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Direct Adaptive Control of LTV Discrete-time Systems with Uncertain Periodic Coefficients*

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***Abstract*— The paper considers the problem of the state-feedback adaptive control of a class of discrete linear time-varying plant with multisinusoidal parameters. The plant is representable in the strict-feedback form, in which the parameters are on the main diagonal of the state matrix. The amplitudes, frequencies, and the phases of the multisinusoids are not known *a priori*, however the maximum numbers of sinusoids (harmonics) in each parameter are available. The problem is solved by design of periodic parameters observers recovering the parameters dynamics, the plant parameterization, and developing a discrete-time adaptive backstepping controller. In order to tune the controller parameters and improve the closed-loop transient performance, an adaptation algorithm with memory regressor extension is applied. The main results of the paper are demonstrated by simulation.**

I. Introduction

The extension of the adaptive control theory to systems with time-varying (TV) parameters is highly challenging and may require nontrivial solutions. At the same time, successful results obtained for uncertain TV systems make it possible to cover a wider class of control systems in various practical applications. In order to compensate the deteriorating effect of variations of uncertain parameters, it is necessary to use some prior information about these variations. If it is just assumed that the uncertain TV parameters are arbitrary but bounded, then robust modifications (e.g., with switching leakage factor or projection) of adaptation algorithms [1], [2], [3], [4, Section 4.1] can be applied to tune the controller parameters. In this case, due to the lack of information about parameters variations, such solutions do not allow to draw control errors to zero. If the uncertain TV parameters are not arbitrary but assumed to be multisinusoidal [5], [6], [7], [8] or polynomial functions of time [9], [10], then the effect of parameters variations can be completely compensated by representing the TV parameters as the outputs of the exosystems and then by suitable integrating the structure, variable, and the parameters (or their estimates) of these models into the structure of controller.

Motivated by numerous practical applications [5], [11], [12], [13], in this paper we consider the problem of adaptive control for a class of discrete linear time-varying (LTV) systems with uncertain multisinusoidal parameters. One of the widely used models with periodic coefficients describing electrical, mechanical (including vibromechanical), hydrodynamical systems, systems with frequency modulation and many others can be represented by the forced Hill's/Mathieu's equation (see [11], [13], [14], [12], [15])

$$y(k) + (\alpha + \beta\psi(k))y(k-1) + y(k-2) = u(k) \qquad (1)$$

with the periodic sequence $\psi(k)$ and constants α and β. Assuming that $\psi(k)$ is representable in the form of truncated Fourier series $\psi(k) = \sum_{i=1}^{N} a_i \sin(\omega_i + \phi_i)$, while $y(k)$ and $y(k-1)$ are measurable, this model belongs to the class of models considered in this paper.

As the literature survey shows (see [5], [6], [12], [16], [17]), the majority of approaches developed mostly for continuous-time case accepts the assumption about unknown amplitudes and phases but known frequencies of the periodic parameters or at least some prior information about the values of these frequencies. In this paper, we relax this assumption by developing TV parameters observers allowing to recover all information about parameters variations.

The paper considers the class of LTV systems representable in the discrete *strict-feedback form*, therefore for design of adaptive controller we develop a scheme of discrete-time backstepping procedure inspired by continuous case that was recently presented in [18] and by the pioneering work [19]. Apart from this, the discrete-time backstepping is a powerful tool in many practical applications [20], [21], [22].

Following to [18], in order to improve transient performance of the closed-loop systems we include the predicted values of the adjustable controller parameters $\hat{\vartheta}(k+i)$ generated by the adaptation algorithm with *memory regressor extension* (MRE) [23]. As shown in [23], the effect of MRE allows us to accelerate the controller parameters tuning and, as a result, to provide rapid tracking error convergence.

Thus, the contribution of the paper consists in design of direct adaptive control for a class of discrete-time LTV systems representable in the strict-feedback form with the following distinguishing features:

- to cope with uncertain amplitudes, phase, and frequencies of harmonics of the multisinusoidal TV parameters, we propose observers that enable parameterization of the plant and further application of the certainty equivalent adaptive control;
- using the parameterized plant, a new iterative procedure of discrete-time adaptive backstepping is proposed;
- discrete-time adaptation algorithm with MRE is introduced to improve the transient performance of the closed-loop system and calculate the predicted values of

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the adjustable parameters necessary for implementation of the backstepping controller.

The proposed solution is motivated by adaptive controller recently presented for continuous-time system in [24]. However, as the experience of analytical computations showed, the discrete solution is quite different from the continuous one especially at the backstepping stage.

Notations: $\mathbb{R}$ ($\mathbb{R}^n$ or $\mathbb{R}^{n\times n}$) is the space of real numbers (n-dimensional vectors or $n\times n$-dimensional matrices); $\mathbb{R}_+$ is the space of real positive numbers; $\mathbb{N}$ is the space of natural numbers including zero; I is an identity matrix; $\mathrm{diag}\{x_i\}$ is the diagonal matrix; $|x|$ is Euclidean norm of a vector x; $||x||_\infty$ is the infinity norm of a vector x; $k \in \mathbb{N}$ is discrete-time step; z is the shift forward operator.

