2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

On Static O'Shea-Zames-Falb Multipliers for Idempotent Nonlinearities

Tsuyoshi Yuno, Shingo Nishinaka, Rin Saeki, and Yoshio Ebihara

***Abstract*— In this paper, we investigate static O'Shea-Zames-Falb (OZF) multipliers for slope-restricted and idempotent nonlinearities. For the analysis of nonlinear feedback systems, the powerful framework of integral quadratic constraint has been frequently employed, where the core of this framework is capturing the behavior of nonlinearities by multipliers. Among them, OZF multipliers are known to be effective for slope-restricted nonlinearities. However, OZF multipliers only grasp the rough slope properties of the nonlinearities, and hence cannot distinguish nonlinearities with the same slope property. To address this issue, we focus on the fact that some nonlinearities that are important in control engineering satisfy *idempotence*. By actively using the idempotence property, we first show that we can enlarge (relax) the set of standard OZF multipliers for slope-restricted nonlinearities. In particular, for slope-restricted and idempotent nonlinearities with slope $[0, 1]$, we can provide a clear understanding on how the set of standard OZF multipliers is enlarged. We finally illustrate the effectiveness of the newly proposed multipliers by numerical examples on stability analysis of nonlinear feedback systems.**

Keywords: idempotent nonlinearities, static O'Shea-Zames-Falb multipliers.

I. Introduction

Recently, it has been recognized that control theoretic approaches are effective for the analysis of optimization algorithms [10], stability analysis of dynamic (recurrent) neural networks (NNs) [14], [4], [5], and performance analysis of feedback control systems driven by NNs [17], [15]. Therefore, there is renewed interest in the analysis of nonlinear feedback systems. In particular, for the analysis of nonlinear feedback systems involving NNs, we have to deal with a huge number of various nonlinear activation functions such as saturation, rectified linear unit (ReLU), hyperbolic tangent (tanh), sigmoid, etc. This poses new challenges for the treatment of nonlinear feedback systems.

For the analysis of nonlinear feedback systems, the powerful framework of integral quadratic constraint (IQC) [11] has been frequently employed, where the core of this framework is capturing the behavior of nonlinearities by multipliers. Among such multipliers, O'Shea-Zames-Falb (OZF) multipliers [12], [18], [16] are known to be effective for slope-restricted nonlinearities. The studies on OZF multipliers have been active, and advanced treatments with convex optimization can be found, e.g., in [8], [1]. The history on the study of OZF multipliers and prominent results to this date are summarized in the tutorial paper [2].

This work was supported by JSPS KAKENHI Grant Number JP 20H04145 and JP21H01354. (Corresponding author: Yoshio Ebihara)

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Even though OZF multipliers are effective for the treatments of slope-restricted nonlinearities, they only grasp the rough slope properties and hence cannot distinguish nonlinearities with the same slope property. To address this issue, in this paper, we focus on the fact that some nonlinearities that are important in control engineering satisfy *idempotence*. Recall that an operator $\phi:\ \mathbb{R} \to \mathbb{R}$ is said to be idempotent if $\phi = \phi \circ \phi$, where $\circ$ stands for the composition. For instance, tanh and a symmetric saturation operator have the same slope property, but in addition the latter also has the idempotence property. By actively using the idempotence property within the framework of OZF multipliers, it is expected that we can enlarge (relax) the set of the multipliers and hence distinguish idempotent and non-idempotent nonlinearities. This is the main issue of this paper, and the novel contributions on OZF multipliers for slope-restricted and idempotent nonlinearities are summarized as follows: (i) for slope-restricted and idempotent nonlinearities, we derive a sequence of novel sets of multiplies indexed by N, each of which comprises the set of standard OZF multipliers for slope-restricted nonlinearities as a special case indexed by $N = 1$ (Theorem 1); (ii) in the sequence of the novel sets of multiplies, we prove that it suffices to take $N = 2$ since we cannot enlarge the set even if we take $N \geq 3$ (Theorem 2); (iii) the novel set indexed by $N = 2$ for slope-restricted with slope $[0, 1]$, idempotent, and non-odd (resp. odd) nonlinearities can be interpreted as a relaxation of the set of the standard OZF multipliers, in the sense that the restriction of the multiplier variables to doubly hyper dominant (resp. doubly dominant) matrices has been relaxed to *right* hyper dominant (resp. *right* dominant) matrices (Theorem 3). These are the main technical results of this paper. We also illustrate the effectiveness of the newly proposed multipliers by numerical examples on stability analysis of nonlinear feedback systems.

We note that it is effective to employ *dynamic* OZF multipliers for nonlinear dynamical system analysis, and hence studies along this direction have been active [8], [1]. However, static multipliers are also important when dealing with nonlinear static systems. In particular, in semidefinite-programming-based reliability verification of feedforward neural networks [7], [13], [6], the crucial step is capturing the behavior of nonlinear activation functions by static multipliers. Therefore, it is preferable to employ enlarged set of static multipliers to obtain less conservative results.

