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Mathematical Modeling of Chattering and the Optimal Design of a Valve*

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***Abstract*— We consider an isothermal flow through two pipes; at the junction, the flow is modified by some devices, for instance a valve. We first provide a general mathematical framework to model the coupling conditions for the flow at both sides of the junction. A key feature in the modeling is the *coherence*; it is related to the chattering, i.e., the rapid switch on and off of a valve, which in turn is linked to the stability of the numerical schemes to find the solutions. We discuss the coherence of some models (for instance, for control valves), we provide numerical simulations showing the chattering, and finally give a procedure to eliminate it from a theoretical point of view.**

I. INTRODUCTION

In this paper we explore the dynamics of isothermal, one-dimensional gas flows through a junction connecting two pipes. The aim is to understand certain mathematical properties of the coupling conditions required at the junction to model the flow. For an overview of flow dynamics on networks, we direct readers to the survey [5].

The essence of coupling conditions lies in their ability to model the presence of various devices between the pipes, such as valves and compressors, or to account for simply different characteristics of the pipes themselves. Coupling conditions can govern aspects such as the continuity of the pressure or of the dynamical pressure; they may potentially determine an increase or a decrease of the pressure; they play crucial roles in optimizing flow rates, and so on.

Our analysis focuses on the Riemann problems (initial-value problems characterized by piecewise constant data that exhibit at most a single jump discontinuity) specifically modeling a coupling at $x = 0$. The way such Riemann problems are solved gives rise to the development of a *c-Riemann solver*. It is noteworthy that for most of c-Riemann solvers, the conservation of momentum is lost, and in these cases they differ from the standard Lax Riemann solver [37].

We deal with three main issues:

- **Uniqueness of the c-Riemann solver:** We delve into whether a specific coupling condition unambiguously identifies a single c-Riemann solver. Our findings indicate that numerous models fail to meet this criterion.
- **Coherence of the c-Riemann solver:** A c-Riemann solver is coherent if any ordered pair of its adjacent values is a fixed point. This property is fundamental both as a test for the validity of the model and for enhancing the robustness of the numerical schemes that one can use to find a c-solution. Our analysis uncovers that coherence fails for some commonly used c-Riemann solvers.
- **Handling supersonic flows:** While most papers only deal with the easier case of subsonic flows, our study extends the discussion to encompass supersonic flows, an area less considered in existing literature [26], [40].

The paper provides a comprehensive examination of various coupling conditions. A general framework to define and study c-Riemann solvers is presented in Section II. The main results in Section III concern a detailed analysis of the continuity conditions (on pressure, dynamic pressure or specific enthalpy) for general two-way flows. Section IV specializes this analysis to the case of one-way flows, allowing however possibly different pressure laws in the pipes. In Section V we consider two types of compressors; we show that if we have a two-way flow then none of the corresponding c-Riemann solvers is uniquely defined, whereas if the flow is one-way then both of them are well defined and coherent. Due to lack of space, we avoid a discussion about the role of supersonic flows in this modelling; we refer the interested reader to the recent preprint [15]. In the concluding Section VII, we first comment on the several valves and related results proved in [13], [14], [16], [17]; then we show that some other modelings proposed in the literature either do not lead to a unique c-Riemann solver or, when this happens, the solver is incoherent.

II. THE MODEL

The gas flow along a pipe is governed by the Euler equations

$$\begin{cases} \rho_t + q_x = 0, \\ q_t + P_x = 0, \end{cases} \tag{1}$$

where $\rho = \rho(t, x)$ is the *density*, $q = q(t, x)$ the *momentum* at time $t \geqslant 0$ and position $x \in \mathbf{R}$, and $P(\rho, q) \doteq q^2/\rho + p(\rho)$ is the *dynamic pressure*. We focus on the isothermal case and then we take as pressure

$$p(\rho) = a^2\rho. \tag{2}$$

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The constant $a > 0$ is the *sound speed*. We also define the *velocity* $v(\rho, q) \doteq q/\rho$. A state $u \doteq (\rho, q)$ is *subsonic* if $|v(u)| < a$, *sonic* if $|v(u)| = a$ and *supersonic* if $|v(u)| > a$. In the phase plane the *sonic curves* are $s_\pm(\rho) \doteq \pm a\,\rho$. We denote $\Omega \doteq \{(\rho, q) \in \mathbf{R}^2 : \rho > 0\}$.

