2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

On Controlling a Coevolutionary Model of Actions and Opinions

Roberta Raineri, Giacomo Como, Fabio Fagnani, Mengbin Ye, and Lorenzo Zino

***Abstract*— We deal with a control problem for a complex social network in which each agent has an action and an opinion, evolving according to a coevolutionary model. In particular, we consider a scenario in which a committed minority —a set of stubborn nodes— aims to steer a population, initially at a consensus, to a different consensus state. Our study focuses on determining the conditions under which such a goal is reached, and ultimately, how to optimally define a minimal committed minority. First, we derive a general monotone convergence result for the controlled coevolutionary model, under mild and general assumptions on the agents' revision sequence. Then, we build on our theoretical result to propose a systematic approach to investigate the research problem.**

I. Introduction

Mathematical modeling of social dynamics has attracted increasing interest within the systems and control community, enabling the development of novel tools to understand, predict, and control collective human behavior [1]–[4]. In particular, a field of growing interest are complex social phenomena that entail individuals deciding on (binary) actions —e.g., *using a disposable cup vs a reusable cup to have a coffee*— on the basis of several factors, including their opinions on the considered action. Empirical evidence and social psychology theories suggest that human decision-making and opinion formation processes are deeply intertwined [5], influencing each other, calling for the development of coevolutionary models of actions and opinions.

A key step toward the development of such model paradigms lies in continuous-opinion discrete-action models, firstly proposed in [6] and then extended along several directions [7], [8]. These models rely on the assumption that the dynamics is driven by the opinion formation process, while actions are a quantization of opinions, limiting the possibility to capture critical features of social systems such as unpopular norms or pluralistic ignorance [9], [10]. Following a different approach, private and expressed opinions have been assumed to coevolve in [11], [12], but without any decision-making process. To address these limitations, a coevolutionary model of actions and opinions was proposed in [13], where an opinion formation process is incorporated within a game-theoretic framework, often used to model decision-making [14], [15]. In this model, improved upon and studied in [16], [17], individuals simultaneously revise their (binary) actions and share their opinions on the support of the actions, accounting for social pressure and consistency.

Here, we build on these efforts using the model from [17] to study the problem of unlocking a paradigm shift in a population. Specifically, we consider a scenario in which a population starts at a consensus and a set of stubborn nodes (committed minority) is introduced to control the remaining ones. Stubborn nodes consistently select the opposite action, and share opinions supporting it, with the goal of steering the entire population to consensus on the opposite action. By studying the behavior of the controlled model, we establish mathematical tools to understand sufficient conditions for the stubborn nodes to achieve their goal. This problem, studied separately for opinion dynamics [2], [18] and game-theoretic models [14], [19], [20], is still unexplored for the coevolutionary model, and is relevant to many real-life applications, from incentivizing social change to guaranteeing robustness of social systems against malicious attacks.

In detail, our main contribution is fourfold. First, we build on [17] to formulate a control problem for the coevolutionary dynamics. Second, we prove convergence of the controlled dynamics under general assumptions on the model parameters and on the agents' revision sequence, extending the convergence results for the uncontrolled dynamics in [17]. Third, we build on these results to propose a systematic approach to study the control problems, establishing an iterative algorithm that is able to determine the final equilibrium reached by the network. Fourth, we demonstrate our approach through two case studies.

II. Model and Problem Statement

Notation. We denote a vector $\boldsymbol{x}$ with bold lowercase font, with x_i its ith entry, and a matrix $\boldsymbol{A}$ with bold capital font, with a_{ij} the jth entry of its ith row. The all-1 column vector is denoted as $\mathbf{1}$, with appropriate dimension depending on the context. Given two vectors $\boldsymbol{x}, \boldsymbol{y}$ with same dimension, we use $\boldsymbol{x} \leq \boldsymbol{y}$ to denote $x_i \leq y_i$, for all entries i.

A. (Uncontrolled) Coevolutionary Model

We consider a population of n individuals, indexed by the set $\mathcal{V} = \{1, 2, \dots, n\}$. Each individual is characterized by a two-dimensional state variable $(x_i(t), y_i(t)) \in \{-1, +1\} \times [-1, +1]$, where t is a discrete time-step. The binary variable $x_i(t) \in \{-1, +1\}$ represents the *action* of individual i at time t. The continuous variable $y_i(t) \in [-1, +1]$ represents

R. Raineri, G. Como, and F. Fagnani are with the Department of Mathematical Sciences, Politecnico di Torino, Torino, Italy ({roberta.raineri,giacomo.como,fabio.fagnani}@polito.it). M. Ye is with the Centre for Optimisation and Decision Science, Curtin University, Perth, Australia (mengbin.ye@curtin.edu.au). L. Zino is with the Department of Electronics and Telecommunications, Politecnico di Torino, Italy (lorenzo.zino@polito.it). G. Como is also with the Department of Automatic Control, Lund University, Sweden. M. Ye was supported by the Western Australian Government through the Premier's Science Fellowship Program. This research was carried on within the framework of the MIUR-funded Progetto di Eccellenza of the Dipartimento di Scienze Matematiche G.L. Lagrange, Politecnico di Torino, CUP: E11G18000350001

i's *opinion* on such action, spanning from $y_i(t) = -1$ (if i is totally in favor of action -1) to $y_i(t) = +1$ (if i fully supports action $+1$). Actions and opinions are gathered in vectors $\boldsymbol{x}(t) \in \{-1, 1\}^n$ and $\boldsymbol{y}(t) \in [-1, 1]^n$, and the joint $2n$-dimensional vector $\boldsymbol{z} := (\boldsymbol{x}(t), \boldsymbol{y}(t)) \in \{-1, 1\}^n \times [-1, 1]^n$ represents the state of the system at time t.

At each time step t, one or more individuals simultaneously revise their state. Specifically, we define a set $\mathcal{R}(t) \subseteq \mathcal{V}$, which denotes the individuals who revise their state at time t, for which we make the following general assumption.