II. Problem Statement

Consider a discrete-time scalar system given by

$$x_i(k+1) = \psi_i(k)x_i(k) + x_{i+1}(k), \tag{2}$$
$$x_n(k+1) = \psi_n(k)x_n(k) + u(k), \tag{3}$$
$$y(k) = x_1(k), \tag{4}$$

where $i = 1, 2, \ldots, n-1$, $x_j \in \mathbb{R}$ ($j = 1, 2, \ldots, n$) are the elements of the state measurable vector $x = [x_1, x_2, \ldots, x_n]^\top$ with the initial condition $x_j(0)$; $y \in \mathbb{R}$ is the plant output, $u \in \mathbb{R}$ is the control signal, $\psi_j(k)$ are the unknown parameters given by the functions

$$\psi_j(k) = \sum_{i=1}^{N_j} a_{j,i} \sin(\omega_{j,i} kT + \varphi_{j,i}), \tag{5}$$

with a priori unknown constant amplitudes $a_{j,i}$, phases $\varphi_{j,i}$, and frequencies $\omega_{j,i}$ of harmonics, however known discrete time interval T and maximum numbers of harmonics N_j [1].

We make the following assumptions regarding the plant.

Assumption 1: The parameters $\psi_j(k)$ are the outputs of the *exosystems*

$$z_j(k+1) = \Gamma_j z_j(k), \quad z_j(0), \tag{6}$$
$$\psi_j(k) = h_j^\top z_j(k), \tag{7}$$

where $z_j \in \mathbb{R}^{m_j}$ are the inaccessible for measurement states of the exosystems, $\Gamma_j \in \mathbb{R}^{m_j \times m_j}$ are the unknown matrices, $h_j \in \mathbb{R}^{m_j}$ are the unknown vectors. ■

The objective is to design a control law that will provide the boundedness of all the closed-loop signals and:

O1. if for all $i = 1, 2, \ldots, n$, some $N_0 \in \mathbb{N}$ and for all $k \geq N_0$, the inequalities $|x_i(k)| \geq x_{0i}$ hold, where $x_{0i} \in \mathbb{R}_+$ are the selected thresholds, then

$$\lim_{k\to\infty} \varepsilon(k) = \lim_{k\to\infty} (y_m(k) - y(k)) = 0. \tag{8}$$

where $y_m(k)$ is a bounded reference signal with known next step n values;

O2. otherwise

$$\lim_{k\to\infty} |\varepsilon(k)| \leq \Delta, \tag{9}$$

where $\Delta \in \mathbb{R}_+$ is a maximum steady-state error, the value of which can be reduced by changing design parameters. ■

[1] Hereafter, we will omit the dependence from k for the sake of brevity and if is not in contrary to the context.

III. Periodic Parameters Observers

To overcome the obstacle of uncertain LTV parameters, we represent the exosystems (6) in the canonical form (see continuous-time representation in [25, Chapter 2])

$$\xi_j(k+1) = G_j \xi_j(k) + l_j \psi_j(k), \tag{10}$$
$$\psi_j(k) = \theta_j^\top \xi_j(k) \tag{11}$$

with the states $\xi_j \in \mathbb{R}^{m_j}$, some initial conditions $\xi_j(0)$, $m_j \times m_j$ Hurwitz matrices

$$G_j = \begin{bmatrix} O_{(m_j-1)\times 1} & I_{(m_j-1)} \\ 0 & O_{1\times(m_j-1)} \end{bmatrix}$$

m_j-dimensional vectors $l_j = [0, 0, \ldots, 0, 1]^\top$ selected so that the pairs (G_j, l_j) are controllable, and the vectors of unknown parameters $\theta_j \in \mathbb{R}^{m_j}$ dependent from the unknown frequencies $\omega_{j,i}$ ($i = 1, 2, \ldots, N_j$).

Replacing (11) in (10), we obtain the autonomous form of exosystem

$$\xi_j(k+1) = \overline{G}_j \xi_j(k), \tag{12}$$

where $\overline{G}_i = G_i + l_i\theta_i^\top$. It can be shown that the form (12), (11), is equivalent to the form (6), (7).