Notation: The set of natural numbers is denoted by $\mathbb{N}$. The set of $n \times m$ real matrices is denoted by $\mathbb{R}^{n\times m}$. We denote by I_n, 0_n, $0_{n,m}$ the $n \times n$ identity matrix, the $n \times n$ zero matrix, and the $n \times m$ zero matrix, respectively. For a matrix A, we write $A \geq 0$ to denote that A is entrywise nonnegative. We denote the set of $n \times n$ real symmetric matrices by $\mathbb{S}^n$. For $A \in \mathbb{S}^n$, we write $A \succ 0$ ($A \prec 0$) to denote that A is positive (negative) definite. For $A \in \mathbb{R}^{n\times n}$, we define $|A|_{\mathrm{d}} \in \mathbb{R}^{n\times n}$ by $|A|_{\mathrm{d},i,i} = A_{i,i}$ and $|A|_{\mathrm{d},i,j} = -|A_{i,j}|$ $(i \neq j)$. For $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, $(*)^T AB$ is a shorthand notation of $B^T AB$. For $v \in \mathbb{R}^n$, we denote by $|v|$ its standard Euclidean norm. The induced norm of a (possibly nonlinear) operator $\Psi : \mathbb{R}^m \to \mathbb{R}^n$ is defined by $\|\Psi\| := \sup_{v\in\mathbb{R}^m\setminus\{0\}} \dfrac{|\Psi(v)|}{|v|}$. For a matrix $M \in \mathbb{R}^{n\times m}$, the norm $\|M\|$ means the induced norm of the linear operator $\mathbb{R}^m \ni v \mapsto Mv \in \mathbb{R}^n$.

Some specific definitions are necessary to describe static OZF multipliers. A matrix $M \in \mathbb{R}^{m\times m}$ is said to be Z-matrix if $M_{ij} \le 0$ for all $i \neq j$. Moreover, M is said to be doubly hyperdominant if it is a Z-matrix and $M\mathbf{1}_m \ge 0$, $\mathbf{1}_m^T M \ge 0$, where $\mathbf{1}_m \in \mathbb{R}^m$ stands for the all-ones-vector. In addition, M is said to be doubly dominant if $|M|_{\mathrm{d}}\mathbf{1}_m \ge 0$, $\mathbf{1}_m^T|M|_{\mathrm{d}} \ge 0$. In this paper we denote by $\mathbb{Z}^m, \mathbb{DHD}^m, \mathbb{DD}^m \subset \mathbb{R}^{m\times m}$ the sets of Z-matrices, doubly hyperdominant, and doubly dominant matrices, respectively. Namely, we define

$$\begin{aligned}\mathbb{DHD}^m &:= \left\{M \in \mathbb{Z}^m :\ M\mathbf{1}_m \ge 0,\ \mathbf{1}_m^T M \ge 0\right\},\\ \mathbb{DD}^m &:= \left\{M \in \mathbb{R}^{m\times m} :\ |M|_{\mathrm{d}}\mathbf{1}_m \ge 0,\ \mathbf{1}_m^T|M|_d \ge 0\right\}.\end{aligned} \tag{1}$$

It is obvious that $\mathbb{DHD}^m \subsetneq \mathbb{DD}^m$.

II. Analysis of Nonlinear Feedback Systems Using IQC

Let us consider the feedback system Σ shown in Fig. 1. Here, G is a linear system described by

$$G : \begin{cases} \dot{x}(t) = Ax(t) + Bw(t), \\ z(t) = Cx(t) + Dw(t) \end{cases} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, and $D \in \mathbb{R}^{m\times m}$. We assume that A is Hurwitz. On the other hand, we have

$$w(t) = \Phi(z(t)) \tag{3}$$

where $\Phi :\ \mathbb{R}^m \to \mathbb{R}^m$ stands for a static nonlinearity. We assume that the feedback system Σ is well-posed. The well-posedness can be assessed according to the property of Φ. If $\|\Phi\| \le 1$, then we can readily conclude the well-posedness if $\|D\| < 1$, even though this is fairly conservative in general.

In this paper, we are interested in the analysis of the feedback system Σ in the framework of IQC with multipliers. Let us focus on the global asymptotic stability (GAS) [9] analysis of the system. In the following, we say that the feedback system Σ is stable if its origin is GAS. The next proposition forms the basis for the stability analysis in the framework of IQC [11], [15].

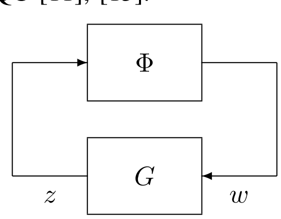

Fig. 1. Nonlinear feedback system Σ.

Proposition 1 ([11]): Let us define $\mathbf{\Pi}^\star \subset \mathbb{S}^{2m}$ by

$$\mathbf{\Pi}^\star := \left\{\Pi \in \mathbb{S}^{2m} :\ \begin{bmatrix}\zeta\\ \Phi(\zeta)\end{bmatrix}^T \Pi \begin{bmatrix}\zeta\\ \Phi(\zeta)\end{bmatrix} \ge 0\ \forall\zeta \in \mathbb{R}^m\right\}. \tag{4}$$

Then, the system Σ is stable if there exist $P \succ 0$ and $\Pi \in \mathbf{\Pi}^\star$ such that

$$\begin{bmatrix}PA + A^T P & PB\\ B^T P & 0\end{bmatrix} + \begin{bmatrix}C & D\\ 0 & I_m\end{bmatrix}^T \Pi \begin{bmatrix}C & D\\ 0 & I_m\end{bmatrix} \prec 0. \tag{5}$$

In addition to the stability, IQC framework allows us to analyze various performance specifications of feedback systems with nonlinearities. In IQC-based analysis conditions such as Proposition 1, it is of prime importance to employ a set of multipliers $\mathbf{\Pi} \subset \mathbf{\Pi}^\star$ that is numerically tractable in solving (5) and captures the input-output properties of the underlying nonlinearities as accurately as possible. This paper addresses this issue and focuses on static OZF multipliers [12], [18], [16] for slope-restricted and idempotent nonlinearities.

