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# Range-only Distributed Formation Control and Network Localization based on Distributed Contracting Bearing Estimators

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*Abstract*— **This paper investigates the distributed formation control problem solely based on range-only measurement, without the availability of bearing or relative bearing information. Without bearing information, local relative position necessary for maintaining rigid formation control cannot be obtained only based on range measurement. Correspondingly, range-only bearing estimator for a single landmark is proposed that can be directly combined with standard distributed rigid formation control law. Asymptotic stability analysis of the desired formation shape is presented and the proposed distributed approach is validated via numerical simulations.**

## I. Introduction

Formation shape control of mobile agents has been extensively studied for the past decades and it has been deployed in various real-world applications [1], such as multi-agent search and rescue, navigation and operation. The original formation control laws are based on the use of global position measurement [2], and they can be grouped into position-based and displacement-based methods. In recent years, new multi-agent applications have imposed some design constraints and limitation on the available on-board sensors. Consequently, formation shape control based on partial measurement information has attracted a renewed research interest [3]. Furthermore, the existing duality between network localization and formation control problems has contributed to this revived interest, as both communities have adopted methods from one another [4].

Bearing-only and angle-based approaches are two dominant formation control methods as far as the partial measurement is concerned. In bearing-only formation control [5], [6], [7], agents are required to achieve desired bearing-based rigid shape only depending on bearing information (commonly sensed by camera). In angle-based formation control [8], [9], [10], angle-based and inner-angle based rigidity are defined to characterize the desired formation shape, while control law is designed by angle measurement between agents or the diameter of neighboring agents represented by inner angle. As an example, angle measurements can be obtained by visual information coming from on-board cameras. Complementary to the aforementioned methods, the potential of range-only approaches has been widely explored in the literature. Unlike the classical distance-based formation control techniques, which rely on relative position measurements from GPS or on-board Lidar, the aim of range-only algorithms is to reach and maintain a desired formation shape using only distance information. The above methods are typically applied when range-only sensors, such as acoustic and radio devices, are deployed. This occurs when GPS is inaccessible (e.g. indoor environments) or when on-board Lidar cannot be used.

There are several existing works on range-only formation control and network localization [11], [12], [13], [14]. In [11], a stop-and-go strategy is presented where each agent is included in cyclical periods with identification and control, while the others must remain stationary when they are not included. In [12], the authors demonstrated that each agent can infer its neighbours' positions in its own local coordinate frame, given a modest number of distance measurements to anchors and communication capabilities between neighbours. The estimated positions can then be directly used in classical distance-based formation shape methods. In [13], a relative localization and affine formation control approach is proposed based on range measurement, odometry measurement and local communication. The approach is a discrete-time scheme and relies on leader-follower structure with directed graph. In [14], the authors propose a novel consensus algorithm based on the simultaneous perturbation stochastic approximation (SPSA) method for distributed tracking under unknown-but-bounded disturbances. The algorithm is applied to estimate relative bearing information in a formation control scenario, where agents can measure the noisy squared distances between each other.

In this paper, we investigate the continuous-time range-only formation control problem based on undirected rigid formation graph. Distinct from the aforementioned works, we propose a scheme for distributed formation control and network localization under a distance-based rigid formation graph, where only range measurements, local odometry measurements and local communication are available to the agents. Note that position measurements are crucial for the agents to determine the direction of movement. In particular, we extend the discrete-time bearing estimator in [15] to the continuous-time case to reconstruct the relative bearings between the agents and a single stationary landmark. Via contraction theory, we perform the convergence analysis for the proposed continuous-time observer. Leveraging this bearing estimator and via local information exchange of the estimated relative positions to the neighbors, we can directly combine the proposed local observer with the standard

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distance-based formation control scheme. Furthermore, an asymptotic stability analysis of the overall systems, consisting of the local bearing estimators and the distributed formation control, is presented.

The paper is organised as follows. Section II presents the preliminaries and problem formulation. The proposed continuous-time range-only estimator and its corresponding convergence analysis are presented in Section III. In Section IV, the range-only distributed formation control law is introduced together with the asymptotic stability analysis. Numerical simulations are provided in Section V, and finally, conclusions are given in Section VI.

## II. Preliminaries & problem formulation

### A. Notation

For a given matrix $A \in \mathbb{R}^{n\times n}$, the corresponding extended matrix is defined by $\bar{A} = A \otimes I_m \in \mathbb{R}^{nm\times pm}$, where $\otimes$ denotes the Kronecker product. With $I_m$, we refer to the $m$-dimensional identity matrix. For a given vector $z \in \mathbb{R}^m$, the diagonal matrix $D_z$ is defined by $D_z = \underset{i=1,\ldots,m}{\text{diag}}\ (z_i)$.