Using the results presented in [24] for continuous-time systems, in a similar way we obtain the TV parameters observers given by

$$\hat{\xi}_i(k) = \zeta_i(k) + N_i(k-1)x_i(k), \tag{13}$$
$$\zeta_i(k+1) = G_i\zeta_i(k) + G_i N_i(k-1)x_i(k) - N_i(k)x_{i+1}(k), \; i = 1, 2, \ldots, n-1, \tag{14}$$
$$\hat{\xi}_n(k) = \zeta_n(k) + N_n(k-1)x_n(k), \tag{15}$$
$$\zeta_n(k+1) = G_n\zeta_n(k) + G_n N_n(k-1)x_n(k) - N_n(k)u(k), \tag{16}$$

where $\hat{\xi}_j$ is the estimate of ξ_j, $j = 1, 2, \ldots, n$,

$$N_j(k) = N_j(x_j(k), \sigma_j(k)) = l_j \frac{x_j(k)}{x_j^2(k) + \sigma_j^2(k)} \tag{17}$$

are the nonlinear functions with σ_j given by

$$\sigma_j(k) = \begin{cases} 0 & \text{if } |x_j(k)| \geq x_{0j}, \\ \sigma_{0j} & \text{otherwise}, \end{cases} \tag{18}$$

$x_{0j}, \sigma_{0j} > 0$ are some constant design parameters.

The observers have the following properties.

Lemma 1: The observers (13)–(16) generate the estimates $\hat{\xi}_j(k)$ that for any initial conditions $\xi_j(0)$, $\zeta_j(0)$

$$\epsilon_j(k) \triangleq \xi_j(k) - \hat{\xi}_j(t) = G_j^k \epsilon_j(0) + (zI - G_j)^{-1} l_i \left[\frac{\sigma_j^2(k)}{x_j^2(k) + \sigma_j^2(k)} \psi_j(k) \right]. \tag{19}$$

The norms $||\epsilon_j(k)||_\infty$ are bounded and can be reduced by reduction of the corresponding gains σ_{0j} and/or x_{0j}. ■

The lemma is proved by calculation of the next step values $\epsilon_j(k+1)$ in view of (19), (10), (13)-(18), and (2)-(4).

In order to design a control law, we will need to get the predicted values of a regressor containing the vectors $\hat{\xi}_i$. To this end, we formulate the following corollary of Lemma 1.

Corollary 1: The estimates $\hat{\xi}_j(k)$ $(j = 1, 2, \ldots, n)$ can be represented in the form

$$\hat{\xi}_j(k) = \overline{G}_j^{n-1} \hat{\xi}_j(k-n+1) + \bar{\epsilon}_j(k), \tag{20}$$

where $\bar{\epsilon}_j(k) = \overline{G}_j^{n-1} \epsilon_j(k-n+1) - \epsilon_j(k)$, $\overline{G}_i$ are the matrices defined in (12). ■

Applying the result of Lemma 1 and Corollary 1 and introducing new notations

$$\vartheta = [\theta_1^\top \overline{G}_1^{n-1}, \theta_2^\top \overline{G}_2^{n-1}, \ldots, \theta_n^\top \overline{G}_n^{n-1}]^\top \in \mathbb{R}^{\sum_{i=1}^{n} m_i},$$
$$\varpi_i = [O_{m_1}^\top, \ldots, O_{m_{i-1}}^\top, \hat{\xi}_i^\top(k-n+1), O_{m_{i+1}}^\top, \ldots, O_{m_n}^\top]^\top,$$

we represent the TV plant parameters in the form

$$\psi_i(k) = \vartheta^\top \varpi_i(k) + \hat{\epsilon}_i(k)$$

with $\hat{\epsilon}_i = \theta_i^\top(\bar{\epsilon}_i + \epsilon_i)$.

As a result, the plant (2)–(4) can be parameterized in the form

$$\begin{cases} x_i(k+1) = \vartheta^\top \varpi_i(k) x_i + \hat{\epsilon}_i(k) x_i + x_{i+1}, \\ x_n(k+1) = \vartheta^\top \varpi_n(k) x_n + \hat{\epsilon}_n(k) x_n + u, \\ y = x_1, \end{cases} \tag{21}$$

where $i = 1, 2, \ldots, n-1$.

Using the form (21), in the next section we propose the adaptive controller.

IV. Control Law

As it is seen from the structure of (21), the uncertainties collected in the vector ϑ are not matched with the control signal u and cannot be compensated directly. To overcome this problem, we apply an iterative procedure of adaptive backstepping. To this end, we introduce the errors $z_1 = y_m - y = y_m - x_1$, $z_i = \alpha_{i-1} - x_i$ $(i = 2, 3, \ldots, n)$, where α_1, α_2,...,α_{n-1} are the virtual control laws given by

$$\alpha_1 = -c_1 z_1(k) - x_1(k)\varpi_1^\top(k)\hat{\vartheta}(k) + y_m(k+1), \tag{22}$$

$$\begin{aligned} \alpha_i &= -c_i z_i(k) \\ &+ \left(\sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(k+1) x_j(k) \varpi_j^\top(k) - x_i(k)\varpi_i^\top(k) \right) \hat{\vartheta}(k) \\ &+ \sum_{j=1}^{i-1} \frac{\partial \alpha_{i-1}}{\partial x_j}(k+1) x_{j+1}(k) + \\ &+ \sum_{j=0}^{i-2} \frac{\partial \alpha_{i-1}}{\partial y_m(k+j)}(k) y_m(k+j+1) + y_m(k+n), \end{aligned} \tag{23}$$

where $i = 2, 3, \ldots, n-1$, $c_j:\ |c_j| < 1$ $(i = 1, 2, \ldots, n-1)$ are constant design parameters, $\hat{\vartheta} \in \mathbb{R}^{\sum_{i=1}^{n} m_i}$ is the vector of adjustable parameters corresponding to the vector ϑ.