III. Static O'Shea-Zames-Falb Multipliers for Idempotent Nonlinearities

A. Basic Results

In this section, we follow the arguments of [8] on OZF multipliers for slope-restricted nonlinearities. For a nonlinearity $\Phi :\ \mathbb{R}^p \to \mathbb{R}^p$, we define $\mathrm{diag}_q(\Phi) :\ \mathbb{R}^{qp} \to \mathbb{R}^{qp}$ by $\mathrm{diag}_q(\Phi) := \mathrm{diag}(\underbrace{\Phi, \ \cdots\ , \ \Phi}_{q})$.

Definition 1: Let $\mu \le 0 \le \nu$. Then a nonlinearity $\phi :\ \mathbb{R} \to \mathbb{R}$ is said to be slope-restricted, in short $\phi \in \mathrm{slope}[\mu, \nu]$, if $\phi(0) = 0$ and $\mu \le \dfrac{\phi(p) - \phi(q)}{p - q} \le \nu$ for all $p, q \in \mathbb{R}$, $p \neq q$.

Definition 2: For $\mu \le 0 \le \nu$, we define

$$\mathbf{\Phi}^m_{\mu,\nu} := \{\Phi :\ \Phi = \mathrm{diag}_m(\phi),\ \phi \in \mathrm{slope}[\mu, \nu]\}. \tag{6}$$

In addition, we define

$$\mathbf{\Phi}^m_{\mathrm{odd}} := \{\Phi :\ \Phi = \mathrm{diag}_m(\phi),\ \phi : \mathbb{R} \to \mathbb{R} \text{ is odd}\}. \tag{7}$$

Under these definitions, the main result of [8] on static OZF multipliers for slope-restricted nonlinearities can be summarized by the next lemma.

Lemma 1: [8] For a given nonlinearity $\Phi \in \mathbf{\Phi}^m_{\mu,\nu}$, we have

$$(*)^T \begin{bmatrix}0_m & M\\ M^T & 0_m\end{bmatrix} \left(\begin{bmatrix}\nu I_m & -I_m\\ -\mu I_m & I_m\end{bmatrix} \begin{bmatrix}\zeta\\ \Phi(\zeta)\end{bmatrix}\right) \ge 0\ \forall\zeta \in \mathbb{R}^m$$

for any $M \in \mathbb{DHD}^m$. In addition, if $\Phi \in \mathbf{\Phi}^m_{\mu,\nu} \cap \mathbf{\Phi}^m_{\mathrm{odd}}$, then this holds for any $M \in \mathbb{DD}^m$.

Remark 1: For systems that include repeated scalar nonlinearities $\Phi \in \mathbf{\Phi}^m_{0,1} \cap \mathbf{\Phi}^m_{\mathrm{odd}}$ in a special fashion, a stability condition using diagonally dominant positive definite matrices is known [3]. We can prove that this condition is a special (restricted) one of the IQC-based condition (5) with standard OZF multipliers in Lemma 1. Details are omitted due to limited space.

B. New Results for Idempotent Nonlinearities

In this paper, we are interested in nonlinearities with idempotence. Note that a nonlinearity $\phi:\ \mathbb{R}\to\mathbb{R}$ is said to be idempotent if $\phi=\phi\circ\phi$ where $\circ$ stands for the composition. The class of idempotent nonlinearities includes asymmetric saturation nonlinearities with any upper and lower limits (see Fig. 2), and the ReLU nonlinearity as a special case of them. These appear very often in control systems and NNs. In fact, the class of idempotent nonlinearities seems to be broader than our intuition, and we can confirm that the nonlinear operator $\phi:\ \mathbb{R}\to\mathbb{R}$ represented by

$$\phi(\zeta)=\begin{cases}\zeta & -b\le\zeta\le a\\ \phi_{a,b}(\zeta) & \zeta<-b,\ a<\zeta\end{cases}$$

is idempotent where $a,b\ge 0$ and $\phi_{a,b}:\ \mathbb{R}\to\mathbb{R}$ is any nonlinear operator satisfying $\phi_{a,b}(\zeta)\in[-b,a]$ $(\forall\zeta\in\mathbb{R})$.

For the ease of description, let us define

$$\mathbf{\Phi}_{\rm id}^{m}:=\{\Phi:\ \Phi=\mathrm{diag}_m(\phi),\ \phi\text{ is idempotent}\}. \tag{8}$$

For a given $N\in\mathbb{N}$, we further define

$$\begin{aligned}
&J_m^{[N]}=\begin{bmatrix}J_{m,11}^{[N]} & J_{m,12}^{[N]}\\ 0_{Nm,m} & J_{m,22}^{[N]}\end{bmatrix}\in\mathbb{R}^{2Nm\times 2m},\\
&J_{m,11}^{[N]}=\begin{bmatrix}0_{(N-1)m,m}\\ I_m\end{bmatrix}\in\mathbb{R}^{Nm\times m},\\
&J_{m,12}^{[N]}=\begin{bmatrix}\mathbf{1}_{N-1}\otimes I_m\\ 0_m\end{bmatrix}\in\mathbb{R}^{Nm\times m},\ J_{m,22}^{[N]}=\mathbf{1}_N\otimes I_m\in\mathbb{R}^{Nm\times m}
\end{aligned}\tag{9}$$

where $\otimes$ stands for the Kronecker product. Note that for $N=1$ we have $J_m^{[1]}=I_{2m}$. We are now ready to state the first main result of this paper.

