear system resulting from the change of basis $\mathbf{z}(t) = \mathbf{x}(t) - \mathbf{x}^*$ yields the same results, provided that $\mathbf{x}^*$ exists uniquely, i.e., when $\rho((I - \Theta)W \otimes C) < 1$ and $-(I_{nm} - ((I_n - \Theta)(I_n - L)) \otimes C)$ Hurwitz. Hence, we need to prove that $(I_n - \Theta)W \otimes C$ is a Schur matrix in DT and $-(I_{nm} - ((I_n - \Theta)(I_n - L)) \otimes C)$ is a Hurwitz matrix in CT. Denoting $D = I_n - \Theta$ and using Lemma 2 in DT, for $W$ ES and a radial matrix, we have that $\rho(DW) < 1$, and $\rho(C) < 1/\rho(DW)$. This leads to $\rho(DW \otimes C) < 1$, which proves that the DT multidimensional SFJ model (1) is stable.

To prove that $\mathbf{v} = \mathbb{1}_n \otimes \mathbf{v_C}$ is the right eigenvector of $Q_W$ corresponding to the eigenvalue 1 when $\lambda(C) = 1$, notice that we have $(DW \otimes C)\mathbf{v} = (D \otimes I_m)\mathbf{v}$, which can be rewritten as $(I_{nm} - DW \otimes C)\mathbf{v} = (\Theta \otimes I_m)\mathbf{v}$, and $(I_{nm} - DW \otimes C)^{-1}(\Theta \otimes I_m)\mathbf{v} = \mathbf{v}$. To prove that $\mathbf{w}^T = \mathbb{1}_n^T D^{-1}\Theta \otimes \mathbf{w_C}^T$ is the left eigenvector of $Q_W$ corresponding to the eigenvalue 1 when $\lambda(C) = 1$, notice the following chain of equalities:

$$(\mathbb{1}_n^T\Theta \otimes \mathbf{w_C}^T)(I_{nm} - WD \otimes C)^{-1} = \mathbb{1}_n^T \otimes \mathbf{w_C}^T.$$

Multiplying from the right by $D^{-1}\Theta \otimes I_m$ gives

$$\begin{aligned}(\mathbb{1}_n^T\Theta \otimes \mathbf{w_C}^T)(I_{nm} - WD \otimes C)^{-1}(D^{-1}\Theta \otimes I_m) &= \mathbf{w}^T,\\ (\mathbb{1}_n^T\Theta \otimes \mathbf{w_C}^T)(D \otimes I_m - DWD \otimes C)^{-1}(\Theta \otimes I_m) &= \mathbf{w}^T,\\ \mathbf{w}^T(I_{nm} - DW \otimes C)^{-1}(\Theta \otimes I_m) &= \mathbf{w}^T.\end{aligned}$$

The proof of CT counterpart follows from the DT one by replacing $W$ with $I_n - L$ and using the property of a Schur matrix $A$ that $A - I$ is Hurwitz. ■

Notice that in the context of the CT case, we use a "rescaled" version of the Laplacian, i.e., instead of $-L$ EEP we use $I_n - L$ ES. The next example shows that, in fact, the case of $-L$ EEP and normal with a stochastic and normal MiDS matrix $C$, may result in an unstable system.

**Example 1** For $-L$ EEP and symmetric

$$L = \begin{bmatrix} 4 & -0.3 & -5.1 & 1.4\\ -0.3 & 2.6 & -2.2 & -0.1\\ -5.1 & -2.2 & 13.3 & -6\\ 1.4 & -0.1 & -6 & 4.7 \end{bmatrix}$$

with a normal MiDS matrix $C = \begin{bmatrix} 0.1 & 0.9\\ 0.9 & 0.1\end{bmatrix}$ and stubbornness values $\Theta = \mathrm{diag}([0.4,\, 0.9,\, 0.8,\, 0.9])$, we notice that $\lambda(-I_{nm} + ((I_n - \Theta)(I_n - L)) \otimes C) = 2.31$ which means that the CT multidimensional model (3) is unstable. ■