Assumption 1 (Revision sequence)**.** *There exists a constant $T < \infty$ such that $\cup_{s=0}^{T-1} \mathcal{R}(t+s) = \mathcal{V}$, for any $t \geq 0$.*

Remark 1. *Assumption 1 generalizes the classical synchronous and asynchronous update rules for dynamics on networks: for synchronous update rules, $\mathcal{R}(t) = \mathcal{V}$ for all t; for asynchronous update rules, $\mathcal{R}(t)$ comprises a single individual, and Assumption 1 is imposed.*

At each time step t, each individual $i \in \mathcal{R}(t)$ updates their state, aiming to maximize a utility function accounting for three contributions: i) an individual's tendency to coordinate with the actions of others; ii) opinions exchanged with peers; and iii) an individual's tendency to act consistently with their own opinion. Following [17], we define the utility that i receives for selecting an action and opinion pair, denoted by $\boldsymbol{\zeta} = (\zeta_x, \zeta_y)$, when the system state is $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{y})$, as

$$u_i(\boldsymbol{\zeta}, \boldsymbol{z}) = \tfrac{\lambda_i(1-\beta_i)}{2} \sum\nolimits_{j \in \mathcal{V}} a_{ij} \big[(1 - x_j)(1 - \zeta_x) + (1 + x_j) \cdot (1 + \zeta_x)\big] - \beta_i(1 - \lambda_i) \sum\nolimits_{j \in \mathcal{V}} w_{ij} (\zeta_y - y_j)^2 - \lambda_i \beta_i (\zeta_x - \zeta_y)^2, \quad (1)$$

where $a_{ij} \in [0, 1]$ and $w_{ij} \in [0, 1]$ are the influence of individual j's action and opinion on individual i, respectively, and $\lambda_i \in (0, 1]$ and $\beta_i \in (0, 1]$ are the weights given to actions and opinions, respectively. The quantities a_{ij} and w_{ij} are gathered into two matrices $\boldsymbol{A}$ and $\boldsymbol{W}$, which we assume to be irreducible and stochastic. Hence, social interactions can be represented by a strongly connected two-layer network $\mathcal{G} = (\mathcal{V}, \mathcal{E}_A, \boldsymbol{A}, \mathcal{E}_W, \boldsymbol{W})$, where $\mathcal{E}_A$ are the edges on the influence layer on which individuals see the actions of others and $\mathcal{E}_A$ are the edges on the communication layer, on which individuals discuss their opinions.

Remark 2. *The utility in Eq. (1) is a specialization of the general case from [17], which contains an additional term to account for individual prejudices, and a parameter to capture possible biases to favor one action over the other.*

In Eq. (1), we enforce $\lambda_i > 0$ and $\beta_i > 0$ to guarantee that the coupling between the two variables is always present. In the limit case in which one of these parameters is equal to 0, the coevolutionary model reduces to simpler (and well-known) dynamics, as commented in the following.

Remark 3. *The utility in Eq. (1) generalizes classical network coordination games [14], [15] (obtained in the limit case $\beta_i \to 0$) and the French-DeGroot opinion dynamics model [2], [21] (in the limit $\lambda_i \to 0$), by coupling the two corresponding utility functions. See, [17] for more details.*

If $i \in \mathcal{R}(t)$, then i revises their action and opinion, at the same time, toward maximizing their utility, according to a joint best-response dynamics, i.e.,

$$(x_i(t+1), y_i(t+1)) \in \text{argmax}_{\boldsymbol{\zeta} \in \{-1,1\} \times [-1,1]} u_i(\boldsymbol{\zeta}, \boldsymbol{z}) \quad (2)$$

with the convention that, when multiple elements $\boldsymbol{\zeta}$ maximize $u_i(\boldsymbol{\zeta}, \boldsymbol{z})$, we set $x_i(t+1) = x_i(t)$ and $y_i(t+1)$ accordingly; see [17] for more details. Individuals $j \notin \mathcal{R}(t)$, do not revise their state, i.e., $(x_j(t+1), y_j(t+1)) = (x_j(t), y_j(t))$. In order to study the dynamics, we use [17] to explicitly derive the update dynamics for the individuals' action and opinions specialized to our scenario, as summarized in the following.

Proposition 1. *Individual $i \in \mathcal{R}(t)$, who follows Eq. (2), updates their state according to*

$$x_i(t+1) = s(\delta_i(\boldsymbol{z}(t))) \quad (3)$$

$$y_i(t+1) = (1 - \lambda_i) \sum\nolimits_{j \in \mathcal{V}} w_{ij} y_j(t) + \lambda_i s(\delta_i(z(t))) \quad (4)$$

where

$$\delta_i(\boldsymbol{z}(t)) = 2\beta_i(1 - \lambda_i) \sum_{j \in \mathcal{V}} w_{ij} y_j(t) + (1 - \beta_i) \sum_{j \in \mathcal{V}} a_{ij} x_j(t) \quad (5)$$

and

$$s(\delta_i(\boldsymbol{z}(t))) = \begin{cases} +1 & \text{if } \delta_i(\boldsymbol{z}(t)) > 0 \\ -1 & \text{if } \delta_i(\boldsymbol{z}(t)) < 0 \\ x_i(t) & \text{if } \delta_i(\boldsymbol{z}(t)) = 0. \end{cases} \quad (6)$$

From Proposition 1, we derive the following observation.