The procedure is finished by design of the actual control

$$u(k) = \alpha_n(k), \tag{24}$$

where α_n is calculated using (23) with $i = n$.

Remark 1: Here and hereafter, the partial derivatives $\frac{\partial \alpha_{i-1}}{\partial x_j}$ and $\frac{\partial \alpha_{i-1}}{\partial y_m(k+j)}$ are used only for compact representation of the resulting control law. This representation is admissible due to the linear dependence of the virtual control laws from the variables x_j and $y_m(k+j)$. ■

Remark 2: The control law (22)–(24) depends on the predicted values $\varpi_j(k+i)$ that are implementable and the predicted adjustable parameters $\hat{\vartheta}(k+i)$ $(i = 1, 2, \ldots, n-1$, $j = 1, 2, \ldots, n)$ that are not measurable, however can be recovered by the adaptation algorithm proposed below. ■

Lemma 2: The error model for the closed-loop system consisting of the parameterized plant (21) and the control law (22)–(24) can be represented as

$$z(k+1) = Az(k) + B(k)\tilde{\vartheta}(k) + D(k)E_z(k)x(k), \tag{25}$$

where $E_z(k) = diag\{\hat{\epsilon}_i(k)\}$ $(i = 1, 2 \ldots, n)$,

$$A = \begin{bmatrix} c_1 & 1 & \cdots & 0 & 0 \\ 0 & c_2 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \ddots & c_{n-1} & 1 \\ 0 & 0 & \cdots & 0 & c_n \end{bmatrix},$$

$$D(k) = \begin{bmatrix} -1 & 0 & \cdots & 0 & 0 \\ \alpha_1^1(k) & -1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \alpha_1^{n-1}(k) & \alpha_2^{n-1}(k) & \cdots & \alpha_{n-1}^{n-1}(k) & -1 \end{bmatrix},$$
$$B(k) = D(k)X(k)W(k),$$

$$W(k) = [\ \varpi_1(k)\ \vdots\ \varpi_2(k)\ \vdots\ \cdots\ \vdots\ \varpi_n(k)\ ]^\top,$$

$\alpha_j^i(k) = \frac{\partial \alpha_i}{\partial x_j}(k+1)$, $X = diag\{x_i\}$. ■

In (25), $D(k) = D(\hat{\vartheta}(k), \ldots, \hat{\vartheta}(k+n-1), \varpi_1(k), \ldots, \varpi_1(k+n-1), \varpi_2(k), \ldots, \varpi_2(k+n-2), \ldots, \varpi_{n-1}(k), \varpi_{n-1}(k+1), \varpi_n(k))$.

The lemma can be proved by evaluation of the next step values $z_1(k+1) = y_m(k+1) - y(k+1)$, $z_i(k+1) = \alpha_{i-1}(k+1) - x_i(k+1)$ in view of (21) and (22)–(24).

By the following lemma we demonstrate the crucial property *input-to-state stability* (ISS) of the model (25).

Lemma 3 (the ISS property): Let $B(k)\tilde{\vartheta}(k)$ be considered as the input of the model (25), while z is its state. Then the model (25) is ISS. ■

Proof: Taking into account the linear dependence of α_i $(i = 1, 2, \ldots, n-1)$ from x_j $(j = 1, 2, \ldots, i)$, from definitions of z_1, z_i we have

$$z(k) = K(k)x(k) - x(k) + \bar{y}_m(k) \tag{26}$$

where $\bar{y}_m(k) = \bar{y}_m(y_m(k), y_m(k+1), \ldots, y_m(k+n))$, $K(k) = K(\hat{\vartheta}(k), \ldots, \hat{\vartheta}(k+n-1), \varpi_1(k), \ldots, \varpi_1(k+$

$n-1), \varpi_2(k), \dots, \varpi_2(k+n-2), \dots, \varpi_{n-1}(k), \varpi_{n-1}(k+1), \varpi_n(k))$ is $n \times n$ matrix defined as

$$K(k) = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ k_{2,1}(k) & 0 & \dots & 0 & 0 \\ k_{3,1}(k) & k_{3,2}(k) & \dots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ k_{n,1}(k) & k_{n,2}(k) & \dots & k_{n,n-1}(k) & 0 \end{bmatrix}.$$

The functions $\bar{y}_m(k)$, $K(k)$ are bounded by definition.

Proceeding with (26), we have

$$z(k) = (K(k) - I)x(k) + \bar{y}_m(k).$$

As a result, (25) is representable in the form

$$\begin{aligned} z(k+1) = \bar{A}(k)z(k) + B(k)\tilde{\vartheta} \\ - D(k)E_z(k)(K(k)-I)^{-1}\bar{y}_m(k), \end{aligned} \tag{27}$$

in which $\bar{A}(k) = \left(A - D(k)E_z(k)(K(k)-I)^{-1}\right)$, while the inverse matrix $(K(k) - I)^{-1}$ exists for any $k \in \mathbb{N}$ ($det(I - K(k)) = (-1)^n \neq 0$).