### B. Graph

An undirected graph $\mathbb{G}$ is defined by the tuple $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{1, 2, \cdots, n\}$ is the *vertex* set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the *edge* set. $|\mathcal{E}|$ denotes the cardinality of the set $\mathcal{E}$. The ordered pair $(i, j) \in \mathcal{E}$ refers to an edge which is represented by an arrow with node $i$ on its tail and node $j$ on its head. We assume throughout the paper that there is no self-loop in $\mathbb{G}$, i.e., $(i, i) \notin \mathcal{E}$ for all $i \in \mathcal{V}$. The set of the neighbours for the vertex $i$ is denoted by $\mathcal{N}_i \triangleq \{j \in \mathcal{V} |\, (i, j) \in \mathcal{E}\}$. Correspondingly, the set of edges associated to the pair $(i, \mathcal{N}_i)$ is denoted by $\mathcal{K}_i := \{k \,|\, (i, j) = \mathcal{E}_k \text{ for some} j \in \mathcal{N}_i\}$. We define the elements of the incidence matrix $B \in \mathbb{R}^{|\mathcal{V}|\times|\mathcal{E}|}$ of $\mathbb{G}$ by

$$b_{ik} \triangleq \begin{cases} +1 & \text{if} \quad i = \mathcal{E}_k^{\text{tail}} \\ -1 & \text{if} \quad i = \mathcal{E}_k^{\text{head}} \\ 0 & \text{otherwise}, \end{cases} \quad . \tag{1}$$

where $\mathcal{E}_k^{\text{tail}}$ and $\mathcal{E}_k^{\text{head}}$ denote the tail and head nodes of the edge $\mathcal{E}_k$, i.e., $\mathcal{E}_k = (\mathcal{E}_k^{\text{tail}}, \mathcal{E}_k^{\text{head}})$.

Given a finite collection of agent's positions $\{p_i\}_{i=1}^n$ in $\mathbb{R}^2$, with $n \geqslant 2$, we define a *configuration* of $\mathbb{G}$ by $p = \left[p_1^\top, \ldots, p_n^\top\right]^\top \in \mathbb{R}^{2n}$. A framework in a 2D-plane, which is denoted by $(\mathbb{G}, p)$, is a combination of an undirected graph $\mathbb{G}$ and a configuration $p$, i.e., every vertex $i \in \mathcal{V}$ is mapped to the point $p_i$ in the configuration.

### C. Infinitesimally rigid framework and distance-based rigid formations

Let us recall the notion of infinitesimally rigid framework in [16], [17] in order to define a desired formation shape. A commonly used relative displacement measurement is given by $z_{ij} = p_i - p_j$ for every $(i, j) \in \mathcal{E}$. From this we use an *edge* function given by $f_{\mathbb{G}}(p) = \text{col}_{(i,j)\in\mathcal{E}}\{\|p_i - p_j\|\}$ to defined a distance-based formation shape. Associated to $f_{\mathbb{G}}(p)$, the rigidity matrix $R(z)$ of the framework $(\mathbb{G}, p)$ is defined by the Jacobian of the edge function $f_{\mathbb{G}}(p)$, which satisfies $R(z) = D_z^\top \bar{B}^\top$. For a given desired formation shape, where the desired distance on every edge is given by $d = \text{col}_{(i,j)\in\mathcal{E}}\{\|z_{ij}^*\|\}$ with $\|z_{ij}^*\|$ the desired distance for the edge $(i, j)$, the set of all equilibrium points that satisfy such distance constraint is $E := \{p | f_{\mathbb{G}}(p) = d\}$. For a given desired distance vector $d$, the corresponding desired framework $(\mathbb{G}, p^*)$, with $p^* \in E$, is said to be *infinitesimally rigid* if the rank of $R(z^*)$ is $2\,|\mathcal{V}| - 3$, for for the 2D case, and $3\,|\mathcal{V}| - 6$, for the 3D one. For an infinitesimally rigid framework $(\mathbb{G}, p^*)$, the admissible infinitesimal displacement is translational and rotational motion [18].