Proposition 2. *The (uncontrolled) coevolutionary dynamics with utility in Eq. (1) has at least two equilibria: $\boldsymbol{x} = \boldsymbol{y} = -\boldsymbol{1}$ and $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{1}$. These are the unique equilibria in which the action vector is at a consensus, i.e., $x_i = x_j$, $\forall\, i, j \in \mathcal{V}$.*

Proof. Without loss of generality, we focus on $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{1}$. First, we show that $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{1}$ is an equilibrium, by observing that, if $\boldsymbol{x}(t) = \boldsymbol{y}(t) = \boldsymbol{1}$, then $\delta_i(t) = 2\beta_i(1 - \lambda_i) + 1 - \beta_i \geq 0$, implying $s(\delta_i(x_i(t), y_i(t))) = +1$, and thus $x_i(t+1) = +1$, $\forall\, i \in \mathcal{V}$. From Eq. (4), we obtain $y_i(t+1) = +1$, yielding the claim. Finally, uniqueness is proved by contradiction. Assume that there exists a consensus equilibrium $(\boldsymbol{1}, \boldsymbol{y}^*)$, with $\boldsymbol{y}^* \neq \boldsymbol{1}$. Let $i = \text{argmin}_{j \in \mathcal{V}} y_j^*$. Clearly, $y_i^* < 1$. Let $\boldsymbol{y}(t) = \boldsymbol{y}^*$. From Eq. (4), we get $y_i(t+1) = (1 - \lambda_i) \sum_{j \in \mathcal{V}} w_{ij} y_j(t) + \lambda_i \geq (1 - \lambda_i) y_i^* + \lambda_i > y_i^*$, which implies that $(\boldsymbol{1}, \boldsymbol{y}^*)$ cannot be an equilibrium, completing the proof. □

B. Controlled dynamics and problem statement

We consider a scenario in which, at time $t = 0$, the population is at a consensus equilibrium. Without any loss in generality, we assume the consensus is on action -1. Based on Proposition 2, all agents also have initial opinion -1, being the unique equilibrium with consensus of actions at -1. Starting from such initial consensus, our goal is to steer the whole system to the opposite consensus state, i.e., to all agents playing action $+1$ (which also implies all opinions are equal to $+1$). To achieve such a goal, we assume that we can control the state of a set of agents $\mathcal{C} \subset \mathcal{V}$ by setting their opinion and action to $+1$ at time $t=1$ and for all the following time-instances, yielding the following assumption.

Assumption 2 (Controlled dynamics)**.** *Given a two-layer network $\mathcal{G}=(\mathcal{V},\mathcal{E}_A,\boldsymbol{A},\mathcal{E}_W,\boldsymbol{W})$ with $\boldsymbol{A}$ and $\boldsymbol{W}$ stochastic and irreducible, and a control set $\mathcal{C}\subseteq\mathcal{V}$, we assume that $x_i(t)=y_i(t)=+1$, $\forall\, i\in\mathcal{C}$ and $\forall\, t\geq 0$, while $x_i(0)=y_i(0)=-1$, $\forall\, i\in\mathcal{U}:=\mathcal{V}\setminus\mathcal{C}$, and all uncontrolled agents $i\in\mathcal{U}$ update their state according to Proposition 1.*

Hereafter, we will refer to a *controlled evolutionary dynamics* as a coevolutionary dynamics with utility function in Eq. (1) and under Assumptions 1 and 2. The goal of the controller, i.e., to lead all the agents to the desired consensus, can be formalized by first defining the objective function

$$\phi(\mathcal{C}):=\mathbb{P}[\exists\, T<\infty : x_i(t)=+1,\forall\, t\geq T,\forall\, i\in\mathcal{V}],\quad (7)$$

i.e., the probability (over the probability space generated by the revision sequence) that all individuals eventually switch definitively their action to $+1$ in finite time when the control set is $\mathcal{C}$. The controller's goal is achieved iff $\phi(\mathcal{C})=1$. Hence, we formalize the following research problems.

Problem 1 (Effectiveness guarantees)**.** *Given a network $\mathcal{G}$, consider a controlled evolutionary dynamics on the network under Assumptions 1 and 2 and specified control set $\mathcal{C}$. Determine whether there holds $\phi(\mathcal{C})=1$.*

Problem 2 (Minimal control set)**.** *Given a network $\mathcal{G}$, consider a controlled evolutionary dynamics on the network under Assumptions 1 and 2 with specified model parameters. Determine the minimal control set $\mathcal{C}$ for which $\phi(\mathcal{C})=1$.*

The problem of controlling the coevolutionary dynamics is inherently complex. In fact, in the limit $\lambda_i\to 0$ and $\beta_i\to 0$ the model simplifies to the French-DeGroot dynamics and network coordination games, respectively. In these cases, Problem 2 is NP-hard [18]–[20]. In the general scenario, the higher complexity of the utility function intuitively suggests that the problem should have at least the same order of complexity, but rigorous proof is left for future research.

III. Main Results

A. Convergence

For the uncontrolled dynamics, a convergence result has been established in [17]. However, such result only applies to scenarios of asynchronous revision sequences, homogeneous parameters (i.e., $\lambda_i=\lambda$, $\beta_i=\beta$, and $\gamma_i=\gamma$ $\forall\, i\in\mathcal{V}$), and two coincident, symmetric layers with self-loops (i.e., $\boldsymbol{W}=\boldsymbol{A}=\boldsymbol{W}^\top=\boldsymbol{A}^\top$ and $a_{ii}>0$, $\forall\, i\in\mathcal{V}$). In the following, instead, we show that such assumptions are not needed to guarantee convergence for the controlled evolutionary dynamics. The proof is reported in Appendix A.

Theorem 1. *Consider a controlled coevolutionary dynamics under Assumptions 1 and 2. Then, there exists an equilibrium $(\boldsymbol{x}^*,\boldsymbol{y}^*)$ such that the action vector $\boldsymbol{x}(t)$ converges to $\boldsymbol{x}^*$ in finite time, and the opinion vector $\boldsymbol{y}(t)$ converges to $\boldsymbol{y}^*$ asymptotically. Moreover, both the opinion and action vectors are monotonically nondecreasing functions of time, i.e., $\boldsymbol{x}(t+1)\geq\boldsymbol{x}(t)$ and $\boldsymbol{y}(t+1)\geq\boldsymbol{y}(t)$, for all $t\geq 0$.*

Theorem 1 not only guarantees that the controlled coevolutionary dynamics converges and that actions converge in finite time, but it also guarantees monotonicity of the trajectory of the state vector $\boldsymbol{z}(t)$. As a consequence, if $i\in\mathcal{U}$ switches to action $+1$ at a certain time, then i will never flip back. This observation will be fundamental to build a systematic approach to study our research problem, as discussed in the next section. Finally, it is worth noticing that Assumption 2 is key for obtaining monotonicity (which then yields convergence). In fact, from a general initial condition, one may observe non-monotone trajectories (see, e.g., [17]).