Theorem 1: For given nonlinearity $\Phi\in\mathbf{\Phi}_{\mu,\nu}^m\cap\mathbf{\Phi}_{\rm id}^m$ and $N\in\mathbb{N}$, we have

$$(*)^T\begin{bmatrix}0_{Nm} & M\\ M^T & 0_{Nm}\end{bmatrix}\left(\begin{bmatrix}\nu I_{Nm} & -I_{Nm}\\ -\mu I_{Nm} & I_{Nm}\end{bmatrix}J_m^{[N]}\begin{bmatrix}\zeta\\ \Phi(\zeta)\end{bmatrix}\right)\ge 0\ \forall\zeta\in\mathbb{R}^m$$

for any $M\in\mathbb{DHD}^{Nm}$. In addition, if $\Phi\in\mathbf{\Phi}_{\mu,\nu}^m\cap\mathbf{\Phi}_{\rm id}^m\cap\mathbf{\Phi}_{\rm odd}^m$, then this holds for any $M\in\mathbb{DD}^{Nm}$.

Proof of Theorem 1: We note that

$$J_m^{[N]}\begin{bmatrix}\zeta\\ \Phi(\zeta)\end{bmatrix}=\begin{bmatrix}\zeta^{[N]}\\ \mathrm{diag}_N(\Phi)(\zeta^{[N]})\end{bmatrix},\ \zeta^{[N]}:=\left.\begin{bmatrix}\Phi(\zeta)\\ \vdots\\ \Phi(\zeta)\\ \zeta\end{bmatrix}\right\}N-1\in\mathbb{R}^{Nm}$$

where we used the fact that $\Phi\in\mathbf{\Phi}_{\rm id}^m$ and hence $\Phi\circ\Phi(\zeta)=\Phi(\zeta)$ holds. Since $\mathrm{diag}_N(\Phi)\in\mathbf{\Phi}_{\mu,\nu}^{Nm}$, the results in Theorem 1 readily follow from Lemma 1. ■

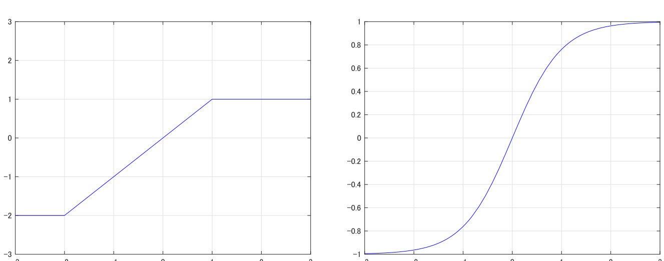

Fig. 2. Input-output plots of asymmetric saturation $\phi_{\rm sat}$ (left) and hyperbolic tangent $\phi_{\tanh}$ (right). We see $\phi_{\rm sat}\in\mathbf{\Phi}_{0,1}$ and $\phi_{\tanh}\in\mathbf{\Phi}_{0,1}$ but $\phi_{\rm sat}$ also satisfies idempotence.

Again, if we let $N=1$, then we have $J_m^{[1]}=I_{2m}$, and hence in this case Theorem 1 reduces to Lemma 1. Therefore, our main interest is whether we can enlarge the set of multipliers by letting $N\ge 2$ in Theorem 1. Here, the fact that Theorem 1 holds for any $N\in\mathbb{N}$ is related to the fact that $\Phi=\underbrace{\Phi\circ\cdots\circ\Phi}_{N}$ holds for any $N\in\mathbb{N}$ (details on this interpretation is omitted due to limited space). However, since the essential point of the idempotence is $\Phi=\Phi\circ\Phi$, it is deduced that we cannot enlarge the set of multipliers by Theorem 1 even if we take $N\ge 3$. To clarify these points, let us define the following sets of multipliers that are readily obtained from Theorem 1:

$$\begin{aligned}\mathbf{\Pi}_{m,\mathbb{DHD}}^{[N]}:=&\Bigg\{\Pi\in\mathbb{S}^{2m}:\ \Pi=(*)^T\begin{bmatrix}0_{Nm} & M\\ M^T & 0_{Nm}\end{bmatrix}\\ &\left(\begin{bmatrix}\nu I_{Nm} & -I_{Nm}\\ -\mu I_{Nm} & I_{Nm}\end{bmatrix}J_m^{[N]}\right),\ M\in\mathbb{DHD}^{Nm}\Bigg\},\end{aligned}\tag{10}$$

$$\begin{aligned}\mathbf{\Pi}_{m,\mathbb{DD}}^{[N]}:=&\Bigg\{\Pi\in\mathbb{S}^{2m}:\ \Pi=(*)^T\begin{bmatrix}0_{Nm} & M\\ M^T & 0_{Nm}\end{bmatrix}\\ &\left(\begin{bmatrix}\nu I_{Nm} & -I_{Nm}\\ -\mu I_{Nm} & I_{Nm}\end{bmatrix}J_m^{[N]}\right),\ M\in\mathbb{DD}^{Nm}\Bigg\}.\end{aligned}\tag{11}$$

Regarding these sets of multipliers, we can obtain the next theorem that validates the intuition stated above.