A. The Riemann problem

For any pair of constant states $u_\ell, u_r \in \Omega$, the Riemann problem for system (1) is the initial-value problem with initial condition

$$u(0, x) = \begin{cases} u_\ell & \text{if } x < 0, \\ u_r & \text{if } x \geqslant 0. \end{cases} \tag{3}$$

Solutions of Riemann problem (1)-(3) are meant in the weak sense. Any smooth discontinuity curve $x = \gamma(t)$ of a piecewise regular weak solution $u = (\rho, q)$ of (1) satisfies the Rankine-Hugoniot conditions

$$(\rho^+ - \rho^-)\,\dot\gamma = q^+ - q^-, \tag{4}$$

$$(q^+ - q^-)\,\dot\gamma = P(\rho^+, q^+) - P(\rho^-, q^-), \tag{5}$$

where $u^\pm(t) \doteq u(t, \gamma(t)^\pm)$ are the traces of u along the discontinuity. Condition (4) ensures the conservation of the mass, while (5) is the conservation of momentum.

We denote the standard Riemann solver of (1) by $\mathcal{RS}_\mathrm{p}$, see [37]. If we denote

$$u_\mathrm{p} \doteq (\rho_\mathrm{p}, q_\mathrm{p}) \doteq \mathcal{RS}_\mathrm{p}[u_\ell, u_r], \quad u_\mathrm{p}^\pm \doteq (\rho_\mathrm{p}^\pm, q_\mathrm{p}^\pm) \doteq u_\mathrm{p}(0^\pm),$$

then the function $(t, x) \mapsto u_\mathrm{p}(x/t)$ is an *entropy solution* of the Riemann problem (1), (3). Recall that $\mathcal{RS}_\mathrm{p}[u_\ell, u_r]$ is well defined for any $(u_\ell, u_r) \in \mathsf{D} \doteq \Omega \times \Omega$ and it is obtained by juxtaposing the 1-Lax wave from u_ℓ to some state $\tilde u \in \Omega$ and the 2-Lax wave from $\tilde u$ to u_r.

B. c-Riemann solvers

For a general coupling at $x = 0$, the associated c-Riemann solver defined in $\mathsf{D}_\mathrm{c} \subseteq \mathsf{D}$ takes the form

$$\mathcal{RS}_\mathrm{c}[u_\ell, u_r](\xi) \doteq \begin{cases} \mathcal{RS}_\mathrm{p}\left[u_\ell, u_\mathrm{c}^-\right](\xi) & \text{if } \xi < 0, \\ \mathcal{RS}_\mathrm{p}\left[u_\mathrm{c}^+, u_r\right](\xi) & \text{if } \xi \geqslant 0, \end{cases} \tag{6}$$

where $(u_\mathrm{c}^-, u_\mathrm{c}^+) = C(u_\ell, u_r)$ for some *coupling function* $C \colon \mathsf{D}_\mathrm{c} \to \mathsf{D}$. The function C is such that $\mathcal{RS}_\mathrm{c}[u_\ell, u_r](0^\pm) = u_\mathrm{c}^\pm$, namely $\mathcal{RS}_\mathrm{p}\left[u_\ell, u_\mathrm{c}^-\right]$ has only waves with negative speeds and viceversa for $\mathcal{RS}_\mathrm{p}\left[u_\mathrm{c}^+, u_r\right]$, see [15]. The traces of the momentum are equal; this common value is denoted by q_c^0. Hence, if we introduce

$$u_\mathrm{c} \doteq (\rho_\mathrm{c}, q_\mathrm{c}) \doteq \mathcal{RS}_\mathrm{c}[u_\ell, u_r],$$

then $u_\mathrm{c}(0^\pm) = u_\mathrm{c}^\pm$ and $u_\mathrm{c}^\pm = (\rho_\mathrm{c}^\pm, q_\mathrm{c}^0)$ for $(u_\ell, u_r) \in \mathsf{D}_\mathrm{c}$.