B. Systematic approach to solve the research problems

Building on Theorem 1, we construct an algorithm to determine the equilibrium $\boldsymbol{z}^*$ reached by the coevolutionary dynamics with control set $\mathcal{C}$. Our procedure is based on the following iterative scheme, summarized in Algorithm 1.

At iteration $k=1$, we initialize the algorithm by defining a set $\mathcal{A}(1)=\mathcal{C}$. At each step of the algorithm k, we construct a candidate equilibrium with action vector $\hat{\boldsymbol{x}}$ with

$$\hat{x}_i=\begin{cases}+1 & \text{if } i\in\mathcal{A}(k),\\ -1 & \text{if } i\notin\mathcal{A}(k),\end{cases}\quad (8)$$

and opinion vector $\hat{\boldsymbol{y}}$, computed by solving the linear system

$$\hat{y}_i=\begin{cases}(1-\lambda_i)\sum_{j\in\mathcal{V}}w_{ij}\hat{y}_j+\lambda_i\hat{x}_i & \text{if } i\in\mathcal{U},\\ 1 & \text{if } i\in\mathcal{C}\end{cases}.\quad (9)$$

To check whether $\hat{\boldsymbol{z}}=(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}})$ is an actual equilibrium, we verify if any of the individuals i who would play action -1 at such candidate equilibrium would switch to $+1$, computing the sign of $\delta_i(\hat{\boldsymbol{z}})$ for all $i\notin\mathcal{A}(k)$. Indeed, from dynamics described in Proposition 1, if all $\delta_i(\hat{\boldsymbol{z}})\leq 0$, then no other individual would switch. Otherwise, all individuals with $\delta_i(\hat{\boldsymbol{z}})>0$ will eventually switch to $+1$. Thus, increasing the iteration index k by 1, we consider a new candidate equilibrium $\mathcal{A}(k)$ where also those individuals are included. This procedure is re-iterated until we get the termination criterion $\mathcal{A}(k)=\mathcal{A}(k-1)$, which implies that no more individuals would change their action. According to this procedure, we get a non-decreasing sequence of sets. When the termination criterion is met (at most in $n-|\mathcal{C}|$ steps), the algorithm returns the final set $\mathcal{A}_f$.

The intuition beyond the algorithm and the convergence result in Theorem 1 leads us to formulate the following conjecture; a formal proof is left for future research.

Conjecture 1. *$\phi(\mathcal{C})=1$ iff $\mathcal{A}_f=\mathcal{V}$; otherwise, $\phi(\mathcal{C})=0$.*

If Conjecture 1 holds true, Algorithm 1 can be used to solve Problem 1: a set $\mathcal{C}$ solves Problem 1 iff the output of Algorithm 1 is $\mathcal{A}_f=\mathcal{V}$. Moreover, the algorithm provides us insights into Problem 2, as we shall discuss in Section IV through two case studies.

Algorithm 1: Equilibrium computation

Data: $\boldsymbol{A}, \boldsymbol{W}, \mathcal{C}, \lambda_i$ and β_i, for all $i \in \mathcal{U}$
Result: $\mathcal{A}_f := \mathcal{A}(k)$, i.e., individuals with $x^* = +1$
$k \leftarrow 1$; $\mathcal{A}(0) \leftarrow \emptyset$; $\mathcal{A}(1) \leftarrow \mathcal{C}$;
while $\mathcal{A}(k) \neq \mathcal{A}(k-1)$ **do**
  Define $\hat{\boldsymbol{x}}$ using Eq. (8);
  Compute $\hat{\boldsymbol{y}}$ by solving Eq. (9) given $\hat{\boldsymbol{x}}$;
  $k \leftarrow k+1$; $\mathcal{A}(k) \leftarrow \mathcal{A}(k-1)$;
  check **for** $i \in \mathcal{V}$ & $i \notin \mathcal{A}(k)$ **do**
    if $\delta_i(\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}) > 0$ **then**
      $\mathcal{A}(k) \leftarrow \mathcal{A}(k) \cup \{i\}$;
    end
  end
end

IV. Case Studies

A. Complete graph

First, we consider a homogeneous complete graph and assume that m individuals are controlled (whose position is irrelevant, due to the network structure and symmetry).

Proposition 3. *Consider a coevolutionary dynamics that satisfies Assumptions 1–2 on a complete graph with n nodes and $w_{ij} = a_{ij} = \frac{1}{n-1}$ for all $i \neq j$. Assume that $|\mathcal{C}| = m$, and define $\gamma = \frac{m}{n-1}$. Then, $\mathcal{A}_f = \mathcal{V}$ iff there holds*

$$2\beta(1-\lambda)\Big(\frac{\gamma-\lambda+\lambda\gamma}{\gamma+\lambda-\lambda\gamma}\Big) + (1-\beta)(2\gamma-1) > 0. \quad (10)$$

Proof. We apply Algorithm 1. We start with $\mathcal{A}(1) = \mathcal{C}$. The corresponding candidate equilibrium $\hat{\boldsymbol{z}}$ has $\hat{\boldsymbol{x}}$ defined using Eq. (8), and $\hat{\boldsymbol{y}}$ with $\hat{y}_j = 1$ for $j \in \mathcal{C}$, while, for the network, all $i \in \mathcal{U}$ has the same $\hat{y}_i$. Hence, Eq. (9) reduces to $\hat{y}_i = (1-\lambda)(\sum_{j\in\mathcal{C}} \frac{1}{n-1} + \sum_{j\in\mathcal{U}\setminus\{i\}} \frac{1}{n-1}\hat{y}_i) - \lambda = (1-\lambda)(\gamma+(1-\gamma)\hat{y}_i) - \lambda$, yielding $\hat{y}_i = \frac{\gamma-\lambda\gamma-\lambda}{\gamma+\lambda-\lambda\gamma}$. Now, for a generic $i \in \mathcal{U}$, we compute $\delta_i(\hat{\boldsymbol{z}})$, obtaining the left-hand side in Eq. (10). If $\delta_i(\hat{\boldsymbol{z}}) > 0$, then $\mathcal{A}(2) = \mathcal{V}$. Otherwise, $\mathcal{A}(2) = \mathcal{A}(1) = \mathcal{C}$. In both cases, the algorithm terminates. □