Theorem 2: For $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[N]}\subset\mathbb{S}^{2m}$ and $\mathbf{\Pi}_{m,\mathbb{DD}}^{[N]}\subset\mathbb{S}^{2m}$ given (10) and (11), respectively, we have

$$\mathbf{\Pi}_{m,\mathbb{DHD}}^{[1]}\subsetneq\mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]},\ \mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]}=\mathbf{\Pi}_{m,\mathbb{DHD}}^{[N]}\ (N\ge 3),\tag{12}$$

$$\mathbf{\Pi}_{m,\mathbb{DD}}^{[1]}\subsetneq\mathbf{\Pi}_{m,\mathbb{DD}}^{[2]},\ \mathbf{\Pi}_{m,\mathbb{DD}}^{[2]}=\mathbf{\Pi}_{m,\mathbb{DD}}^{[N]}\ (N\ge 3).\tag{13}$$

For the proof of this theorem, we need the next two lemmas. The proofs of these lemmas are omitted due to limited space.

Lemma 2: For given $m,N\in\mathbb{N}$ with $N\ge 2$, let us define

$$E_m^{[N]}:=\begin{bmatrix}e_1\otimes I_m\\ I_{Nm}\end{bmatrix}\in\mathbb{R}^{(N+1)m\times Nm}\tag{14}$$

where $e_1\in\mathbb{R}^{1\times N}$ stands for the first row of I_N. Then, for $J_m^{[N]}\in\mathbb{R}^{2Nm\times 2m}$ defined by (9), we have

$$(I_2\otimes E_m^{[N]})J_m^{[N]}=J_m^{[N+1]}.\tag{15}$$

Lemma 3: For given $m,N\in\mathbb{N}$ with $N\ge 2$, let us define

$$\begin{aligned}\overline{\mathbb{DHD}}^{Nm}:=&\big\{M\in\mathbb{R}^{Nm\times Nm}:\\ &M=E_m^{[N]T}\overline{M}E_m^{[N]},\ \overline{M}\in\mathbb{DHD}^{(N+1)m}\big\},\end{aligned}\tag{16}$$

$$\begin{aligned}\overline{\mathbb{DD}}^{Nm}:=&\big\{M\in\mathbb{R}^{Nm\times Nm}:\\ &M=E_m^{[N]T}\overline{M}E_m^{[N]},\ \overline{M}\in\mathbb{DD}^{(N+1)m}\big\}.\end{aligned}\tag{17}$$

Then, we have $\overline{\mathbb{DHD}}^{Nm}=\mathbb{DHD}^{Nm}$ and $\overline{\mathbb{DD}}^{Nm}=\mathbb{DD}^{Nm}$.

Proof of Theorem 2: We first prove $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[1]}\subsetneq\mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]}$ in (12). The proof for $\mathbf{\Pi}_{m,\mathbb{DD}}^{[1]}\subsetneq\mathbf{\Pi}_{m,\mathbb{DD}}^{[2]}$ in (13) follows in the same way. To this end, by focusing on (10), let us define $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD},\mathrm{sub}} \subsetneq \mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}$ by

$$\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD},\mathrm{sub}} := \left\{ \Pi \in \mathbb{S}^{2m} :\ \Pi = (*)^T \begin{bmatrix} 0_{2m} & M \\ M^T & 0_{2m} \end{bmatrix} \left(\begin{bmatrix} \nu I_{2m} & -I_{2m} \\ -\mu I_{2m} & I_{2m} \end{bmatrix} J_m^{[2]} \right),\ M = \begin{bmatrix} 0_m & 0_m \\ 0_m & M_0 \end{bmatrix},\ M_0 \in \mathbb{DHD}^m \right\}.$$

Then, for the proof of $\mathbf{\Pi}^{[1]}_{m,\mathbb{DHD}} \subsetneq \mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}$, it suffices to show that $\mathbf{\Pi}^{[1]}_{m,\mathbb{DHD}} = \mathbf{\Pi}^{[2]}_{m,\mathbb{DHD},\mathrm{sub}}$. Since

$$\begin{bmatrix} 0_{2m} & M \\ M^T & 0_{2m} \end{bmatrix} = \begin{bmatrix} 0_m & 0_m \\ I_m & 0_m \\ 0_m & 0_m \\ 0_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & M_0 \\ M_0^T & 0_m \end{bmatrix} \begin{bmatrix} 0_m & 0_m \\ I_m & 0_m \\ 0_m & 0_m \\ 0_m & I_m \end{bmatrix}^T$$

holds for $M = \begin{bmatrix} 0_m & 0_m \\ 0_m & M_0 \end{bmatrix}$, and since

$$\begin{bmatrix} 0_m & 0_m \\ I_m & 0_m \\ 0_m & 0_m \\ 0_m & I_m \end{bmatrix}^T \begin{bmatrix} \nu I_{2m} & -I_{2m} \\ -\mu I_{2m} & I_{2m} \end{bmatrix} J_m^{[2]} = \begin{bmatrix} 0_m & \nu I_m & 0_m & -I_m \\ 0_m & -\mu I_m & 0_m & I_m \end{bmatrix} \begin{bmatrix} 0_m & I_m \\ I_m & 0_m \\ 0_m & I_m \\ 0_m & I_m \end{bmatrix} = \begin{bmatrix} \nu I_m & -I_m \\ -\mu I_m & I_m \end{bmatrix}$$

holds, we see that $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD},\mathrm{sub}}$ can be characterized equivalently as

$$\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD},\mathrm{sub}} := \left\{ \Pi \in \mathbb{S}^{2m} :\ \Pi = (*)^T \begin{bmatrix} 0_m & M_0 \\ M_0^T & 0_m \end{bmatrix} \begin{bmatrix} \nu I_m & -I_m \\ -\mu I_m & I_m \end{bmatrix},\ M_0 \in \mathbb{DHD}^m \right\}.$$

This clearly shows that $\mathbf{\Pi}^{[1]}_{m,\mathbb{DHD}} = \mathbf{\Pi}^{[2]}_{m,\mathbb{DHD},\mathrm{sub}}$ holds.