Using the framework $(\mathbb{G}, p)$, let each agent position $p_i$ evolves in $\mathbb{R}^d$ according to $\dot{p}_i = u_i$, for all $i = 1, \ldots, n$, where $u_i \in \mathbb{R}^d$ is the velocity control input. Using the infinitesimally rigid framework, distributed distance-based formation control methods have been widely studied in the literature, see for example [18], [19] and the references therein. In these works, the distance formation error in the $k$-th edge is defined by $e_k = \|z_k\|^l - d_k^l$ with $l \geq 1$ be any positive integer number. Associated to every edge $k$, the distributed control design method amounts to finding a positive definite function $V_k(e_k)$, so that the local gradient-based control law for each agent $i$ is given by $u_i = -\sum_{k\in\mathcal{K}_i} \frac{\partial V_k(e_k)}{\partial p_i}^\top$. In particular, if $V_k = \frac{1}{2}\|e_k\|^2$ is chosen, one can obtain the closed-loop system dynamics of the whole system in the following compact form

$$\begin{aligned} \dot{p} &= -\bar{B} D_z D_{\bar{z}} e = -R(z)^\top D_{\bar{z}} e \\ \dot{e} &= l D_{\bar{z}} D_z^\top \bar{B}^\top \dot{p} = -l D_{\bar{z}} R(z) R(z)^\top D_{\bar{z}} e \end{aligned} \tag{2}$$

where $e, \bar{z} \in \mathbb{R}^{|\mathcal{E}|}$ are the stacked vectors of $e_k$'s and $\|z_k\|^{l-2}$'s respectively, for all $k \in 1, \ldots, |\mathcal{E}|$. The above distance-based formation control law guarantees exponential stability, as studied in [18], [19], [20].

### D. Contraction theory

Consider a time-varying nonlinear system

$$\dot{x} = f(x, t) \tag{3}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state and $f(x, t)$ is a continuously differentiable function. Following the theoretical framework in [21], the system (3) is said to be *contracting* if for any two trajectories $x_1(t)$ and $x_2(t)$ of (3), starting from two different initial states $x_1(0) \neq x_2(0) \in \mathcal{X}$, there exist positive constants $c$ and $\alpha$, such that

$$\|x_1(t) - x_2(t)\| \leq c e^{-\alpha t} \|x_1(0) - x_2(0)\|, \quad \forall t \geq 0. \tag{4}$$

This contraction property can be established via the variational system of (3) given by

$$\delta \dot{x} = \frac{\partial f}{\partial x}(x, t) \delta x \tag{5}$$

where $\delta x$ denotes the variational state.

*Lemma 2.1 (Lohmiller and Slotine in [21]):* If the variational system (5) is uniformly stable in $\mathcal{X}$, then (3) is contracting.

### E. Problem formulation

In this paper, we consider a multi-agent system and a static landmark. In particular, for each agent $A_i$, with $i = 1, \dots, n$, its dynamics is given by

$$A_i : \begin{cases} \dot{p}_i = u_i \\ y_i = \|l^* - p_i\| , \end{cases} \tag{6}$$

where $p_i, u_i \in \mathbb{R}^2$ denote the position of the agent $i$ and its the velocity input, respectively, $l^* \in \mathbb{R}^2$ is the true position of a static landmark (which is unknown to the agent) and $y_i \in \mathbb{R}_+$ is the available local measurement of relative distance between the agent $i$ and the landmark. We define the relative position vector between $A_i$ and the landmark by

$$p_i^l := p_i - l^* = y_i \left[\begin{smallmatrix} \cos\theta_i \\ \sin\theta_i \end{smallmatrix}\right] \tag{7}$$

where $\theta_i \in \mathbb{R}$ is the true relative bearing. By definition, $\theta_i$ is given by

$$\theta_i = \tan^{-1} \frac{[l^* - p_i]_2}{[l^* - p_i]_1} \tag{8}$$

where $[.]_1$ and $[.]_2$ denote the first and second components of a 2D vector, respectively.

In addition, we denote $z_k \in \mathbb{R}^2$ and $e_k \in \mathbb{R}$ to represent the relative position vector and the distance error for the edge $\mathcal{E}_k = (i, j)$, and they are defined by

$$\left.\begin{aligned} z_k &= p_i - p_j \\ e_k &= \|z_k\|^2 - d_k^2 \end{aligned}\right\}$$

where $d_k \in \mathbb{R}$ is the desired distance between agent $i$ and agent $j$ of the edge $\mathcal{E}_k$. Additionally, the stacked vector of the relative positions can be compactly written as

$$z = \bar{B}^\top p. \tag{9}$$

As given in the Introduction, there are two design problems that are simultaneously treated in this paper. Firstly, we design an estimator to reconstruct the relative bearing between each agent $i$ and a static landmark. Secondly, we develop a control law which allows the agents to maintain a distance-based rigid formation shape using the estimates obtained from the above observers. The main advantage of our method is that it only requires range measurements, therefore eliminating the need to deploy bearing sensors, which are used in many current studies [2], [19], [22].