The function $(t, x) \mapsto \mathcal{RS}_\mathrm{c}\left[u_\ell, u_r\right](x/t)$ is an entropy solution to (1) in $x < 0$ and $x > 0$ and satisfies condition (4) at $x = 0$. On the contrary, u_c may not satisfy condition (5) at $x = 0$, hence conservation of momentum may be lost.

An important property of $\mathcal{RS}_\mathrm{c}$ is that it admits at most four waves, included the stationary discontinuity at $\xi = 0$; the maximal number is reached only for supersonic flows. Indeed, if the flow is subsonic at $\xi = 0$, then $\mathcal{RS}_\mathrm{c}$ can involve up to three waves; in this case u_c consists of a 1-wave in $\xi < 0$ and a 2-wave in $\xi > 0$.

A c-Riemann solver $\mathcal{RS}_\mathrm{c}$, defined on a set D_c, is *coherent* at $(u_\ell, u_r) \in \mathsf{D}_\mathrm{c}$ if the traces $u_\mathrm{c}^\pm$ satisfy

$$(u_\mathrm{c}^-, u_\mathrm{c}^+) \in \mathsf{D}_\mathrm{c} \qquad \text{and} \qquad C(u_\mathrm{c}^-, u_\mathrm{c}^+) = (u_\mathrm{c}^-, u_\mathrm{c}^+).$$

The *coherence domain* CH of $\mathcal{RS}_\mathrm{c}$ is the set of all pairs $(u_\ell, u_r) \in \mathsf{D}_\mathrm{c}$ where $\mathcal{RS}_\mathrm{c}$ is coherent. The set $\mathsf{CH}^\complement \doteq \mathsf{D}_\mathrm{c} \setminus \mathsf{CH}$ is the *incoherence domain*. Coherence may be thought as a stability property: if we apply the c-Riemann solver to the traces $u_\mathrm{c}^\pm$ at $\xi = 0$ of a solution (indeed, to any ordered pair of adjacent values of a solution), then we obtain a solution with the same traces at $\xi = 0$. On the contrary, for instance if the coupling represents a valve, see Figure 1, the incoherence of a c-Riemann solver is understood as modeling the chattering [16, Section 6] and may yield analytical and numerical instabilities, see Fig. 2.

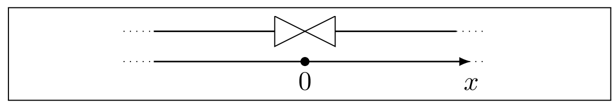

Fig. 1: Schematic diagram to illustrate the pipeline and valve configuration.

The chattering, the rapid opening and closing of valves, causes loud vibration noises and damages the valve components [7], [35], [34], [18]. Addressing its causes and prevention is then essential for system safety and functionality.

The Riemann solver $\mathcal{RS}_\mathrm{p}$ is coherent in D, see [13]. It is possible to give sufficient conditions on the coupling function C in order that the corresponding c-Riemann solver is coherent, see [15].

III. CONTINUITY CONDITIONS

We now review some popular coupling conditions and investigate their coherence; we refer to [15] for details.