We next prove $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}} = \mathbf{\Pi}^{[N]}_{m,\mathbb{DHD}}$ $(N \geq 3)$ in (12). The proof for $\mathbf{\Pi}^{[2]}_{m,\mathbb{DD}} = \mathbf{\Pi}^{[N]}_{m,\mathbb{DD}}$ $(N \geq 3)$ in (13) follows in the same way. For $N \geq 2$, recall from (10) that

$$\mathbf{\Pi}^{[N+1]}_{m,\mathbb{DHD}} = \left\{ \Pi \in \mathbb{S}^{2m} :\ \Pi = (*)^T \begin{bmatrix} 0_{(N+1)m} & M \\ M^T & 0_{(N+1)m} \end{bmatrix} \left(\begin{bmatrix} \nu I_{(N+1)m} & -I_{(N+1)m} \\ -\mu I_{(N+1)m} & I_{(N+1)m} \end{bmatrix} J_m^{[N+1]} \right),\ M \in \mathbb{DHD}^{(N+1)m} \right\}.$$

Then, since $J_m^{[N+1]} = (I_2 \otimes E_m^{[N]}) J_m^{[N]}$ holds from Lemma 2, and since

$$\begin{bmatrix} \nu I_{(N+1)m} & -I_{(N+1)m} \\ -\mu I_{(N+1)m} & I_{(N+1)m} \end{bmatrix} (I_2 \otimes E_m^{[N]}) = (I_2 \otimes E_m^{[N]}) \begin{bmatrix} \nu I_{Nm} & -I_{Nm} \\ -\mu I_{Nm} & I_{Nm} \end{bmatrix}$$

holds, we see

$$\mathbf{\Pi}^{[N+1]}_{m,\mathbb{DHD}} = \left\{ \Pi \in \mathbb{S}^{2m} :\ \Pi = (*)^T \begin{bmatrix} 0_{Nm} & E_m^{[N]T} M E_m^{[N]} \\ * & 0_{Nm} \end{bmatrix} \left(\begin{bmatrix} \nu I_{Nm} & -I_{Nm} \\ -\mu I_{Nm} & I_{Nm} \end{bmatrix} J_m^{[N]} \right),\ M \in \mathbb{DHD}^{(N+1)m} \right\}.$$

Then, by using (16), this can be rewritten equivalently as

$$\mathbf{\Pi}^{[N+1]}_{m,\mathbb{DHD}} = \left\{ \Pi \in \mathbb{S}^{2m} :\ \Pi = (*)^T \begin{bmatrix} 0_{Nm} & M \\ M^T & 0_{Nm} \end{bmatrix} \left(\begin{bmatrix} \nu I_{Nm} & -I_{Nm} \\ -\mu I_{Nm} & I_{Nm} \end{bmatrix} J_m^{[N]} \right),\ M \in \overline{\mathbb{DHD}}^{Nm} \right\}.$$

From Lemma 3, this clearly shows that $\mathbf{\Pi}^{[N]}_{m,\mathbb{DHD}} = \mathbf{\Pi}^{[N+1]}_{m,\mathbb{DHD}}$ $(N \geq 2)$. This completes the proof. ■

IV. Interpretation on the Enlargement of the Set of Multipliers

In the preceding section, we have clarified that for $\Phi \in \mathbf{\Phi}^m_{\mu,\nu} \cap \mathbf{\Phi}^m_{\mathrm{id}}$ we can employ the novel set of multiplier $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}$ defined by (10), and for $\Phi \in \mathbf{\Phi}^m_{\mu,\nu} \cap \mathbf{\Phi}^m_{\mathrm{id}} \cap \mathbf{\Phi}^m_{\mathrm{odd}}$ we can employ $\mathbf{\Pi}^{[2]}_{m,\mathbb{DD}}$ defined by (11). As proved, the novel set of multiplier $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}$ comprises the known set of multipliers $\mathbf{\Pi}^{[1]}_{m,\mathbb{DHD}}$ in the way that

$$\mathbf{\Pi}^{[1]}_{m,\mathbb{DHD}} = \mathbf{\Pi}^{[2]}_{m,\mathbb{DHD},\mathrm{sub}} \subsetneq \mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}. \tag{18}$$

Similarly, we can show that

$$\mathbf{\Pi}^{[1]}_{m,\mathbb{DD}} = \mathbf{\Pi}^{[2]}_{m,\mathbb{DD},\mathrm{sub}} \subsetneq \mathbf{\Pi}^{[2]}_{m,\mathbb{DD}} \tag{19}$$

holds where

$$\mathbf{\Pi}^{[2]}_{m,\mathbb{DD},\mathrm{sub}} := \left\{ \Pi \in \mathbb{S}^{2m} :\ \Pi = (*)^T \begin{bmatrix} 0_{2m} & M \\ M^T & 0_{2m} \end{bmatrix} \left(\begin{bmatrix} \nu I_{2m} & -I_{2m} \\ -\mu I_{2m} & I_{2m} \end{bmatrix} J_m^{[2]} \right),\ M = \begin{bmatrix} 0_m & 0_m \\ 0_m & M_0 \end{bmatrix},\ M_0 \in \mathbb{DD}^m \right\}.$$