## V. Concatenation of multidimensional SFJ models

Similarly to the concatenated SFJ model for a single opinion [13], the multidimensional concatenated case involves a sequence of discussion events, each encapsulated by a multidimensional SFJ model, either DT or CT. The opinion vector, now multidimensional, is denoted as $\mathbf{x}(s,t) \in \mathbb{R}^{nm}$, where $s$ represents the event index and $t$ denotes time within each event. The dynamics governing the SFJ model at event $s$ in the multidimensional setting can be expressed as:

$$\mathbf{x}(s,\infty) = Q\,\mathbf{x}(s,0), \quad Q \in \{Q_W, Q_L\}. \tag{5}$$

The concatenation principle remains consistent with the single-dimensional case: $\mathbf{x}(s,0) = \mathbf{x}(s-1,\infty)$, which signifies that the opinion state at the end of one discussion event becomes the starting point for the subsequent event. The initial opinions at the start of the first discussion are given by $\mathbf{x}(0,\infty)$. In essence, the multidimensional concatenated SFJ model operates similarly to its single-dimensional counterpart, but now accommodates multiple topics and (5) can be rewritten as

$$\mathbf{x}(s,\infty) = Q\,\mathbf{x}(s-1,\infty), \quad Q \in \{Q_W, Q_L\}. \tag{6}$$

The following example shows the importance of finding sufficient conditions for the concatenated multidimensional case; as can be seen, the topics independently converge to consensus, i.e., $\rho(P) = 1$ with $P \in \{P_W, P_L\}$. However, in the multidimensional concatenated case, we have $\rho(Q) > 1$, which indicates unstable behavior.

**Example 2** For an ES matrix

$$W = \begin{bmatrix} 0.1 & 0.8 & 0.7 & -0.6\\ -0.2 & -0.3 & 0.7 & 0.8\\ 0.8 & 0.1 & -0.8 & 0.9\\ 0.5 & 0.4 & 0.4 & -0.3 \end{bmatrix},$$

with topic dependencies (MiDS matrix) $C = \begin{bmatrix} 0.1 & 0.9\\ 0.9 & 0.1\end{bmatrix}$ which is a normal matrix, and stubbornness values $\Theta = \mathrm{diag}([0.7,\, 0.9,\, 0.4,\, 0.6])$, we notice that the matrix $Q_W$ has spectral radius greater than 1, i.e., $\rho(Q_W) = 1.0064$, even though the single multidimensional SFJ model converges to $\mathbf{x}^* = Q_W\mathbf{x}(0)$. Also, the concatenation of independent topics are themselves stable, i.e., $\rho(P_W) = 1$. ■

**Theorem 2** *Under Assumption 1, if*
(i) *[DT case:] $W$ is ES and radial matrix*
(ii) *[CT case:] $I_n - L$ is ES and radial matrix,*

*then the concatenated multidimensional SFJ model* (6) *converges to the consensus value $\mathbf{x}^* = \overline{\mathbf{w}}^T\mathbf{x}(0)\overline{\mathbf{v}}$ if $1 \in \Lambda(C)$ is a strictly dominant eigenvalue with either (i) $r(C) = 1$ ($C$ a radial matrix) or (ii) $C \succ 0$, where $\overline{\mathbf{w}} = \mathbf{w}/\|\mathbf{w}\|_1$ with $\mathbf{w} = (I_n - \Theta)^{-1}\Theta\mathbb{1}_n \otimes \mathbf{w_C}$, $\overline{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|_\infty$ with*

$\mathbf{v} = \mathbb{1}_n \otimes \mathbf{v_C}$*, and* $\mathbf{w_C}$, $\mathbf{v_C}$ *are the left and right eigenvectors of* $C$ *corresponding to the eigenvalue* 1.

Before proving this Theorem, we need the following Lemma whose proof is omitted as it is similar to the proof of Lemma 4 in [13].