A. Continuity of the pressure

The condition

$$p(\rho_\mathrm{c}^+) = p(\rho_\mathrm{c}^-), \tag{7}$$

expresses the continuity of the pressure at $x = 0$, see [1, (15b)], [4, (6)], [6, (A.2)], [12, (4.5)], [20, (8)], [22, (87)], [23, (11)], [24, (2.4)], [26, (2a)], [28, § 2.4], [30, (4.1)], [31, (8b)], [32, (C2)], [36, (3)], [38, (18)], [39, (34)], [42, (3.1)]. By (2), condition (7) implies the continuity of the density at $x = 0$ and so $u_\mathrm{c}^- = u_\mathrm{c}^+$. Then, the requirement (7) hides a smoothness assumption on the flow at $x = 0$. The resulting c-Riemann solver $\mathcal{RS}_\mathrm{c}$ coincides with $\mathcal{RS}_\mathrm{p}$ in its domain of definition $\mathsf{D}_\mathrm{c} \doteq \{(u_\ell, u_r) \in \mathsf{D} : \rho_\mathrm{p}^+ = \rho_\mathrm{p}^-\}$, which is strictly contained in the domain of definition D of $\mathcal{RS}_\mathrm{p}$. Moreover, $\mathcal{RS}_\mathrm{c}$ is coherent in D_c because $\mathcal{RS}_\mathrm{p}$ is coherent in D.

B. Continuity of the dynamic pressure

The coupling condition

$$P(\rho_c^+, q_c^0) = P(\rho_c^-, q_c^0), \tag{8}$$

expresses the continuity of $P(u_c)$ at $x = 0$, where P is the dynamic pressure. It corresponds to the second Rankine-Hugoniot condition (5) at $x = 0$. If the flow is subsonic, then $u_c^- = u_c^+$ and therefore $u_c \equiv u_p$; as a consequence, coherence is granted. However, for a general flow, condition (8) does not select a unique c-Riemann solver, see [15].

In order to overcome the non-uniqueness of the c-Riemann solver, one can add to condition (8) the coupling inequality

$$F(\rho_c^+, q_c^0) \leqslant F(\rho_c^-, q_c^0), \tag{9}$$

for $F(\rho, q) \doteq q\left(\frac{q^2}{2\rho^2} + a^2 \ln(\rho)\right)$ being the *flow of the energy density*. The c-Riemann solver corresponding to (8), (9) coincides with $\mathcal{RS}_p$; as a consequence, the c-Riemann solver implicitly defined by (8), (9) is well defined and coherent in D.

C. Continuity of the specific enthalpy

The coupling condition

$$\mathcal{E}(u_c^+) = \mathcal{E}(u_c^-) \quad \text{for} \quad \mathcal{E}(u) \doteq \frac{v(u)^2}{2} + a^2 \ln(\rho), \tag{10}$$

is the continuity of the *Bernoulli invariant* [43] or of the *specific stagnation enthalpy* [21]. As in case B, condition (10) does not select a unique coupling function C in D.

IV. ONE WAY FLOWS

This section deals with the two couplings introduced in [2]. We consider one-way flows; moreover, the flow in the two pipes $x < 0$ and $x > 0$ is ruled by the pressure laws $p_1(\rho) \doteq a_1^2\,\rho$ and $p_2(\rho) \doteq a_2^2\,\rho$, respectively. We denote the function $\tilde{u}$ introduced above and corresponding to the sonic velocity $a = a_1$ by $\tilde{u}_{a_1}$; an analogous notation is exploited for other quantities.

The first coupling is obtained by imposing that the flow along the outgoing pipe is not supersonic and by maximizing q_c^0. The resulting c-Riemann solver is unique and coherent.