**Distributed Range-Only Formation Control with Bearing Estimation Problem:**

Q1. For each agent $i$, design a distributed estimator

$$\dot{\hat{\theta}}_i = f_i(\hat{\theta}_i, y_i, u_i), \qquad \forall i \in \mathcal{V} \tag{10}$$

with continuously differentiable function $f_i$ such that the bearing estimation error $e_i^\theta := \hat{\theta}_i - \theta_i$ converges to zero as $t \to \infty$.

Q2. For an infinitesimally rigid framework $(\mathbb{G}, p)$, linked to agents with bearing estimator (10), design a distributed control law $u_i = \sum_{(i,j)=\mathcal{E}_k} k_i(\hat{\theta}_i, \hat{\theta}_j, y_i, d_k)$ for all $i \in \mathcal{V}$ such that $e_k \to 0$ as $t \to \infty$ for all $\mathcal{E}_k \in \mathcal{E}$.

## III. Relative bearing estimator for each agent

In this section, we present the design of the distributed bearing estimator for each agent that addresses Q1. Since the convergence of the observer to a single landmark is independent for each agent, we present our first result for the single agent case. Subsequently, we will use this distributed bearing estimator in the context of distributed formation control.

Firstly, with $v$ we denote a unit vector that gives the direction from the agent to the static landmark and with $\hat{v}$ its estimation, where the estimated relative angle is $\hat{\theta}$. Let $w$ be a unit vector orthogonal to $v$, i.e.

$$v = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}; \quad w = \begin{bmatrix} \sin\theta \\ -\cos\theta \end{bmatrix} \tag{11}$$

As before, $\hat{w}$ denotes the estimation of $w$ where $\hat{\theta}$ replaces $\theta$. As in Q1, the estimated angle error is defined as $e^\theta = \theta - \hat{\theta}$. Using these notations, the proposed relative bearing estimator is given by

$$\begin{aligned} \dot{\hat{\theta}} &= \frac{\langle u, \hat{\omega}\rangle}{y} + \gamma \operatorname{sign}(\langle u, \hat{\omega}\rangle)(\dot{y} - \dot{\hat{y}}) \\ \dot{\hat{y}} &= -\langle u, \hat{v}\rangle \end{aligned} \tag{12}$$

where $v$ and $w$ are as in (11), $\gamma$ is a positive estimator gain and $\dot{y}$ is the rate of change of distance.

### A. Contraction analysis

*Proposition 3.1:* Suppose that $u$ and $\hat{\omega}$ are such that there exists $c \in (0, 1)$ satisfying $\| \langle u, \hat{\omega}\rangle \| \geq c \|u\|$. If the estimator gain $\gamma > \frac{1}{cy}$ then (12) is contracting.

Proof. Since the sign function is non-differentiable, the proof concerns two cases. We first analyse the case when $\operatorname{sign}(\langle u, \hat{\omega}\rangle) = 1$ and thus $\langle u, \hat{\omega}\rangle \geq c \|u\|$. The variational system of (12) is given by

$$\delta\dot{\hat{\theta}} = \underbrace{\left(\frac{\langle u, \hat{v}\rangle}{y} - \gamma \langle u, \hat{\omega}\rangle\right)}_{\frac{\partial f}{\partial \hat{\theta}}} \delta\hat{\theta}. \tag{13}$$

Since $\frac{\partial f}{\partial \hat{\theta}}$ is a scalar, in order to apply Lemma 2.1, we need to verify that $\frac{\partial f}{\partial \hat{\theta}} < 0$. Following (13), this implies that we need to verify that $\gamma > \frac{\langle u, \hat{v}\rangle}{y\langle u, \hat{\omega}\rangle}$. Since $-\|u\| \leq \langle u, \hat{v}\rangle \leq \|u\|$, we conclude that

$$\gamma > \frac{1}{cy} \geq \frac{\|u\|}{y\langle u, \hat{\omega}\rangle} \geq \frac{\langle u, \hat{v}\rangle}{y\langle u, \hat{\omega}\rangle} \tag{14}$$

so that $\frac{\partial f}{\partial \hat{\theta}} < 0$. Hence by Lemma 2.1, the estimator (12) is contracting.