**Lemma 5** *Consider*

(i) *[DT case:]* $W$ *is ES and radial matrix*

(ii) *[CT case:]* $I_n - L$ *is ES and radial matrix,*

*and* $\Theta$ *such that Assumption 1 is obeyed. Whenever* $(I_{nm} - (I_n - \Theta)\, W \otimes C)^{-1}$ *in DT and* $(I_{nm} - (I_n - \Theta)\,(I_n - L) \otimes C)^{-1}$ *in CT exist, then the condition* $\rho(Q) \leq 1$*, where* $Q \in \{Q_W, Q_L\}$ *(with* $Q_W$ *and* $Q_L$ *defined in* (2) *and* (4)*), is equivalent to either of the following*

(i) *[DT case:]* $(I_{nm} - \Theta^{-1} \otimes I_m)(I_{nm} - W \otimes C)$ *having none of the eigenvalues in the interior of the unit disk centered at 1:*

$$\left|\lambda\left(I_{nm} - (I_{nm} - \Theta^{-1} \otimes I_m)(I_{nm} - W \otimes C)\right)\right| \geq 1.$$

(ii) *[CT case:]* $(I_{nm} - \Theta^{-1} \otimes I_m)(I_{nm} - (I_n - L) \otimes C)$ *having none of the eigenvalues in the interior of the unit disk centered at 1:*

$$\left|\lambda\Big(I_{nm} - (I_{nm} - \Theta^{-1} \otimes I_m)(I_{nm} - (I_n - L) \otimes C)\Big)\right| \geq 1.$$

**Proof of Theorem 2.** Using Theorem 1, we find that $\rho((I_n - \Theta) W \otimes C) < 1$, ensuring that $-I_{nm} + (I_n - \Theta) W \otimes C$ is Hurwitz. Consequently, Lemma 5 is applicable as the assumptions are satisfied. Let $D = -(I_{nm} - \Theta^{-1} \otimes I_m)$ and $F = I_{nm} - W \otimes C$. When $1 \in \Lambda(C)$ is a simple and strictly dominant eigenvalue with either $C$ a radial matrix or $r(C) = 1$, it follows that $F \succeq 0$ using Lemma 1 with $\mathrm{corank}(F) = 1$ given that $1 \in \Lambda\,(W \otimes C)$ is a simple and strictly dominant eigenvalue. Note that, since $0 < \theta_i < 1\ \forall i$, we also get $D \geq 0$ and $D \succ 0$. Using arguments similar to those in the proof of Theorem 2 in [13], we can prove that $\lambda(-DF) \in \mathcal{LHP}_c$ with $\mathrm{corank}(DF) = 1$ which leads to the presence of exactly one eigenvalue of $-DF$ on the unit disk centered at 1. Using Lemma 5 and Theorem 1 where we have $Q_W \mathbf{v} = \mathbf{v}$, establishes that $1 \in \Lambda(Q_W)$ is indeed simple and satisfies $1 > |\lambda(Q_W)|$ for all $\lambda\,(Q_W) \neq 1$.

Denote $\overline{D} = (I_n - \Theta)^{1/2}(I_n - W)(I_n - \Theta)^{1/2} \otimes C$. When $C \succ 0$, notice that $Q_W$ is similar to $(\Theta^{1/2} \otimes I_m)(I_{nm} + \overline{D})^{-1}(\Theta^{1/2} \otimes I_m)$ with the transformation matrix $T = \Theta^{1/2}(I_n - \Theta)^{-1/2} \otimes I_m$. Since $\overline{D} \succeq 0$, we get that $r((I_{nm} + \overline{D})^{-1}) < 1$ and consequently $\rho(Q_W) \leq 1$. Notice that corank of $\overline{D}$ is 1, which implies that $1 \in \Lambda(Q_W)$ is a simple and strictly dominant eigenvalue.

The same arguments holds for the CT case, by replacing $W$ by $I_n - L$. Notice, from Theorem 1, that $Q\mathbf{v} = \mathbf{v}$ and $\mathbf{w}^T Q = \mathbf{w}^T$ implies that the consensus value is given by $\mathbf{x}^* = \overline{\mathbf{w}}^T \mathbf{x}(0) \overline{\mathbf{v}}$ where $\overline{\mathbf{w}} = \mathbf{w}/\|\mathbf{w}\|_1$ with $\mathbf{w} = (I - \Theta)^{-1}\,\Theta \mathbb{1}_n \otimes \mathbf{w_C}$, and $\overline{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|_\infty$ with $\mathbf{v} = \mathbb{1}_n \otimes \mathbf{v_C}$. This completes the proof. ∎

**Remark 1** In our simulations, we have not encountered a matrix $C$ with $\rho(C) \leq 1$ that leads to $\rho(Q) > 1$ for $W$ (or $I_n - L$) ES and radial matrix. However, we have yet to provide a formal proof for this observation. Furthermore, notice that $\rho(C) > 1$ may also lead to convergence to an equilibrium point for the single multidimensional SFJ model (1) or (3), but the concatenation of such multidimensional SFJ model will be unstable as $\rho(Q) > 1$. The conditions on the matrix $C$, i.e. $C$ a radial matrix or $C \succ 0$, are practically relevant as in the radial matrix there will be some symmetry in the relationship among topics, and $C \succ 0$ implies higher self weights in topic dependency.