Since all the elements of $E_z(k) = diag\{\hat{\epsilon}_i(k)\}$ are inverse proportional to the corresponding squares $x_i^2(k)$ (see Lemma 1 and the definitions of $\hat{\epsilon}_i(k)$ and $\bar{\epsilon}_i(k)$), then $D(k)E_z(k)(K(k)-I)^{-1}\bar{y}_m(k)$ is bounded and

$$\lim_{|x|\to\infty} \bar{A}(x,k) = \lim_{|z|\to\infty} \bar{A}((K(k)-I)^{-1}(z-\bar{y}_m(k)), k) = A.$$

As a result, there exists such sufficiently large value of $||z(k)||_\infty < \infty$ that the system (27) is asymptotically stable due to stability of the matrix A. This completes the proof. ■

Remark 3: As seen from the proof of Lemma 3, in the considered discrete-time case the role of nonlinear damping terms (see [26]) are played by nonlinear functions $\epsilon_j(k)$ given by (19) and suitably involved into the matrix function $E_z(k)$ in (25). ■

V. Adaptation algorithm

To generate the vector of adjustable parameters $\hat{\vartheta}(k)$ and its predicted estimates $\hat{\vartheta}(k+1), \hat{\vartheta}(k+2), \dots, \hat{\vartheta}(k+n-1)$ and then achieve the objective (8), (9), we need to represent the model (25) in a static form. To this end, we formulate the swapping lemma.

Lemma 4: Let us introduce the swapping filters

$$\zeta_z(k+1) = A\zeta_z(k) + B(k)\hat{\vartheta}(k), \tag{28}$$

$$Z(k+1) = AZ(k) + B(k) \tag{29}$$

and the *augmented error*

$$\bar{z}(k) = z(k) + \zeta_z(k). \tag{30}$$

Then the vector $\bar{z}$ can be represented as the output of the static error model

$$\bar{z}(k) = Z(k)\vartheta + \bar{\epsilon}_z(k), \tag{31}$$

where $\bar{\epsilon}_z$ is the vector function satisfying the equation

$$\bar{\epsilon}_z(k+1) = A\bar{\epsilon}_z(k) + D(k)E_z(k)x(k) \tag{32}$$

with the initial condition $\bar{\epsilon}_z(0) = z(0) + \zeta_z(0) - Z(0)\vartheta$. ■

The lemma is proved by evaluation of the next step value $\bar{\epsilon}_z(k+1) = z(k+1) + \zeta_z(k+1) - Z(k+1)\vartheta$ in view of (25), (28), and (29).

Corollary 2: If the conditions of the problem objective *O1* hold, then the term $\bar{\epsilon}_z(k)$ exponentially tends to zero as $k \to \infty$. Otherwise, it is bounded for any $k \geq 0$. ■

The corollary is followed from (32), definitions of ϵ_i and $\hat{\epsilon}_i$, and expressions (18), (19).

To design an adaptation algorithm generating $\hat{\vartheta}(k)$ together with $\hat{\vartheta}(k+1), \hat{\vartheta}(k+2), \dots, \hat{\vartheta}(k+n-1)$, we use the result presented in [23] and modified for discrete systems. To obtain the algorithm, we left-multipy (31) by $Z^\top$. Then we introduce an asymptotically stable transfer function

$$L(z) := \prod_{i=1}^{n-1} \frac{1-d_i}{z-d_i} \tag{33}$$

so that $||L(z)||_\infty < 1$, where $0 \leq d_i < 1$ are the function poles, and applying it to the regression (31) obtain the *memory extended regression* model [2]

$$Y(k) = \Omega(k)\vartheta + \epsilon_L(k) \tag{34}$$

with the memory extended output $Y = L(z)\left[Z^\top \bar{z}\right]$, the memory extended regressor matrix $\Omega = L(z)\left[Z^\top Z\right]$, $\epsilon_L = L(z)[Z^\top \bar{\epsilon}_z]$[3].

Extended regression (34) allows us to apply robust adaptation algorithm with memory regressor extension [4]

$$\hat{\vartheta}(k+1) = (1 - \bar{\sigma}(k))\hat{\vartheta}(k) + \gamma R^{-1}(k)\Omega(k)E(k), \tag{35}$$

where $E := Y - \Omega\hat{\vartheta} = \Omega\tilde{\vartheta} + \epsilon_L$ is the memory extended error, $\gamma \in (0,1)$ is the adaptation gain, $\bar{\sigma}(k)$ is the leakage factor defined as

$$\bar{\sigma}(k) = L(z)[\sigma_\theta(\hat{\vartheta}(k))], \tag{36}$$

$$\sigma_\theta(\hat{\vartheta}) = \begin{cases} 0 & \text{if } |\hat{\vartheta}| < \vartheta^*, \\ \sigma_{\theta 0} & \text{if } |\hat{\vartheta}| \geq \vartheta^*, \end{cases} \tag{37}$$

$\vartheta^* \geq 2|\vartheta|$ is a large enough constant, $R = \rho I_n + \Omega^2$ is the normalization matrix, $\rho \in \mathbb{R}_+$ is some small constant, $\sigma_{\theta 0} \in [0, \frac{1}{2}(1-\gamma))$ is a constant parameter.