Remark 4. *Subject to Conjecture 1, Proposition 3 solves both Problems 1 and 2. In fact, given $\mathcal{C}$, Problem 1 is solved for those values of λ and β that satisfy Eq. (10). On the other hand, given $|\mathcal{C}| = m$, the choice of the nodes to control is irrelevant, and re-writing Eq. (10) as a condition on γ, we can ultimately determine the minimum number of nodes to be controlled to guarantee $\phi(\mathcal{C}) = 1$, solving Problem 2.*

Figure 1 illustrates our findings and the discussion in Remark 4. In fact, for each control set (specifically, for a given value of γ), we associate its corresponding contour curve. For all values of the parameters (λ, β) above the curve, we guarantee that $\phi(\mathcal{C}) = 1$, solving Problem 1. Alternatively, fixing the parameters (λ, β), the color intensity identifies the minimum fraction of nodes that we need to control to obtain $\phi(\mathcal{C}) = 1$, solving Problem 2. Observe that, given that Eq. (10) does not depend explicitly on n, but only on the fraction of controlled nodes through the parameter γ, the network size has no direct impact on our findings.

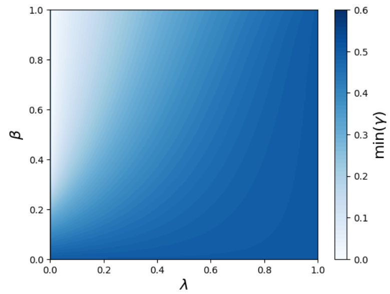


Fig. 1: Results for a complete graph. The color intensity represents the cardinality of the minimal control set $\gamma = |\mathcal{C}|/(n-1)$ that solves Problem 2.

B. Star graph

We consider now a star graph, which is made by a center node (with label 1) connected to $n-1$ leaf nodes, and is an instance of a hub-and-spoke model, used in telecommunication systems and distribution markets. Also in this case, the network's symmetry simplifies the problem: all leaves are identical to one another thus it only matters how many of them are controlled. Moreover, if the center is controlled, then there are no interactions between any two leaves, since leaves only interact with the center, whose state is fixed. Hence, two control strategies are relevant for a star: i) to control the center; or ii) to control m leaves.

First, we consider the case in which we control the center. Our result is presented as a set of two conditions on λ and β, which can be used to provide a direct solution to Problem 1.

Proposition 4. *Consider a coevolutionary dynamics that satisfies Assumptions 1 and 2 on a star graph $\mathcal{G}$ with n nodes with $w_{1i} = a_{1i} = \frac{1}{n-1}$ and $w_{ii} = a_{ii} = \frac{n-2}{n-1}$ for $i \in \mathcal{V}\setminus\{1\}$. Assume that $1 \in \mathcal{C}$. Then, there holds $\mathcal{A}_f = \mathcal{V}$ iff*

$$\begin{cases} \beta > \frac{(n-3)(1-2\lambda+\lambda n)}{n^2\lambda(2\lambda-1)+n(3-6\lambda^2-\lambda)+(4\lambda^2+4\lambda-5)}; \\ n^2\lambda(2\lambda-1)+n(3-6\lambda^2-\lambda)+(4\lambda^2+4\lambda-5) > 0. \end{cases} \quad (11)$$

Proof. We start with $\mathcal{A}(1) = \mathcal{C}$ and compute the corresponding candidate equilibrium $\hat{\boldsymbol{z}}$. Due to symmetry, uncontrolled leaves are indistinguishable and we consider an arbitrary $j \in \mathcal{U}$, for which Eq. (9) becomes $\hat{y}_j = (1-\lambda)(\frac{1}{n-1} + \frac{n-2}{n-1}\hat{y}_j) - \lambda$, yielding $\hat{y}_j = \frac{1-\lambda n}{1-2\lambda+\lambda n}$ for all $j \in \mathcal{U}$. Then, we impose $\delta_j(\hat{\boldsymbol{z}}) = 2\beta(1-\lambda)[\frac{(n-2)(1-\lambda n)}{1-2\lambda+\lambda n} + 1] - (1-\beta)(n-3) > 0$. Re-writing it in terms of parameter β, and distinguishing two cases depending on the sign of the denominator obtained, we get two sets of conditions: one yields Eq. (11), the second case (obtained if the denominator is negative) would lead to a negative (and thus infeasible) bound for β. If $\delta_j(\hat{\boldsymbol{z}}) > 0$, then $\mathcal{A}(2) = \mathcal{V}$. Otherwise, $\mathcal{A}(2) = \mathcal{A}(1) = \mathcal{C}$. □

Now, we consider the case in which we control a set $\mathcal{C}$ that does not contain the center. In this case, the problem becomes more complicated since $\mathcal{U}$ comprises not only leaves but also the center, with different dynamics. For this reason, we will see that we need to perform two iterations of Algorithm 1.

Proposition 5. *Consider a coevolutionary dynamics that satisfies Assumptions 1–2 on a star graph with n nodes with $w_{1i} = a_{1i} = \frac{1}{n-1}$ and $w_{ii} = a_{ii} = \frac{n-2}{n-1}$ for $i \in \mathcal{V}\backslash\{1\}$. Assume $1 \notin \mathcal{C}$ and $\gamma = \frac{|\mathcal{C}|}{n-1}$. Then, there holds $\mathcal{A}_f = \mathcal{V}$ iff*

$$\gamma > \frac{\lambda[2\lambda^2\beta n+((1-3n)\beta-1)\lambda+\beta n(n-1)+n+4]}{(1-\lambda)^2(\beta(1-n)+2\beta\lambda-1)} \tag{12}$$

$$\begin{aligned}&2\beta(n-1)(1-\lambda^2)(1-\lambda)\gamma - 2\beta\lambda^2(n-1)(1+\lambda-n)\\&-2\beta\lambda(n-1)(n-2)+(1-\beta)(3-n)(1-\lambda)^2\gamma\\&+(1-\beta)(3-n)\lambda(n-\lambda)>0.\end{aligned} \tag{13}$$