**Remark 2** In order to simplify the analysis of the multidimensional case, in this paper, for the CT case, we assume the Laplacian is "rescaled", $L_n$, which is such that $I_n - L_n$ is ES. When we compare $-L$ EEP to its "rescaled" version, $L_n = \varrho L$ where $0 < \varrho \leq 1$, both matrices possess identical left and right eigenvectors corresponding to the eigenvalue 0. Consequently, a stable system represented by $\dot{\mathbf{x}}(t) = -L\mathbf{x}(t)$ and its "rescaled" counterpart $\dot{\mathbf{x}}(t) = -L_n \mathbf{x}(t)$ converge to the same value ($\mathbf{x}^* = \mathbf{w}_L^T \mathbf{x}(0) \mathbb{1}_n$ with $\mathbf{w}_L^T L = 0$) when starting from the same initial conditions. However, in the SFJ model, $\dot{\mathbf{x}}(t) = -(\Theta + (I_n - \Theta)L)\mathbf{x}(t) + \Theta \mathbf{x}(0)$, the opinions converge to different values: $\mathbf{x}^* = (\Theta + (I_n - \Theta)L)^{-1}\Theta \mathbf{x}(0) = P_L \mathbf{x}(0)$ and $\mathbf{x}^* = (\Theta + (I_n - \Theta)L_n)^{-1}\Theta \mathbf{x}(0) = P_{L_n} \mathbf{x}(0)$. Nonetheless, in the concatenated SFJ model, $\mathbf{x}(s+1) = P\mathbf{x}(s)$ $P \in \{P_L, P_{L_n}\}$, the opinions converge to the identical value: $\mathbf{x}^* = \overline{\mathbf{w}}^T \mathbf{x}(0) \mathbb{1}_n$ where $\overline{\mathbf{w}} = \mathbf{w}/\|\mathbf{w}\|_1$ and $\mathbf{w} = (I_n - \Theta)^{-1}\Theta \mathbf{w}_L$. Similarly, consider the concatenation of the multidimensional SFJ model with the dynamics $\mathbf{x}(s+1) = Q\mathbf{x}(s)$ where $Q \in \{Q_L, Q_{L_n}\}$. When $1 \in \Lambda(Q_L)$ is strictly dominant and simple, the opinions converge to $\mathbf{x}^* = \overline{\mathbf{w}}^T \mathbf{x}(0) \overline{\mathbf{v}}$ where $\overline{\mathbf{w}} = \mathbf{w}/\|\mathbf{w}\|_1$ with $\mathbf{w} = (I_n - \Theta)^{-1}\Theta \mathbf{w}_L \otimes \mathbf{w_C}$, $\overline{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|_\infty$ with $\mathbf{v} = \mathbb{1}_n \otimes \mathbf{v_C}$, and $\mathbf{w_C}$, $\mathbf{v_C}$ are the left and right eigenvectors of $C$ corresponding to the eigenvalue 1.

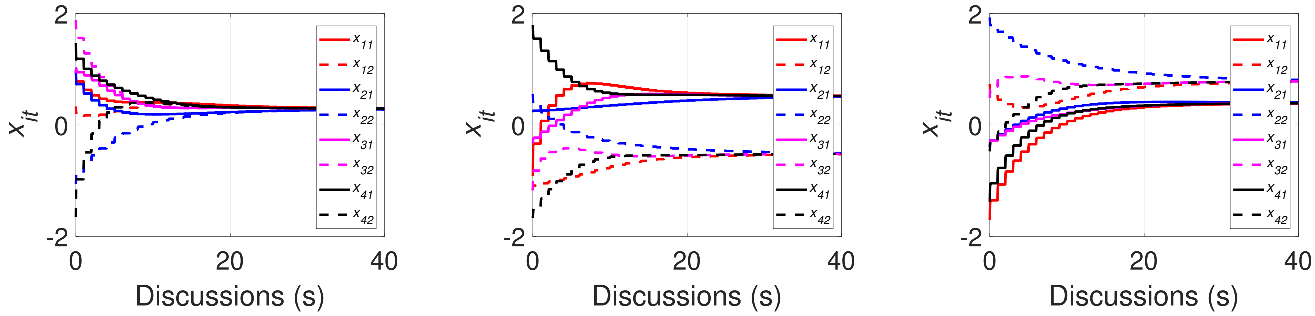
**Example 3** Consider an ES and radial matrix