Due to the choice of the transfer function $L(z)$, we are able to calculate the predicted values $\hat{\vartheta}(k+l+1)$ ($l = 0, 1, \dots, n-1$) via the following recursive procedure:

$$\begin{aligned} \hat{\vartheta}(k+l+1) = (1 - \bar{\sigma}(k+l))\hat{\vartheta}(k+l) \\ + \gamma R^{-1}(k+l)\Omega(k+l)E(k+l), \end{aligned} \tag{38}$$

[2]The term "memory regressor extension" was taken from [23], where the improvement of the adaptation process was achieved by using past values of the regressor that are collected by the filter $L(z)$.

[3]If $d_i = 0$, the algorithm becomes $(n\quad 1)$-step-ahead predictor considered in [27].

[4]Extended form of robust σ-modification of the gradient adaptation algorithm that can be found, e.g. in [28], [29, Section 3.8]

in which $E(k+l) = Y(k+l) - \Omega(k+l)\hat{\vartheta}(k+l)$,

$$\bar{\sigma}(k+l) = \frac{d(1)z^l}{d(z)}[\sigma_\theta(\hat{\vartheta}(k))],\ R(k+l) = \rho I_n + \Omega^2(k+l),$$

$$Y(k+l) = \frac{d(1)z^l}{d(z)}\left[Z^\top(k)\bar{z}(k)\right],$$

$$\Omega(k+l) = \frac{d(1)z^l}{d(z)}\left[Z^\top(k)Z(k)\right].$$

After replacement of (34) in (35), we obtain the following parametric error model:

$$\tilde{\vartheta}(k+1) = \tilde{\vartheta}(k) + \bar{\sigma}(k)\hat{\vartheta}(k) - \gamma R^{-1}(k)\Omega(k)E(k). \quad (39)$$

Now, formulate the main result of the paper.

Proposition 1: Under Assumption 1, The adjustable backstepping controller (22)–(24) together with the TV parameters observers (13)–(16), the swapping filters (28), (29), the augmented error (30), and the robust adaptation algorithm with MRE (35)–(38) ensures the following properties of the closed-loop system:

P1.1 the boundedness of all the signals in the system;

P1.2 control objective in accordance with (8) and (9). The maximum steady-state error Δ can be decreased by decreasing $\max\limits_{j=1,2,\ldots,n}\{\sigma_{0j}\}$;

P1.3 if the conditions of the objective O1 are satisfied, and

$$\Omega^2(k) \succeq \lambda_\Omega(k)I_n \quad (40)$$

holds for some positive function $\lambda_\Omega(k) \notin \mathcal{L}_1$[5], then $|\tilde{\vartheta}(k)|$ approaches zero asymptotically as $k \to \infty$;

P1.4 if the conditions of the objective O1 are satisfied, $Z \in$ PE, then $|\tilde{\vartheta}(k)|$ converges to zero exponentially fast as $k \to \infty$, and the rate of this convergence can be increased by increasing γ. □□□

Proof: Select the Lyapunov function candidate $V(k) = \frac{1}{\gamma}\tilde{\vartheta}^\top(k)\tilde{\vartheta}(k)$ and calculate the difference $\Delta V = V(k+1) - V(k)$ (omitting dependence on k) in view of (39) and (34):

$$\begin{aligned}
\Delta V &= \frac{1}{\gamma}\Big(\tilde{\vartheta}^\top\tilde{\vartheta} + \bar{\sigma}^2\hat{\vartheta}^\top\hat{\vartheta} + \gamma^2 E^\top\Omega R^{-2}\Omega E + 2\bar{\sigma}\tilde{\vartheta}^\top\hat{\vartheta} \\
&\quad -2\gamma E^\top R^{-1}E + 2\gamma\epsilon_L(k)R^{-1}(k)E - 2\gamma\bar{\sigma}\hat{\vartheta}^\top R^{-1}\Omega E\Big) \\
&\quad - \frac{1}{\gamma}\tilde{\vartheta}^\top\tilde{\vartheta} = -E^\top R^{-1}\left(I_n + \frac{1}{2\gamma}I_n - R^{-1}\Omega^2\right)E \\
&\quad + 2\frac{\bar{\sigma}}{\gamma}\tilde{\vartheta}^\top\hat{\vartheta} + \bar{\sigma}^2\left(\frac{1}{\gamma} + \frac{1}{1-\gamma}\right)\hat{\vartheta}^\top\hat{\vartheta} \\
&\quad - \left|\frac{\bar{\sigma}}{\sqrt{1-\gamma}}\hat{\vartheta} - \sqrt{1-\gamma}R^{-1}\Omega E\}\right|^2 \\
&\quad - \left|R^{-\frac{1}{2}}\left(\sqrt{\left(\frac{2}{2-\gamma}\right)}\epsilon_L - \sqrt{\left(\frac{2-\gamma}{2}\right)}E\right)\right|^2 \\
&\quad + \frac{2}{2-\gamma}\epsilon_L^\top R^{-1}\epsilon_L \leq -\frac{1}{2\gamma}E^\top R^{-1}E + 2\frac{\bar{\sigma}}{\gamma}|\vartheta|\left|\hat{\vartheta}\right| \\
&\quad + \frac{\bar{\sigma}}{\gamma}\left(\frac{\bar{\sigma}}{1-\gamma} - 2\right)\left|\hat{\vartheta}\right|^2 + \frac{2}{2-\gamma}\epsilon_L^\top R^{-1}\epsilon_L
\end{aligned}$$