The inclusion relationship (18) (resp. (19)) shows how the set of the multipliers has been enlarged by $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}$ over $\mathbf{\Pi}^{[1]}_{m,\mathbb{DHD}}$ (resp. $\mathbf{\Pi}^{[2]}_{m,\mathbb{DD}}$ over $\mathbf{\Pi}^{[1]}_{m,\mathbb{DD}}$). For general μ, ν such that $\mu \leq 0 \leq \nu$, further interpretation of this enlargement seems hardly available. However, for $\mu = 0$ and $\nu = 1$, we can obtain much clearer interpretation as shown below. Even under the restriction to $\mu = 0$ and $\nu = 1$, we note that $\mathbf{\Phi}^m_{0,1} \cap \mathbf{\Phi}^m_{\mathrm{id}}$ and $\mathbf{\Phi}^m_{0,1} \cap \mathbf{\Phi}^m_{\mathrm{id}} \cap \mathbf{\Phi}^m_{\mathrm{odd}}$ still include important nonlinearities such as asymmetric saturation nonlinearities with any upper and lower limits and symmetric saturation nonlinearities, respectively.

To clarify this interpretation, let us define the sets of *right* hyperdominant matrices and *right* dominant matrices by

$$\begin{aligned} \mathbb{RHD}^m &:= \{ M \in \mathbb{Z}^m :\ M \mathbf{1}_m \geq 0 \}, \\ \mathbb{RD}^m &:= \{ M \in \mathbb{R}^{m \times m} :\ |M|_{\mathrm{d}} \mathbf{1}_m \geq 0 \}. \end{aligned} \tag{20}$$

It is clear that $\mathbb{RHD}^m \subsetneq \mathbb{RD}^m$. In addition, in comparison with $\mathbb{DHD}^m$ and $\mathbb{DD}^m$ given by (1), we emphasize that $\mathbb{DHD}^m \subsetneq \mathbb{RHD}^m$ and $\mathbb{DD}^m \subsetneq \mathbb{RD}^m$ hold. By using these sets of matrices, we can characterize the sets of multipliers $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}$ and $\mathbf{\Pi}^{[2]}_{m,\mathbb{DD}}$ for $\mu = 0$ and $\nu = 1$ concisely as summarized in the next theorem.

Theorem 3: Let us consider the sets of multipliers $\mathbf{\Pi}^{[2]}_{m,\mathbb{DHD}}$ and $\mathbf{\Pi}^{[2]}_{m,\mathbb{DD}}$ given by (10) and (11), respectively. Suppose

$\mu = 0$ and $\nu = 1$. Then, these sets of multipliers are equivalently characterized as

$$\mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]} = \mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]},$$
$$\mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]} := \left\{\Pi \in \mathbb{S}^{2m} : \ \Pi = (*)^T \begin{bmatrix} 0_m & M \\ M^T & 0_m \end{bmatrix} \begin{bmatrix} I_m & -I_m \\ 0_m & I_m \end{bmatrix},\ M \in \mathbb{RHD}^m \right\}, \tag{21}$$

$$\mathbf{\Pi}_{m,\mathbb{DD}}^{[2]} = \mathbf{\Pi}_{m,\mathbb{RD}}^{[1]},$$
$$\mathbf{\Pi}_{m,\mathbb{RD}}^{[1]} := \left\{\Pi \in \mathbb{S}^{2m} : \ \Pi = (*)^T \begin{bmatrix} 0_m & M \\ M^T & 0_m \end{bmatrix} \begin{bmatrix} I_m & -I_m \\ 0_m & I_m \end{bmatrix},\ M \in \mathbb{RD}^m \right\}. \tag{22}$$

Remark 2: This theorem shows that, when dealing with a nonlinearity $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id}^m$ (resp. $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id}^m \cap \mathbf{\Phi}_{\rm odd}^m$), the restriction $M \in \mathbb{DHD}^m$ (resp. $M \in \mathbb{DD}^m$) in the known set of OZF multiplies $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[1]}$ (resp. $\mathbf{\Pi}_{m,\mathbb{DD}}^{[1]}$) is relaxed to $M \in \mathbb{RHD}^m$ (resp. $M \in \mathbb{RD}^m$).

Remark 3: When dealing with a nonlinearity $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id}^m$ (resp. $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id}^m \cap \mathbf{\Phi}_{\rm odd}^m$) with the novel set of OZF multipliers $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]}$ (resp. $\mathbf{\Pi}_{m,\mathbb{DD}}^{[2]}$) directly in Theorem 2, we have to employ the matrix variable $M \in \mathbb{DHD}^{2m}$ (resp. $M \in \mathbb{DD}^{2m}$) whose size is $2m$. However, Theorem 3 shows that it suffices in fact to employ $M \in \mathbb{RHD}^m$ (resp. $M \in \mathbb{RD}^m$) whose size is m. Namely, by Theorem 3, we can achieve considerable reduction of the computational complexity.