$$W = \begin{bmatrix} 0.7 & 0 & -0.1 & 0.4 \\ 0 & 0.8 & 0 & 0.2 \\ 0.4 & 0.2 & 0.7 & -0.3 \\ -0.1 & 0 & 0.4 & 0.7 \end{bmatrix}$$

with $\Theta = \mathrm{diag}([0.1, 0.7, 0.1, 0.6])$, arbitrary initial conditions $\mathbf{x}(0)$, and MiDS matrices $C$ as follows

$$C_1 = \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix}, C_2 = \begin{bmatrix} 0.6 & -0.4 \\ -0.3 & 0.7 \end{bmatrix}, C_3 = \begin{bmatrix} 0.6 & 0.2 \\ 0 & 1 \end{bmatrix},$$

Note that with MiDS matrix $C_1$ (row stochastic, i.e. with $\mathbf{v_{C_1}} = \mathbb{1}_2$), consensus is achieved on all topics (See Fig. 1a). In the case of $C_2$, we observe a bipartite consensus with respect to the topics (See Fig. 1b), while $C_3$ exhibits an unusual behavior. In $C_3$, the opinions related to topic 2 reach a consensus independently and the steady state value of topic 1 depends on the steady state value of topic 2 as per the




(a) (b) (c)

Fig. 1: Simulation results for Example 3 with different MiDS matrix $C$. (a) Consensus for MiDS matrix $C_1$, (b) Bipartite consensus with respect to topics for MiDS matrix $C_2$, and (c) Independent consensus for topic 2, while topic 1 converges to 50% of steady state value of topic 2 for MiDS matrix $C_3$.

matrix $C_3$, i.e., $\mathbf{x}^* = \overline{\mathbf{w}}^T\mathbf{x}(0)\overline{\mathbf{v}}$ with $\overline{\mathbf{w}} = \mathbf{w}/\|\mathbf{w}\|_1$, $\mathbf{w} = \Theta(I_4 - \Theta)^{-1}\mathbb{1}_4 \otimes \mathbf{w}_{\mathbf{C_3}}$, $\overline{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|_\infty$, $\mathbf{v} = \mathbb{1}_4 \otimes \mathbf{v}_{\mathbf{C_3}}$. For the matrix $C_3$, the right eigenvector corresponding to the eigenvalue 1 is given by $\mathbf{v}_{\mathbf{C_3}} = [0.5,\, 1]^T$, meaning that the steady state value of topic 1 will be half of the steady state value attained by topic 2 (See Fig. 1c). The initial values of topic 1 do not play a role in the consensus values as $\mathbf{w}_{\mathbf{C_3}} = [0,\, 1]^T$. In the case of MiDS matrix $C$ where $\rho(C) < 1$, all opinions converge to zero. Lastly, when the matrix $C = I$, it results in independent SFJ models for each topic. ∎

## VI. Conclusion

In this paper, we extend the results of the FJ model with antagonism to a multidimensional case, where more than one topic is considered at any time. We provide sufficient conditions for the single multidimensional SFJ model to converge to an equilibrium point, that is, $\rho((I_n - \Theta)W)\rho(C) < 1$. We further investigate the concatenation of such multidimensional SFJ models and find that if $1 \in \Lambda(C)$ is simple, strictly dominant and either $C$ is radial, or $C \succ 0$, or $r(C) = 1$, then the concatenated model converges to different types of consensus depending on the left and right eigenvectors of $C$. The different consensus types include common consensus value for all topics, bipartite consensus with respect to topics, as well as more complicated forms of partial consensus. If $\rho(C) > 1$ the concatenated model diverges.