[5]In the simplest case, λ_Ω represents the minimum eigenvalue of the matrix Ω.

Since $||L(z)||_\infty < 1$, then $\bar{\sigma}(k) \leq \sigma_{\theta 0}$ and hence

$$\begin{aligned}
\Delta V &\leq -\frac{1}{2\gamma}E^\top R^{-1}E \\
&\quad + \frac{\sigma_{\theta 0}}{\gamma}\left|\hat{\vartheta}\right|\left(2|\vartheta| + \left(\frac{\sigma_{\theta 0}}{1-\gamma} - 2\right)\left|\hat{\vartheta}\right|\right) + \frac{2}{2-\gamma}\epsilon_L^\top R^{-1}\epsilon_L.
\end{aligned}$$

For the case, when $|\hat{\vartheta}| \geq 2|\vartheta|$,

$$\begin{aligned}
&\frac{\sigma_{\theta 0}}{\gamma}\left|\hat{\vartheta}\right|\left(2|\vartheta| + \left(\frac{\sigma_{\theta 0}}{1-\gamma} - 2\right)\left|\hat{\vartheta}\right|\right) \leq \\
&\frac{\sigma_{\theta 0}}{\gamma}\left(1 + \left(\frac{\sigma_{\theta 0}}{1-\gamma} - 2\right)\right)\left|\hat{\vartheta}\right|^2 \leq -\frac{\sigma_{\theta 0}}{2\gamma}\left|\hat{\vartheta}\right|^2
\end{aligned}$$

if $0 < \gamma < 1$ and $0 < \sigma_{\theta 0} \leq \frac{1}{2}(1-\gamma)$.

Finally, we obtain

$$\Delta V \leq -\frac{1}{2\gamma}E^\top R^{-1}E - \frac{\sigma_{\theta 0}}{2\gamma}\left|\hat{\vartheta}\right|^2 + \frac{2}{2-\gamma}\epsilon_L^\top R^{-1}\epsilon_L. \quad (41)$$

Since $\epsilon_L(k) \in \mathcal{L}_\infty$ by definition and due to result of Corollary 2, then $\tilde{\vartheta}(k) \in \mathcal{L}_\infty$, and $E^\top(k)R^{-1}(k)E(k)$ tends to a residual set with the boundary defined by $\frac{4\gamma}{2-\gamma}\epsilon_L^\top(k)R^{-1}\epsilon_L(k)$. Since R is strictly positive definite and $|\epsilon_L| \in \mathcal{L}_\infty$ (due to the properties of the stable dynamic systems (19), (29), due to the inputs of (19) that are inverse proportional to $x_j^2 + \sigma_j^2$), we have

$$\begin{aligned}
0 < \lambda_{Rmin}(k)|E(k)|^2 &\leq \frac{4\gamma}{2-\gamma}\epsilon_L^\top(k)R^{-1}(k)\epsilon_L(k) \\
&\leq \lambda_{Rmax}(k)|\epsilon_L(k)|^2,
\end{aligned}$$

where $\lambda_{Rmin}(k)$ and $\lambda_{Rmax}(k)$ is the minimum and maximum eigenvalue of $R^{-1}(k)$, respectively. Hence, $|E(k)| \leq r||\epsilon_L(k)||_\infty$, where $r \in \mathbb{R}_+$ is a constant. Following the properties of linear discrete systems and taking into account (29), the definition of $B(k)$, the boundedness of $\tilde{\vartheta}(k)$, $\varpi_i(k)$, and $\epsilon_L(k)$, we prove the boundedness of $\Omega(k)$. Therefore, following the result of Lemma 3, we have: $\varepsilon(k), z(k), x(k), u(k) \in \mathcal{L}_\infty$. Since the magnitude of $\epsilon_L(k)$ can be decreased by decreasing the value $\max\limits_{j=1,2,\ldots,n}\{\sigma_{0j}\}$, the norms $||E(k)||_\infty$ and $||\varepsilon||_\infty$ are reduced by this value as well. This completes the proof of Property P1.1 and Objective *O2* and the inequality (9).