Proof. We set $\mathcal{A}(1) = \mathcal{C}$, and compute the candidate equilibrium $\hat{\boldsymbol{z}}$. By symmetry, all uncontrolled leaves will have the same value $\hat{y}_j = \hat{y}_\ell$. Hence, Eq. (9) reduces to

$$\begin{aligned}\hat{y}_1 &= (1-\lambda)\left[\gamma + (1-\gamma)\hat{y}_\ell\right] - \lambda\\ \hat{y}_\ell &= (1-\lambda)\left[\tfrac{1}{n-1}\hat{y}_1 + \tfrac{n-2}{n-1}\hat{y}_\ell\right] - \lambda,\end{aligned} \tag{14}$$

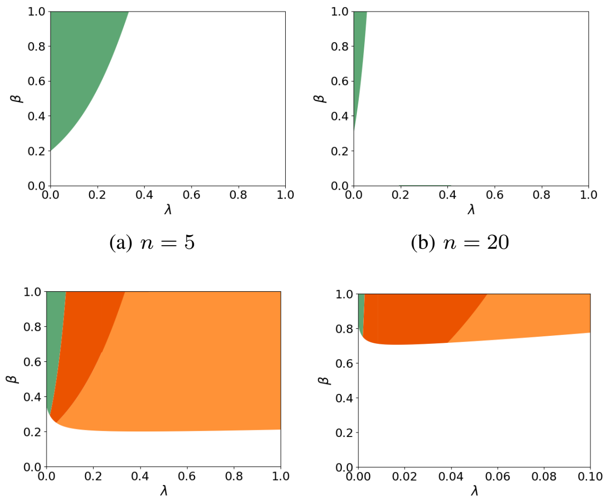
yielding $\hat{y}_1 = \frac{(1-\lambda)\gamma-\lambda}{\lambda+\gamma(1-\lambda)}$ and $,\hat{y}_\ell = \frac{2\lambda-\lambda^2-n\lambda-(1-\lambda)^2\gamma}{\lambda(\lambda-n)-\gamma(1-\lambda)^2}$. Then, we observe that $\delta_1(\hat{\boldsymbol{z}}) \geq \delta_\ell(\hat{\boldsymbol{z}})$. Hence, if $\delta_1(\hat{\boldsymbol{z}}) \leq 0$, necessarily also $\delta_\ell(\hat{\boldsymbol{z}}) \leq 0$. For this reason, we focus on computing under which conditions $\delta_1(\hat{\boldsymbol{z}}) = 2\beta(1-\lambda)(\frac{1}{n-1} + \frac{n-2}{n-1}\hat{y}_\ell) + (1-\beta)(\frac{1}{n-1} - \frac{n-2}{n-1}) > 0$. Substituting $\hat{y}_1$ and $\hat{y}_\ell$, rewriting the inequality in terms of γ, we ultimately get Eq. (12). If the condition is not satisfied, we get $\mathcal{A}(2) = \mathcal{A}(1)$ and the algorithm terminates. Otherwise, $1 \in \mathcal{A}(2)$. We must then check $\delta_\ell(\hat{\boldsymbol{z}})$. However, due to the monotonicity of Eq. (5) with respect to $\boldsymbol{z}$ and given that Theorem 1 guarantees that an individual i will never flip back, we can postpone this check to the next iteration.

We re-iterate with $\mathcal{A}(2) = \{1\} \cup \mathcal{A}(1)$, by solving Eq. (9) and inserting the solution into the condition $\delta_\ell(\bar{\boldsymbol{z}}) > 0$. This ultimately yields Eq. (13), with computations omitted. If the condition is not satisfied, we get $\mathcal{A}(3) = \mathcal{A}(2)$ and the algorithm terminates; otherwise, $\mathcal{A}(3) = \mathcal{V}$. □

Remark 5. *Eq. (13) is always at least as restrictive as Eq. (11). In fact, both conditions are obtained by imposing $\delta_\ell > 0$ for a generic uncontrolled leaf when the center has action $+1$. However, in Eq. (11) the center has opinion $+1$, while in Eq. (13) the opinion of the center may be smaller. Monotonicity of δ with respect to $\boldsymbol{z}$ yields the claim.*

Subject to Conjecture 1, Propositions 4–5 fully characterize the controlled coevolutionary dynamics on a star. Specifically, to solve Problem 1, if $1 \in \mathcal{C}$, then we need to check Eq. (11); if $1 \notin \mathcal{C}$, we need to check Eqs. (12)–(13). For Problem 2, if Eq. (11) is satisfied, then $\mathcal{C} = \{1\}$ is the minimal control set, otherwise it is not possible to control the system, as a consequence of Remark 5. An interesting scenario is where one wants to solve Problem 2, but the center is not controllable. Here, the minimal number of leaves to control is determined by $m = \lceil \bar{\gamma}(n-1) \rceil$, where $\bar{\gamma}$ is the minimum value of γ that satisfies Eqs. (12)– (13).

Figure 2 shows a numerical representation of our results, highlighting in green the areas of the parameter space in which Problem 1 is solved. The upper panels refer to controlling the center; the lower ones to controlling a single leaf. In the orange areas, controlling a single leaf is enough



(a) $n = 5$ (b) $n = 20$ (c) $n = 5$ (d) $n = 20$

Fig. 2: Results for a star graph (a,b) controlled in the center and (c,d) in a leaf ($\gamma = 1/(n-1)$), for different network sizes. The green areas represent the parameters for which Problem 1 is solved.

to switch the center, but not other leaves. The darker areas depict the difference between the conditions in Proposition 4 for $1 \in \mathcal{C}$ and those in Proposition 5 for $1 \notin \mathcal{C}$, illustrating Remark 5: controlling the center is always more effective.