On the other hand, assume the continuity of the pressure at $\xi = 0$, i.e., (7), and, among all the c-Riemann solvers satisfying such condition, assume that $\mathcal{RS}_c$ maximizes the flow across $\xi = 0$. Also this coupling provides a unique c-Riemann solver, but it is not coherent for some choices of a_1 and a_2, see Fig. 2. There, the parameters $a_1 = 1$, $a_2 = 2$, $q_\ell = 0$, $u_r = (10, 24)$ and $\rho_\ell \approx 13.47$ are chosen so that the pair (u_ℓ, u_r) is an incoherent Riemann datum. For the numerical scheme, we use the Glimm scheme [9], [46], applied to c-Riemann solvers and described in [14], where the validation was performed by comparisons with available exact solutions and has shown that the scheme is of order one in space. In Fig. 2, the two pictures above show the correct solution, corresponding to $\mathcal{RS}_c[u_\ell, u_r]$, which is obtained when the traces $u_c^\pm$ are computed *once and for all* at the initial time. The two pictures below present the case where the traces are *updated at each time step*. The procedure of updating $u_c^\pm$ at each time step, which is usual in numerical schemes for general initial-value problems, has the serious drawback of introducing nearby incoherent states. The solution obtained in this way substantially differs from the right one: a small rarefaction wave in $\xi < 0$ and the jump at $\xi = 0$ are substituted by a large rarefaction wave, and the plateau in $\xi > 0$ is higher, longer and contains oscillations, that appear independently of the values of the discretization parameters. This demonstrates the instability of the scheme in this case, caused by the incoherence of $\mathcal{RS}_c$. On the contrary, the scheme shows no oscillations for coherent Riemann data. Moreover, computing once and for all the traces or at each time step leads to the same solutions. This is illustrated in Fig. 3, where the coherent initial datum has the same values as before except for $\rho_\ell = 13.5$.

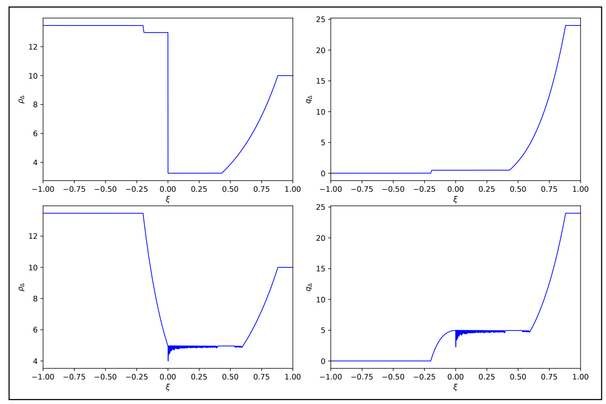

Fig. 2: Numerical illustration of the incoherence of the second c-Riemann solver considered in Section IV. The profiles are computed at time $t = 0.2$. Above: traces $u_c^\pm$ are computed once and for all at the initial time. Below: the traces are updated at each time step.

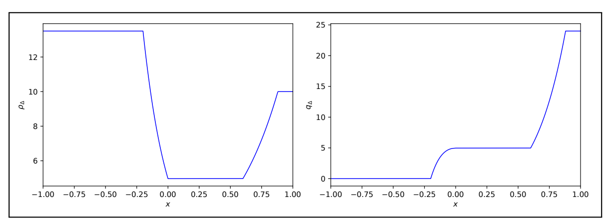

Fig. 3: Numerical illustration of the second c-Riemann solver considered in Section IV with coherent initial datum. The profiles are computed at time $t = 0.2$. The solution computed with updating the traces at each time step and the one computed only once and for all, coincide.

V. COMPRESSORS

In the previous sections we investigated the well-posedness and coherence of several couplings at the intersection of two pipes. Such couplings are rather general, but they are understood to model "free" flows through the junction. From now on, we focus on couplings where external forces, modeling the action of some device or friction effects, take place. We begin with compressors.

A compressor is powered by the gas flowing through it; however, the gas consumption is usually neglected in the modeling. The purpose of a compressor is twofold. On the one hand, it increases the pressure and then the density of the gas, i.e., $p(\rho_{\rm c}^-) < p(\rho_{\rm c}^+)$ and $\rho_{\rm c}^- < \rho_{\rm c}^+$. On the other hand, it reduces the outlet velocity, i.e., $v(\rho_{\rm c}^-) > v(\rho_{\rm c}^+)$, because $u_{\rm c}^-$ and $u_{\rm c}^+$ have the same flow $q_{\rm c}^0$ but $\rho_{\rm c}^- < \rho_{\rm c}^+$.