If the conditions of Objective *O1* are satisfied, then according to Corollary 2 $\bar{\epsilon}_z(k) \to 0$ exponentially fast as $k \to \infty$. As a result, $|\hat{\vartheta}(k)| \leq \vartheta^*$, $\sigma_{\theta 0} = 0$, and $\bar{\sigma}(k) \to 0$ exponentially fast as $k \to \infty$. In this case, from (41) it follows that

$$\Delta V \leq -\frac{1}{2\gamma}E^\top(k)R^{-1}(k)E(k) + \frac{2}{2-\gamma}\epsilon_L^\top(k)R^{-1}(k)\epsilon_L(k).$$

It follows from the later that $E^\top(k)R^{-1}(k)E(k) \in \mathcal{L}_2$ and $E(k) \to 0$ as $k \to \infty$ [27], [29]. As a result, $\varepsilon(k) \to 0$ as $k \to \infty$ according to Objective *O1* and the equality (8). This completes the proof of Property P1.2.

The proofs of Properties P1.3 and P1.4 for the case of Objective *O1* can be found in [23] in the framework of analysis of the error model

$$\tilde{\vartheta}(k+1) = \tilde{\vartheta}(k) - \gamma R^{-1}(k)\Omega(k)E(k)$$

or [25, Chapter 3] (for continuous-time case). ■

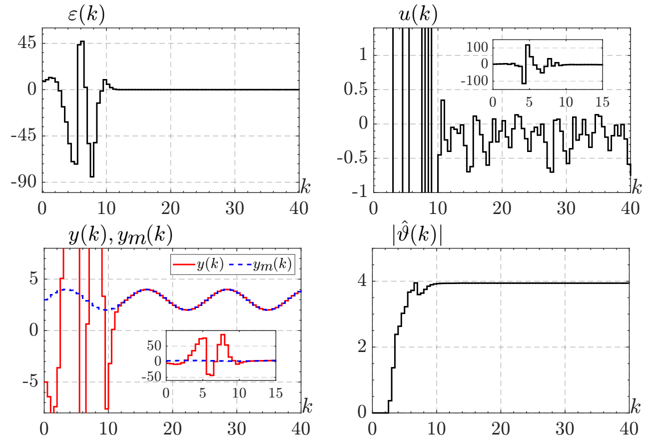

Fig. 1. Transients in the system closed-loop by the adaptive backstepping controller

VI. SIMULATION RESULTS

Consider the third order unstable plant (2)-(4)

$$\begin{aligned}
x_1(k+1) &= \psi_1(k)x_1(k) + x_2(k),\\
x_2(k+1) &= \psi_2(k)x_2(k) + x_3(k),\\
x_3(k+1) &= \psi_3(k)x_3(k) + u(k),\\
y(k) &= x_1(k)
\end{aligned}$$

with the initial condition $x(0) = [-5, 0, 0]^\top$ and the TV parameters $\psi_1(k) = 0.1\sin(2kT + 1) + 1.2$, $\psi_2(k) = 0.2\sin(3kT+2)+0.4$, and $\psi_3(k) = 0.3\sin(4kT+3)+0.5$ with a priori unknown amplitudes, phases, frequencies, and the biases. The sample time T is set to 0.5s.

Problem is to design a control that drives the plant output to the reference $y_m(k) = 3.0 + 1.0\sin(0.5kT)$ with all the closed-loop signals bounded.

To design the observers of the TV parameters $\psi_1(k)$, $\psi_2(k)$, and $\psi_3(k)$, we select $m_1 = m_2 = m_3 = 3$,

$$G_1 = G_2 = G_3 = \begin{bmatrix} 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}, \quad l_1 = l_2 = l_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}.$$

The initial conditions for the TV parameters observers (13)–(18) are set to zero, while their design parameters are given by $\varkappa_{01} = \varkappa_{02} = \varkappa_{03} = 0.001$, $\sigma_{01} = \sigma_{02} = \sigma_{03} = 0.001$.

For the backstepping procedure implemented in three steps, in (23) we choose $c_1 = -0.1$, $c_2 = c_3 = 0.2$.

For the robust adaptation algorithm with MRE (35)-(37), we use: the first order filter $L(z) = \frac{1-d_0}{z+d_0}$ with $d_0 = 0.7$; adaptation gain $\gamma = 0.5$; parameter $\rho = 0.1$; leakage factor $\sigma_{\theta 0} = 0.01$; threshold $\vartheta^* = 10$. The initial conditions $\hat{\vartheta}(0)$ of the algorithm are set to zero.

As shown by Fig.1, despite the instability of the plant and the influence of uncertain sinusoidal parameters, the adaptive controller provides the boundedness of $y(k), u(k), \hat{\vartheta}(k)$ and the convergence of the tracking error to zero.

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