Analogously, for the case where $\operatorname{sign}(\langle u, \hat{\omega}\rangle) = -1$, i.e. $\langle u, \hat{\omega}\rangle \leq -c\|u\|$, (13) becomes

$$\delta\dot{\hat{\theta}} = \left(\frac{\langle u, \hat{v}\rangle}{y} + \gamma \langle u, \hat{\omega}\rangle\right) \delta\hat{\theta} \tag{15}$$

Following the same reasoning as above, we can derive that that $\frac{\partial f}{\partial \hat{\theta}} < 0$ is ensured as long as the observer gain $\gamma$ satisfies the following inequality

$$\gamma > \frac{1}{cy} \geq -\frac{\|u\|}{y\langle u, \hat{\omega}\rangle} \geq -\frac{\langle u, \hat{v}\rangle}{y\langle u, \hat{\omega}\rangle} \tag{16}$$

This concludes the proof. □

### B. Convergence analysis

In this subsection, we presents an analysis concerning the conditions under which the contractivity of the estimator (12) implies that the estimated angle error landmark position $l(t)$ converges to the true position $l^*$ as $t \to \infty$. We separate the analysis into two different cases: curved motion and straight motion of the mobile agent.

*Proposition 3.2:* Assume that the hypotheses in Proposition 3.1 hold for certain $c, \gamma > 0$ so that (12) is contracting. If the agent is not moving on a straight trajectory, then the estimated landmark position $l$ converges to the true landmark position $l^*$ as $t \to \infty$, i.e. $e^{\theta}(t) \to 0$.

PROOF. According to the construction of the estimator, one of the admissible trajectories of (12) is the case when $e^{\theta} = 0$. In this case, the corresponding steady-state trajectory $\theta_{ss}(t)$ satisfies

$$\begin{cases} \dot{\theta}_{ss}(t) = \dfrac{\left\langle u(t), \begin{bmatrix} \sin\theta_{ss}(t) \\ -\cos\theta_{ss}(t) \end{bmatrix} \right\rangle}{d(t)} \\ \dot{y}(t) = \left\langle u(t), \begin{bmatrix} \cos\theta_{ss}(t) \\ \sin\theta_{ss}(t) \end{bmatrix} \right\rangle \end{cases} \tag{17}$$

which is invariant, i.e. $l_{ss}(t) = l^*$. By the contraction property of (12), all trajectories converge exponentially to each other, i.e. $\theta(t) \to \theta_{ss}(t)$ as $t \to \infty$ when the initial error $e^{\theta}(0) \neq 0$.

In the following, we will prove by contradiction that the steady-state trajectory $\theta_{ss}(t)$ is unique when the mobile agent is not moving on a straight trajectory. Let $\theta'_{ss}(t) \neq \theta_{ss}(t)$ be another steady-state trajectory satisfying (17) and the trajectory $\theta'_{ss}(t)$ corresponds to another static landmark $l^{**}$. Since the error is zero for both trajectories, this implies that

$$\dot{y}(t) = \left\langle u(t), \begin{bmatrix} \cos\theta_{ss}(t) \\ \sin\theta_{ss}(t) \end{bmatrix} \right\rangle = \left\langle u(t), \begin{bmatrix} \cos\theta'_{ss}(t) \\ \sin\theta'_{ss}(t) \end{bmatrix} \right\rangle. \tag{18}$$

In other words,

$$\left\langle u(t), \begin{bmatrix} \cos\theta_{ss}(t) \\ \sin\theta_{ss}(t) \end{bmatrix} - \begin{bmatrix} \cos\theta'_{ss}(t) \\ \sin\theta'_{ss}(t) \end{bmatrix} \right\rangle = 0 \tag{19}$$

Based on the relationship between the landmark position and the distance measurement, we could obtain

$$\begin{aligned} l^* &= y(t) \begin{bmatrix} \cos\theta_{ss}(t) \\ \sin\theta_{ss}(t) \end{bmatrix} + p(t) \\ l^{**} &= y(t) \begin{bmatrix} \cos\theta'_{ss}(t) \\ \sin\theta'_{ss}(t) \end{bmatrix} + p(t) \end{aligned} \tag{20}$$