We consider below both two-way and one-way flows; the compressor is located at $x = 0$. In both two-way cases, we show that the coupling conditions do not select a unique coupling function C. On the contrary, in the one-way cases, we prove the uniqueness of C and the coherence of the corresponding c-Riemann solver.

A. Two-way flows

A first modeling assumes that the ratio between the incoming and outgoing pressures is constant. If the compressor is switched on, then this ratio is greater than one if the flow at $x = 0$ is positive, otherwise it is less than one; if the compressor is switched off then this ratio equals one. The corresponding coupling condition can be written as

$$p(\rho_{\rm c}^+) = \big(1 + K_p(q_{\rm c}^0)\big)\, p(\rho_{\rm c}^-). \tag{11}$$

Here, if the compressor is switched on then either $K_p(q_{\rm c}^0) > 0$ in the case $q_{\rm c}^0 > 0$ or $K_p(q_{\rm c}^0) \in (-1, 0)$ otherwise; if the compressor is switched off then $K_p(q_{\rm c}^0) = 0$, see [20]. Observe that if $K_p(q_{\rm c}^0) = 0$ then condition (11) reduces to (7), which has already been considered. Then, below we assume $K_p(q_{\rm c}^0) \neq 0$.

In general K_p depends on time; we assume K_p to be piecewise constant since we are interested in the corresponding c-Riemann solver. More precisely, we focus on

$$K_p(q_{\rm c}^0) = \begin{cases} K_p^- & \text{if } q_{\rm c}^0 \leqslant 0, \\ K_p^+ & \text{if } q_{\rm c}^0 > 0, \end{cases} \qquad -1 < K_p^- < 0 < K_p^+. \tag{12}$$

For this modeling, one can show that conditions (11) and (12) do not select a unique c-Riemann solver. Moreover, it is possible to construct a counterexample which avoids null flows at $x = 0$ (which makes the modeling not completely meaningful) and only involves subsonic flows.

A different modeling considers the coupling condition

$$|q_{\rm c}^0| \left(\big(p(\rho_{\rm c}^+)/p(\rho_{\rm c}^-)\big)^{\operatorname{sign}(q_{\rm c}^0)\,\kappa} - 1 \right) = K, \tag{13}$$

at $x = 0$, where $\kappa \in [2/7, 2/5]$ is a parameter that depends on the gas under consideration and $K \geqslant 0$ is the *compressor power* [23]. Also in this case, for fixed $K \geqslant 0$, condition (13) does not select a unique c-Riemann solver.

B. One-way flows

Consider the coupling condition (11), (12), and assume the flow is one-way, say $q \geqslant 0$. Then the coupling condition is

$$p(\rho_{\rm c}^+) = (1 + K_p^+)\, p(\rho_{\rm c}^-), \tag{14}$$

for a constant $K_p^+ > 0$, see [29], [38]. Then the coupling condition (14) selects a unique coupling function C. Moreover, the c-Riemann solver $\mathcal{RS}_{\rm c}$ corresponding to (14) is coherent.

Now, consider the coupling condition (13) and assume $K > 0$. Then the corresponding coupling condition is

$$q_{\rm c}^0 \left(\big(p(\rho_{\rm c}^+)/p(\rho_{\rm c}^-)\big)^{\kappa} - 1 \right) = K, \tag{15}$$

see [8], [19], [25], [27], [31], [32], [44], [45]. In [19] and [25] the authors let κ vary in $[2/7, 2/5]$ and $[1/3, 3/5]$, respectively. By the assumptions we have

$$\rho_{\rm c}^+ > \rho_{\rm c}^-, \qquad\qquad q_{\rm c}^0 > 0. \tag{16}$$

Therefore (15) is equivalent to

$$p(\rho_{\rm c}^+) = \left(1 + \frac{K}{q_{\rm c}^0}\right)^{1/\kappa} p(\rho_{\rm c}^-).$$

In this case the coupling condition (15) selects a unique coupling function and the corresponding c-Riemann solver is coherent.