V. Conclusion

We proposed and studied a control problem for social networks by incorporating a committed minority in a coevolutionary model of actions and opinions [17]. We established a general convergence result and, leveraging it, we designed an algorithm to determine whether the committed minority is able to steer the population to the desired state. Our results outline several research directions. First, we assumed that both action and opinion of the individuals can be controlled. Our results should be extended to scenarios with partial ability to control nodes. Second, a formal proof for Conjecture 1 is missing, as well as a method to solve Problem 2 for general networks. A promising approach could be to extend the algorithms developed for similar problems in [20], [22].

References

[1] N. E. Friedkin, "The Problem of Social Control and Coordination of Complex Systems in Sociology: A Look at the Community Cleavage Problem," *IEEE Control Syst. Mag.*, vol. 35, no. 3, pp. 40–51, 2015.

[2] A. V. Proskurnikov and R. Tempo, "A tutorial on modeling and analysis of dynamic social networks. Part I," *Annu. Rev. Control*, vol. 43, pp. 65–79, 2017.

[3] ——, "A tutorial on modeling and analysis of dynamic social networks. Part II," *Annu. Rev. Control*, vol. 45, pp. 166–190, 2018.

[4] C. Bernardo, C. Altafini, A. Proskurnikov, and F. Vasca, "Bounded confidence opinion dynamics: A survey," *Automatica*, vol. 159, p. 111302, Jan. 2024.

[5] S. Gavrilets and P. J. Richerson, "Collective action and the evolution of social norm internalization," *Proc. Natl. Acad. Sci. USA*, vol. 114, no. 23, pp. 6068–6073, 2017.

[6] A. C. R. Martins, "Continuous opinions and discrete actions in opinion dynamics problems," *Int. J. Mod. Phys. C*, vol. 19, no. 04, pp. 617–624, 2008.

[7] F. Ceragioli and P. Frasca, "Consensus and disagreement: The role of quantized behaviors in opinion dynamics," *SIAM J. Control Optim.*, vol. 56, no. 2, pp. 1058–1080, 2018.

[8] K. Tang, Y. Zhao, J. Zhang, and J. Hu, "Synchronous coda opinion dynamics over social networks," in *40th Chinese Control Conf.*, 2021, pp. 5448–5453.
[9] D. A. Prentice and D. T. Miller, "Pluralistic Ignorance and Alcohol Use on Campus: Some Consequences of Misperceiving the Social Norm," *J. Pers. Soc. Psychol.*, vol. 64, no. 2, pp. 243–256, 1993.
[10] R. Willer, K. Kuwabara, and M. W. Macy, "The False Enforcement of Unpopular Norms," *Am. J. Sociol.*, vol. 115, pp. 451–490, 2009.
[11] P. Duggins, "A psychologically-motivated model of opinion change with applications to american politics." *J. Artif. Soc. Soc. Simul.*, vol. 20, no. 1, 2017.
[12] M. Ye, Y. Qin, A. Govaert, B. D. Anderson, and M. Cao, "An influence network model to study discrepancies in expressed and private opinions," *Automatica*, vol. 107, pp. 371–381, 2019.
[13] L. Zino, M. Ye, and M. Cao, "A two-layer model for coevolving opinion dynamics and collective decision-making in complex social systems," *Chaos*, vol. 30, no. 8, p. 083107, 2020.
[14] A. Montanari and A. Saberi, "The spread of innovations in social networks," *Proc. Natl. Acad. Sci. USA*, vol. 107, no. 47, pp. 20 196–20 201, 2010.
[15] M. O. Jackson and Y. Zenou, "Games on Networks," in *Handbook of Game Theory with Economic Applications*. Elsevier, 2015, vol. 4, ch. 3, pp. 95–163.
[16] L. Zino, M. Ye, and M. Cao, "A coevolutionary model for actions and opinions in social networks," in *Proc. 59th IEEE Conf. Decis. Control.*, 2020, pp. 1110–1115.
[17] H. D. Aghbolagh, M. Ye, L. Zino, Z. Chen, and M. Cao, "Coevolutionary dynamics of actions and opinions in social networks," *IEEE Trans. Autom. Control.*, vol. 68, no. 12, pp. 7708–7723, 2023.
[18] Y. Yi, T. Castiglia, and S. Patterson, "Shifting opinions in a social network through leader selection," *IEEE Trans. Control Netw. Syst.*, vol. 8, no. 3, pp. 1116–1127, 2021.
[19] D. Kempe, J. Kleinberg, and E. Tardos, "Maximizing the Spread of Influence through a Social Network," in *Proc. 9th ACM SIGKDD Int. Conf. Knowledge Discovery and Data Mining*, 2003, pp. 137–146.
[20] G. Como, S. Durand, and F. Fagnani, "Optimal targeting in super-modular games," *IEEE Trans. Autom. Control.*, vol. 67, no. 12, pp. 6366–6380, 2022.
[21] J. R. Marden, G. Arslan, and J. S. Shamma, "Cooperative control and potential games," *IEEE Trans. Syst. Man Cybern. B*, vol. 39, no. 6, pp. 1393–1407, 2009.
[22] S. Messina, G. Como, S. Durand, and F. Fagnani, "Optimal intervention in non-binary super-modular games," *IEEE Control Syst. Lett.*, vol. 7, pp. 2353–2358, 2023.
[23] R. G. Bartle, *The Elements of Real Analysis*. Wiley, 1976.

APPENDIX

A. Proof of Theorem 1

The proof deploys over three steps: i) prove that $\boldsymbol{y}(t)$ is monotonically non-decreasing subject to the monotonicity of $\boldsymbol{x}(t)$; ii) prove that $\boldsymbol{x}(t)$ is non-decreasing; and iii) combine the two results, yielding the claim. For the sake of readability, we denote $\delta_i(t) := \delta_i(\boldsymbol{z}(t))$.