According to (19) and (20), we have

$$\langle u(t), l^* - l^{**} \rangle = 0 \tag{21}$$

If $\theta'_{ss}(t) \neq \theta_{ss}(t)$, it is obvious that $l^* \neq l^{**}$. Therefore, (21) is satisfied only when $u(t)$ is orthogonal to the vector $l^* - l^{**}$. In the 2D plane, this means that the estimated static landmark position $l^*$ mirrors another possible static landmark position $l^{**}$ with respect to the axis $u(t)$. In our case, the mobile agent is not moving on a straight line implying that the direction of $u(t)$ will change. Because $l^*$ is static, another possible landmark position $l^{**}$ will change according to the mirror relationship. This leads to a contradiction that $l^{**}$ is static. Thus, we conclude that $\theta_{ss}(t)$ is unique and corresponds to $l^*$, and $l(t)$ converges to $l^*$, i.e. $e^{\theta}(t) \to 0$ with $t \to \infty$. □

*Proposition 3.3:* Assume that the hypotheses Proposition 3.1 hold for certain $c, \gamma > 0$ so that (12) is contracting. If the agent is moving on a straight trajectory, then the estimated landmark position $l$ converges either to the true landmark position $l^*$ or to another point $l^{**}$ (mirrored to $l^*$ with respect to the axis collinear with $u(t)$) as $t \to \infty$.

PROOF. The proof of the proposition follows the same reasoning as the proof of *Proposition* 3.3, where we conclude that there are two distinct points $l^*$ and $l^{**}$, that satisfy (21). This means that the estimated static landmark position $l^*$ mirrors another possible static landmark position $l^{**}$ with respect to the axis $u(t)$. If the agent is moving on a straight trajectory, the axis of $u(t)$ remains unchanged, and we could obtain the other static landmark position $l^{**}$ by the mirror relationship.

Moreover, from geometrical considerations we can conclude that when $l(t)$ converges to the other steady-state point $l^{**}$, then the estimated error converges to $\alpha(t)$ defined as

$$\alpha(t) = 2\cos^{-1}\left(\frac{\langle u(t), l^* - p(t)\rangle}{\|l^* - p(t)\|\|u(t)\|}\right) \tag{22}$$

□

## IV. Range-only formation control with the relative bearing estimator

Equip with the proposed bearing estimator in (12), we can solve Q2. Consider a multi-agent system with a static landmark as in (6). The relative positions between neighbors, the formation framework and distance errors are as discussed in Section II.E.

Recall the definition of the relative position between agent $i$ and the static landmark in (7). By assuming that all the local coordinate frames are aligned with each other, the relative position vector for the $k$-th edge can be given by

$$z_k = p_j^l - p_i^l = y_j \begin{bmatrix} \cos\theta_j \\ \sin\theta_j \end{bmatrix} - y_i \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \end{bmatrix} = \|z_k\| \begin{bmatrix} \cos\theta_k^l \\ \sin\theta_k^l \end{bmatrix} \tag{23}$$

where $\theta_k^l$ denotes the angle of relative position vector for the $k$-th edge. Therefore, by deploying (12) to reconstruct the bearing information, the estimated states can be given by

$$\begin{aligned} \hat{p}_i^l &= y_i \begin{bmatrix} \cos\hat{\theta}_i \\ \sin\hat{\theta}_i \end{bmatrix} \\ \hat{z}_k^l &= \hat{p}_j^l - \hat{p}_i^l = y_j \begin{bmatrix} \cos\hat{\theta}_j \\ \sin\hat{\theta}_j \end{bmatrix} - y_i \begin{bmatrix} \cos\hat{\theta}_i \\ \sin\hat{\theta}_i \end{bmatrix} \\ &= \|z_k\| \begin{bmatrix} \cos\hat{\theta}_k^l \\ \sin\hat{\theta}_k^l \end{bmatrix} \end{aligned} \tag{24}$$

where $\hat{p}_i^l$ denotes the estimated positions between each agent and the landmark, while $\hat{z}_k^l$ is the relative position vector for the edge $k$. It is obvious that when both $\hat{\theta}_i$ and $\hat{\theta}_j$ converge to $\theta_i$ and $\theta_j$ respectively, then $\hat{\theta}_k^l$ converges to $\theta_k^l$.