VI. Valves

As well as compressors, valves are an important ingredient in gas networks. In this section, we first briefly recall some recent results concerning their coherence and its physical meaning. Then we show that some simplified models usually exploited for flow optimizations in networks are not coherent.

A. Coherence and chattering

Pressure-relief valves are considered in [13]. A detailed study is conducted on a valve, which is closed if the jump of absolute value of the pressure at $x = 0$ does not exceed a threshold value $M > 0$; otherwise, it is open. The velocities of the flows are general: no assumption of subsonicity is done. In such a case, the c-Riemann solver is not coherent and its coherence domain is explicitly provided. The lack of coherence has been interpreted in that paper as modeling the phenomenon of *chattering*, the rapid and repeated opening and closing of the valve; see [13], [14], [16], [17] and references therein for more information on this phenomenon.

The case of one-way valves is discussed in [16]. As an example of the general framework treated there, it was considered the case of a valve that aims at keeping a fixed outgoing flow $q_* > 0$; when this is not possible, then the valve shuts. Such valves are known as *pressure independent characterized control valves*. The main results are an explicit characterization of the coherence domain of the c-Riemann solver, and a discussion of the invariant domains. Also the case of a valve with a non-zero reaction time $\tau > 0$ is considered and an explicit example is constructed to show how the incoherence of a c-Riemann solver leads to a chattering behaviour, i.e., a rapid switch on and off, of the corresponding valve, see Fig. 4. In that figure we use a wave-front tracking scheme where each continuous rarefaction wave is approximated by three jump waves; we refer to [16, §6] for details. Furthermore, it is considered the following control problem: for any given $\Omega_\ell \subseteq \Omega$, determine the values of the control parameter q_* such that the restriction of the corresponding c-Riemann solver to $\Omega_\ell \times \Omega$ is coherent.

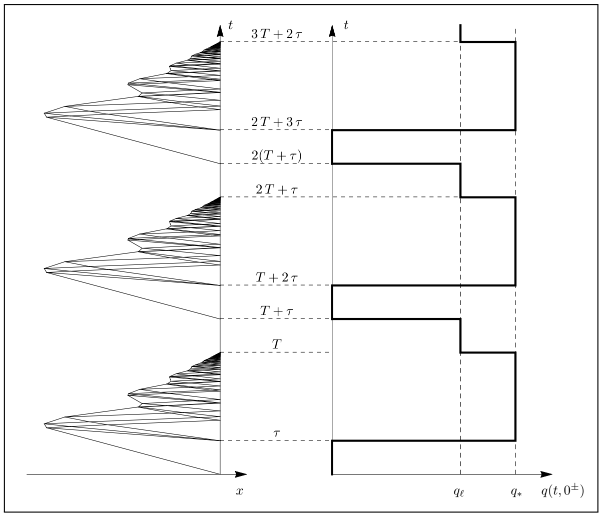

Fig. 4: Left: the approximate solution $(t, x) \mapsto u(t, x)$ for $t \in [0, 3T + 2\tau]$ and $x < 0$. Right: the flow through the valve.

The paper [17] deals with flux-maximizing valves. Moreover, the flow is imposed to occur within prescribed bounds of pressure and flow; this requirement clearly corresponds to the existence of invariant domains. Within this framework, three kinds of valves are described, which differ for their action; two of them lead to a coherent solver, the third one does not.

How to "remove" the chattering of valve? A theoretical answer to this issue is given in [14]: one can modify the c-Riemann solver in the incoherent domain CH^{c} in order that it becomes coherent in the whole of its domain $\mathsf{D}_{\rm c}$ and the new c-Riemann solver differs from the old one only for the states that led the old solver to lose coherence. An example of this procedure is shown in [14] for the (incoherent) c-Riemann solver considered in [16]. Moreover, for incoherent initial data, the new solver selects the unique solution that maximizes the flow through the valve among all c-Riemann solvers. Several numerical simulations are also provided.