Lemma 1. *Consider a controlled coevolutionary dynamics under Assumptions 1 and 2. Assume that $\boldsymbol{x}(t) \geq \boldsymbol{x}(t-1)$, $\forall\, t \leq \tau$. Then, $\boldsymbol{y}(t) \geq \boldsymbol{y}(t-1)$, $\forall\, t \leq \tau$.*

Proof. We prove the inequality component-wisely. The statement holds trivially for all $i \in \mathcal{C}$. For individuals in $\mathcal{U}$, we prove the statement by induction. Consider an arbitrary $i \in \mathcal{U}$. At $t = 1$, the inequality holds true. In fact, if $i \in \mathcal{R}(0)$, then $y_i(1) = (1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij} y_j(0) + \lambda_i s(\delta_i(t))) \geq -(1-\lambda_i) - \lambda_i = -1 = y_i(0)$, where we bound $y_j(0) \geq -1$ $\forall\, j \in \mathcal{V}$. If $i \notin \mathcal{R}(0)$, then $y_i(1) = y_i(0)$.

We prove now the induction step, by demonstrating that, if $\boldsymbol{y}(t) \geq \boldsymbol{y}(t-1)$ for a generic $t < \tau$, then $\boldsymbol{y}(t+1) \geq \boldsymbol{y}(t)$. If $i \notin \mathcal{R}(t)$, then $y_i(t+1) = y_i(t)$, yielding the claim. If $i \in \mathcal{R}(t)$, we distinguish two cases. If i does not change action at time t, then $s(\delta_i(t)) = s(\delta_i(t-1))$. Using the inductive hypothesis it follows that $y_i(t+1) = (1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij} y_j(t) + \lambda_i s(\delta_i(t)) \geq (1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij} y_j(t-1) + \lambda_i s(\delta_i(t-1)) = y_i(t)$, where in the first term we bound $y_j(t) \geq y_j(t-1)$, using the assumption that the inequality holds at time t. On the other hand, if i at time t changes action, then the change is necessarily from $x_i(t-1) = -1$ to $x_i(t) = +1$ (because of the monotonicity assumption on $\boldsymbol{x}$). Then, $+1 = s(\delta_i(t)) \geq s(\delta_i(t-1))$. Under this hypothesis, and with $\lambda_i > 0$, the inequality may become strict. In both cases $y_i(t+1) \geq y_i(t)$, $\forall\, i \in \mathcal{U}$. Finally, combining the base case $t = 1$ and the induction step, yields the claim. □

Lemma 2. *Consider a controlled coevolutionary dynamics under Assumptions 1 and 2. Then, $\boldsymbol{x}(t+1) \geq \boldsymbol{x}(t)$, $\forall\, t \geq 0$.*

Proof. We prove the inequality component-wisely. Since the statement holds trivially for all $i \in \mathcal{C}$, we focus on $i \in \mathcal{U}$ and we proceed by contradiction. Assume that $\boldsymbol{x}(t)$ is not monotonically nondecreasing. Then, there exists a finite time $\hat{t} := \inf\{t : \exists\, i \in \mathcal{U} \text{ such that } x_i(t) < x_i(t-1)\}$. By definition, at time $\hat{t}$ there is necessarily $i \in \mathcal{U}$ for which $x_i(\hat{t}-1) = +1$ and $x_i(\hat{t}) = -1$. The model definition indicates that $\delta_i(\hat{t}-1) < 0$. Moreover, since $x_i(\hat{t}-1) = +1$, there must be a time $\tilde{t} \in \{\hat{t}-1-T, \dots, \hat{t}-2\}$ in which $i \in \mathcal{R}(\tilde{t})$ and $\delta_i(\tilde{t}) \geq 0$. Hence, $\delta_i(\hat{t}-1) < \delta_i(\tilde{t})$. Let us use the definition of $\delta_i(t)$ to rewrite the condition $\delta_i(\hat{t}-1) < \delta_i(\tilde{t})$ and prove that such relation is impossible. In fact, we get

$$2\beta_i(1-\lambda_i){\textstyle\sum_{j\in\mathcal{V}}} w_{ij} y_j(\tilde{t}) + (1-\beta_i){\textstyle\sum_{j\in\mathcal{V}}} a_{ij} x_j(\tilde{t}) > 2\beta_i(1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij} y_j(\hat{t}-1) + (1-\beta_i)\sum_{j\in\mathcal{V}} a_{ij} x_j(\hat{t}-1). \quad (15)$$

Given that Eq. (15) involves only the values of $\boldsymbol{x}$ and $\boldsymbol{y}$ up to time $\hat{t}-1$ and being $\hat{t}$ the first time instant in which $\boldsymbol{x}(t)$ has decreased, then for any $t \leq \hat{t}-1$, the action vector $\boldsymbol{x}(t)$ is monotonically non-decreasing. Hence by Lemma 1, $\boldsymbol{y}(\tilde{t}) \leq \boldsymbol{y}(\hat{t}-1)$, and so $\beta_i(1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij} y_j(\tilde{t}) \leq 2\beta_i(1-\lambda_i)\sum_{j\in\mathcal{V}} w_{ij} y_j(\hat{t}-1)$. In other words, the first term on the LHS of Eq. (15) is not larger than the corresponding term on the right-hand side (RHS). Hence, a necessary condition for Eq. (15) to hold is that the second term on the LHS is larger than the corresponding one on the RHS, i.e., $\sum_{j\in\mathcal{V}} a_{ij} x_j(\tilde{t}) > \sum_{j\in\mathcal{V}} a_{ij} x_j(\hat{t}-1)$. However, this inequality would be satisfied iff it exists at least an individual j who has changed action from $+1$ to -1 at some time between $\tilde{t}+1$ and $\hat{t}-1$, which is impossible by the definition of $\hat{t}$. Hence, Eq. (15) cannot hold and so $\hat{t}$ cannot exist, yielding the contraddiction and thus the claim. □

Finally, by combining Lemmas 1 and 2, we conclude that the whole state vector $\boldsymbol{z}(t) = (\boldsymbol{x}(t), \boldsymbol{y}(t))$ is a monotonically non-decreasing function of t. Moreover, both vectors $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are bounded by Proposition 1. Hence, by the monotone convergence theorem [23], both sequences $\boldsymbol{y}(t)$ and $\boldsymbol{x}(t)$ admit a limit. Finally, being $\boldsymbol{x}(t)$ a discrete (and finite) sequence, its convergence should necessary occur in finite time, while convergence of the vector $\boldsymbol{y}(t)$ is asymptotic.