In general, the standard distance-based control law is designed as follows

$$\dot{p} = -\bar{B} D_z D_{\tilde{z}} e \tag{25}$$

where $e, \tilde{z} \in \mathbb{R}^{|\mathcal{E}|}$ are the stacked vectors of $e_k$'s and $\|z_k\|^{m-2}$'s respectively, for all $k \in 1, \ldots, |\mathcal{E}|$. In our formation control law with bearing estimator, we substitute $z$ with $\hat{z}^l$ so that the above law becomes

$$\dot{p} = -\bar{B} D_{\hat{z}^l} D_{\tilde{z}} e \tag{26}$$

The dynamics of the distance error for the overall closed-loop system can thus be derived as follows:

$$\dot{e} = -\frac{1}{m} D_{\tilde{z}} D_z^\top \bar{B}^\top \bar{B} D_{\hat{z}^l} D_{\tilde{z}} e \tag{27}$$

*Proposition 4.1:* Consider the closed-loop system (27) with graph $\mathbb{G}$ under the distributed control law (26). If the agents do not move in a straight line and the formation in the desired shape is infinitesimally and minimally rigid, then the system is locally asymptotically stable.

PROOF. First, to simplify the analysis, let us rewrite (2) as

$$\begin{aligned} \dot{e} &= -\frac{1}{m} D_{\tilde{z}} D_z^\top \bar{B}^\top \bar{B} D_{\hat{z}^l} D_{\tilde{z}} e \\ &= -\frac{1}{m} D_{\tilde{z}} R(z) R(z)^\top D_{\tilde{z}} e + \frac{1}{m} P(z) D_{\hat{z}^l - z} D_{\tilde{z}} e \end{aligned} \tag{28}$$

where $R(z) = D_z^\top \bar{B}^\top$, $P(z) = D_{\tilde{z}} D_z^\top \bar{B}^\top \bar{B}$ and $D_{\hat{z}^l - z}$ is a diagonal matrix with diagonal elements $\|z_k\| \begin{bmatrix} \cos\hat{\theta}_k^l - \cos\theta_k^l \\ \sin\hat{\theta}_k^l - \sin\theta_k^l \end{bmatrix}$. According to [23], both the vector $\tilde{z}$ and the matrix $R(z)R(z)^\top$ can be rewritten as a smooth function of $e$ and it is positive-definite in the neighborhood of $e = 0$.

To prove local stability of the equilibrium point $e = 0$ of (27), we choose $V = \frac{1}{2m}\|e\|^2$ as a Lyapunov function, which satisfies

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= -e^\top R(z) R(z)^\top e + e^\top P(z) D_{\hat{z}^l - z} D_{\tilde{z}} e \\ &\le -\lambda_{\min} \|e\|^2 + \delta_{\max} \|P(z)\|_2 \|D_{\tilde{z}}\|_2 \|e\|^2 \end{aligned} \tag{29}$$

where $\|P(z)\|_2$, $\|D_{\tilde{z}}\|_2$ are smooth functions of $e$ and $\delta_{\max} = \max\left\{\|z_k\|\sqrt{(\cos\hat{\theta}_k^l - \cos\theta_k^l)^2 + (\sin\hat{\theta}_k^l - \sin\theta_k^l)^2}\right\}$, representing the maximum singular value of $D_{\hat{z}^l - z}$. According to Proposition 3.1 and Proposition 3.2, $\hat{\theta}_k^l$ will converge to $\theta_k^l$ exponentially fast. Hence, $\delta_{\max}(t) \to 0$ exponentially fast as $t \to \infty$. Consequently, there exists a $\tau > 0$ such that when $t > \tau$ then $\delta_{\max} < \frac{\lambda_{\min}}{\|P(z)\|_2 \|D_{\tilde{z}}\|_2}$, so that $\dot{V}(t) < 0$ for all $t > \tau$. Therefore, we can conclude that the closed-loop system (27) is locally asymptotically stable. □

*Remark 4.1:* In most cases, the distributed formation control law will result in non-straight trajectories. The initial conditions, where the agents' motion in a formation control is a linear one, correspond to the scaled version of the desired shape. Therefore, the assumption of agents not moving in a straight line in Proposition 4.1 is a mild one.

*Remark 4.2:* In Proposition 4.1, we show the convergence of the agents to the desired shape without having the convergence of the bearing estimators to the real bearing. In other words, it is possible that the bearing estimators reach a steady-state when the desired formation has been reached. This can also be seen from the foregoing stability analysis, where when $\delta_{\max} < \frac{\lambda_{\min}}{\|P(z)\|_2 \|D_{\tilde{z}}\|_2}$ has been satisfied because $\delta_{\max}$ has become sufficiently small, then we have $\dot{V}(t) < 0$ for the rest of the time. This result shows that the convergence of the bearing estimator to the real bearing measurement for each agent is not necessary for the multi-agent system to converge to the desired formation. Furthermore, according to Proposition 3.1, the convergence of the bearing estimators can be guaranteed through group maneuvers once the desired formation shape has been achieved.