## References


[1] H. Noorazar, "Recent advances in opinion propagation dynamics," *European Physical Journal Plus*, vol. 135, no. 6, 2020.

[2] A. Fontan and C. Altafini, "A signed network perspective on the government formation process in parliamentary democracies," *Scientific Reports*, vol. 11, no. 1, p. 5134, 2021.

[3] Q. Zha, G. Kou, H. Zhang, H. Liang, X. Chen, C.-C. Li, and Y. Dong, "Opinion dynamics in finance and business: a literature review and research opportunities," *Financial Innovation*, vol. 6, pp. 1–22, 2020.

[4] F. Amblard and G. Deffuant, "The role of network topology on extremism propagation with the relative agreement opinion dynamics," *Physica A: Statistical Mechanics and its Applications*, vol. 343, pp. 725–738, 2004.

[5] X. Yin, H. Wang, P. Yin, and H. Zhu, "Agent-based opinion formation modeling in social network: A perspective of social psychology," *Physica A: Statistical Mechanics and its Applications*, vol. 532, p. 121786, 2019.

[6] M. H. DeGroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.

[7] N. E. Friedkin and E. C. Johnsen, "Social positions in influence networks," *Social Networks*, vol. 19, no. 3, pp. 209–222, 1997.

[8] Y. Tian and L. Wang, "Opinion dynamics in social networks with stubborn agents: An issue-based perspective," *Automatica*, vol. 96, pp. 213–223, 2018.

[9] C. Bernardo, L. Wang, F. Vasca, Y. Hong, G. Shi, and C. Altafini, "Achieving consensus in multilateral international negotiations: The case study of the 2015 Paris agreement on climate change," *Science Advances*, vol. 7, no. 51, p. eabg8068, 2021.

[10] L. Wang, C. Bernardo, Y. Hong, F. Vasca, G. Shi, and C. Altafini, "Consensus in concatenated opinion dynamics with stubborn agents," *IEEE Transactions on Automatic Control*, vol. 68, no. 7, pp. 4008–4023, 2023.

[11] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo, "Opinion dynamics and the evolution of social power in influence networks," *SIAM review*, vol. 57, no. 3, pp. 367–397, 2015.

[12] A. Fontan, L. Wang, Y. Hong, G. Shi, and C. Altafini, "Multi-agent consensus over time-invariant and time-varying signed digraphs via eventual positivity," *IEEE Transactions on Automatic Control*, vol. 68, no. 9, pp. 5429–5444, 2023.

[13] M. A. Razaq and C. Altafini, "Propagation of stubborn opinions on signed graphs," in *2023 62nd IEEE Conference on Decision and Control (CDC)*. IEEE, 2023, pp. 491–496.

[14] C. Altafini and G. Lini, "Predictable dynamics of opinion forming for networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 60, no. 2, pp. 342–357, 2014.

[15] Q. Zhou and Z. Wu, "Multidimensional Friedkin-Johnsen model with increasing stubbornness in social networks," *Information Sciences*, vol. 600, pp. 170–188, 2022.

[16] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, "Novel multidimensional models of opinion dynamics in social networks," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2270–2285, 2017.

[17] D. A. Gubanov, I. V. Petrov, and A. G. Chkhartishvili, "Multidimensional model of opinion dynamics in social networks: polarization indices," *Automation and Remote Control*, vol. 82, pp. 1802–1811, 2021.

[18] H. Yang, J. Cao, Y. Yuan, and J. Wang, "Modulus consensus for time-varying heterogeneous opinion dynamics on multiple interdependent topics," *IEEE Transactions on Automatic Control*, vol. 68, no. 11, pp. 6913–6920, 2023.

[19] M. Ye, M. H. Trinh, Y.-H. Lim, B. D. Anderson, and H.-S. Ahn, "Continuous-time opinion dynamics on multiple interdependent topics," *Automatica*, vol. 115, p. 108884, 2020.

[20] G. He, Z. Ci, X. Wu, and M. Hu, "Opinion dynamics with antagonistic relationship and multiple interdependent topics," *IEEE Access*, vol. 10, pp. 31 595–31 606, 2022.

[21] G. Shi, C. Altafini, and J. S. Baras, "Dynamics over signed networks," *SIAM Review*, vol. 61, no. 2, pp. 229–257, 2019.

[22] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge University Press, 2012.