A partial conclusion of the results of the above papers is that the mechanism that leads to the loss of coherence, and then possibly triggers chattering, is hard to understand. Indeed, such a behavior strongly depends on the type of valve under consideration and establishing general criteria is not yet clear.

B. Control valves in optimization

An important issue in gas networks concerns the optimization of flows, depending on the presence of compressors, valves and other devices in the network [28], [45]. The complexity of the problem essentially requires that the flows are constant in each pipe and variations only occur at the junctions. In this subsection we briefly comment on how valves are modeled in such a framework and show that such oversimplified modelings are not coherent.

A *control valve* can be modeled by the coupling conditions

$$\begin{aligned} p(\rho_{\rm c}^+) &= p(\rho_{\rm c}^-) - \Delta && \text{if the valve is active,}\\ p(\rho_{\rm c}^+) &= p(\rho_{\rm c}^-) && \text{if the valve is in bypass mode,}\\ q_{\rm c}^0 &= 0 && \text{if the valve is closed,} \end{aligned}$$

where $\Delta > 0$ is a fixed parameter. In general the pressure difference Δ depends on time; we assume that it is constant.

In the case the control valve is in bypass mode, then $\mathcal{RS}_{\rm c} \equiv \mathcal{RS}_{\rm p}$. If the valve is always active, then the situation is more delicate. Indeed, the coupling condition

$$p(\rho_{\rm c}^+) = p(\rho_{\rm c}^-) - \Delta \tag{17}$$

does not select a unique c-Riemann solver.

A simple way to fix the above drawback consists in imposing a maximization property of the flow across the coupling: among all the c-Riemann solvers satisfying (17), we choose the one that maximizes the flow across $x = 0$. Such a c-Riemann solver is unique, but it is not coherent.

In [45] the authors consider a *control valve without remote access*. Such a valve is designed to keep the outgoing pressure below a given threshold, $p(\rho_{\rm c}^+) \leqslant p^{\rm set}$. The valve is in bypass mode if $p(\rho_{\rm c}^-) \leqslant p^{\rm set}$, it automatically closes when $p(\rho_{\rm c}^-) > p^{\rm set} + \Delta$, where Δ is a constant; otherwise the control valve is active, reducing the pressure by some amount $\delta = \delta(u_\ell, u_r) \in (0, \Delta]$ so that $p(\rho_{\rm c}^+) = p(\rho_{\rm c}^-) - \delta = p^{\rm set}$. As a result we have

$$\begin{aligned} p(\rho_{\rm c}^+) &= p(\rho_{\rm c}^-) && \text{if } p(\rho_{\rm c}^-) \leqslant p^{\rm set}, && \text{(18a)}\\ p(\rho_{\rm c}^+) &= p^{\rm set} && \text{if } p^{\rm set} < p(\rho_{\rm c}^-) \leqslant p^{\rm set} + \Delta, && \text{(18b)}\\ q_{\rm c}^0 &= 0 && \text{if } p(\rho_{\rm c}^-) > p^{\rm set} + \Delta. && \text{(18c)} \end{aligned}$$

The c-Riemann solver corresponding to a control valve without remote access corresponding to (18) is uniquely determined but it is not coherent.

VII. Conclusions

We proposed a general framework for the mathematical modeling of isothermal fluid flows through a junction. We showed that several models occurring in the applied literature have serious drawbacks: lack of well-posedness of the Riemann problem, non-uniqueness of the solver and incoherence. The latter condition, in particular, leads to instabilities in numerical simulations and, if a valve is present at the junction, is responsible for the chattering of the valve. We also propose a theoretical way to modify a valve to eliminate the chattering. We provide numerical simulations that confirm our results. This research paves to way to several related problems: the modeling in the isentropic case or for the full Euler system, the study of chattering in networks, and the analysis of the control of the flows.

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