## V. Numerical simulations

### A. Simulation setup

For the simulation setup, we consider an *infinitesimally and minimally* formation of three agents with incidence matrix $B$ given by $B = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$. The three agents and the static landmark are initialized at $p_1(0) = \begin{bmatrix} 10.85 \\ -2.25 \end{bmatrix}, p_2(0) = \begin{bmatrix} 3.61 \\ -17.40 \end{bmatrix}, p_3(0) = \begin{bmatrix} 16.98 \\ 4.09 \end{bmatrix}, l^* = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$. Additionally, the estimated angles between each agent and the static landmark are initialized as $\left\{\hat{\theta}_1(0) = \frac{\pi}{3}, \hat{\theta}_2(0) = \frac{7\pi}{36}, \hat{\theta}_3(0) = \frac{\pi}{6}\right\}$. For the desired formation shape, the desired distance set for the three edges is set as $\{d_1 = 10, d_2 = 10, d_3 = 10\}$. The numerical simulations were conducted using MATLAB 2023a.

### B. Simulation results

Let us denote $e_k = \|z_k\|^2 - d_k^2$ as the distance error for $k$-th edge where $k \in \{1, 2, 3\}$, and denote $e_i^\theta = \hat{\theta}_i^l - \theta_i^l$ as the estimated angular error between $i$-th agent and the landmark, with $i \in \{1, 2, 3\}$. Figure 1 shows the trajectories of the three agents under the proposed distributed control law, the true landmark position, and the estimated positions of the landmark for each agent. In the figure, the agents are able to maintain the desired triangular shape, while estimating the real position of the landmark. The distance errors of the edges and the estimated angle errors for each agent are presented in Figure 2 and 3, respectively. Figure 2 illustrates that all distance errors converge to zero. According to Figure 3, the estimated angle errors are bounded reflecting the convergence property emphasized in Remark 4.2.

## VI. Conclusion

In this paper, we have proposed a range-only distributed formation control law for a multi-agent system. To obtain accurate relative bearing estimates for each agent, a continuous-time bearing estimator has been designed. By applying results from contraction theory, we establish the exponential convergence of the proposed observer. In addition, the asymptotic stability analysis for the closed-loop

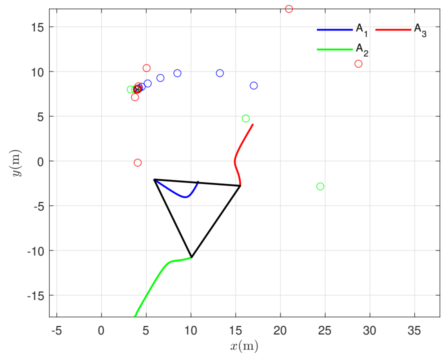


Fig. 1: Trajectories and estimated positions of the landmark for the three agents with the distributed control law; (■, ■, ■) = (agent1, agent2, agent3), × represents the real position of the landmark and ∘ the estimated positions of the landmark by the agents.

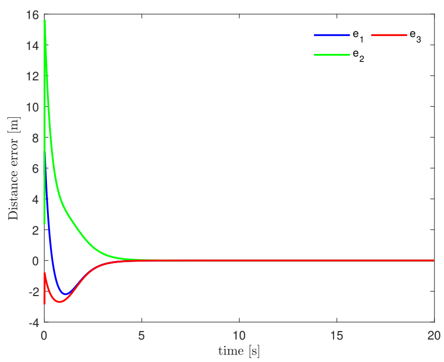


Fig. 2: Distance error trajectories $e_k(t)$ for each edge $k$ of the multi-agent formation.

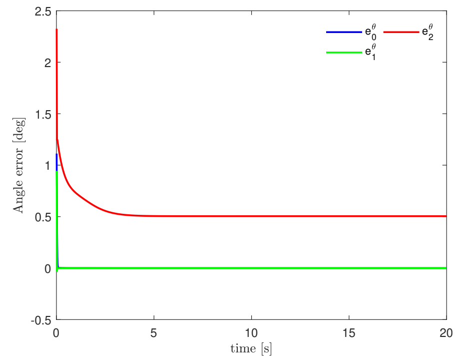


Fig. 3: Estimation angle error trajectories $e_i^\theta(t)$ of the relative bearing between each agent $i$ and the static landmark.

multi-agent system is presented, ensuring that the agents are able to achieve the desired rigid formation shape. Numerical simulation are carried out to validate our theoretical results.

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