Remark 4: We note that the proof of Theorem 3 given below strongly relies on the matrix structure

$$\begin{bmatrix} \nu I_m & -I_m \\ -\mu I_m & I_m \end{bmatrix} = \begin{bmatrix} I_m & -I_m \\ 0_m & I_m \end{bmatrix} \tag{23}$$

that holds for $\mu = 0$ and $\nu = 1$.

For the proof of Theorem 3, the next lemma plays an important role. The proof of this lemma is omitted due to limited space.

Lemma 4: Let us define

$$\overline{\mathbb{RHD}}^m := \left\{ M = \begin{bmatrix} 0_m \\ I_m \end{bmatrix}^T \overline{M} \begin{bmatrix} I_m \\ I_m \end{bmatrix} : \ \overline{M} \in \mathbb{DHD}^{2m} \right\}, \tag{24}$$

$$\overline{\mathbb{RD}}^m := \left\{ M = \begin{bmatrix} 0_m \\ I_m \end{bmatrix}^T \overline{M} \begin{bmatrix} I_m \\ I_m \end{bmatrix} : \ \overline{M} \in \mathbb{DD}^{2m} \right\}. \tag{25}$$

Then, we have $\overline{\mathbb{RHD}}^m = \mathbb{RHD}^m$ and $\overline{\mathbb{RD}}^m = \mathbb{RD}^m$.

Proof of Theorem 3: To prove (21), we write down explicitly $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]}$ by (10) as

$$\begin{aligned} &\mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]} \\ &= \left\{ \Pi \in \mathbb{S}^{2m} : \ \Pi = (*)^T \begin{bmatrix} 0_{2m} & \overline{M} \\ \overline{M}^T & -\overline{M} - \overline{M}^T \end{bmatrix} \begin{bmatrix} 0_m & I_m \\ I_m & 0_m \\ 0_m & I_m \\ 0_m & I_m \end{bmatrix}, \right. \\ &\qquad \left. \overline{M} \in \mathbb{DHD}^{2m} \right\} \\ &= \left\{ \Pi \in \mathbb{S}^{2m} : \ \Pi = (*)^T \begin{bmatrix} 0_m & M \\ M^T & 0_m \end{bmatrix} \begin{bmatrix} I_m & -I_m \\ 0_m & I_m \end{bmatrix}, \right. \\ &\qquad \left. M = \begin{bmatrix} 0_m \\ I_m \end{bmatrix}^T \overline{M} \begin{bmatrix} I_m \\ I_m \end{bmatrix},\ \overline{M} \in \mathbb{DHD}^{2m} \right\} \end{aligned} \tag{26}$$

where (23) enables us to arrive at this concise expression. Then, from Lemma 4, we see that $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[2]} = \mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]}$ holds. The proof for (22) follows in the same way. Indeed, from (11), we see that $\mathbf{\Pi}_{m,\mathbb{DD}}^{[2]}$ can be rewritten as (26) with $\mathbb{DHD}^{2m}$ being replaced by $\mathbb{DD}^{2m}$. Therefore, from Lemma 4, we see that $\mathbf{\Pi}_{m,\mathbb{DD}}^{[2]} = \mathbf{\Pi}_{m,\mathbb{RD}}^{[1]}$ holds. ■

V. Numerical Examples

Let us consider the case where the coefficient matrices of the system G given by (2) are

$$A = \begin{bmatrix} 0.16 & 0.49 \\ -0.41 & -0.56 \end{bmatrix},\ B = \begin{bmatrix} a & -0.13 & 0.28 & 0.43 \\ -0.32 & b & -0.42 & 0.28 \end{bmatrix}, \\ C = \begin{bmatrix} -0.09 & 0.06 \\ -0.39 & 0.06 \\ -0.33 & 1.92 \\ -1.17 & 1.77 \end{bmatrix}, \quad D = 0_{4,4}. \tag{27}$$

For the cases $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id}^m$ and $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id}^m \cap \mathbf{\Phi}_{\rm odd}^m$ where $m = 4$, we examined the stability of the feedback system Σ with respect to the variation of the parameters a and b over $a \in [-1.5, 4.5]$ and $b \in [-2.5, 2.5]$.

Even though this example is an academic and small-size one, the problem setup is chosen to simulate the typical situation of the stability analysis of RNNs and NN-driven control systems. When we employ an RNN as a model of a complex dynamical system, we typically incorporate a large number of activation functions to imitate the complex behavior. In addition, for a plant subject to unknown nonlinearities, we often employ a dynamical NN as a controller again with a large number of activation functions to approximate the unknown nonlinearities. In such cases, the system of interest will be described by (2) and (3) where $m > n$ as in (27) (in this case, $n = 2$ and $m = 4$). In addition, in the learning phase of an RNN as a model of a complex dynamical system or a dynamical NN controller, the weights of NNs are updated along decent directions with respect to predefined objective functions. In such cases, it is preferable to see beforehand how far we can shift the current weights along the decent direction while maintaining the stability. Such weights appear as the elements of the coefficient matrices of the system G in (2). With these in mind, the problem setup here has been chosen.

Assuming $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id}^m$ that includes asymmetric saturation nonlinearities as special cases, we tested the LMI (5) with the set of known OZF multipliers $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[1]} \subset \mathbf{\Pi}^\star$ given by (10) and newly proposed $\mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]} \subset \mathbf{\Pi}^\star$ given by (21), where $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[1]} \subsetneq \mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]}$ holds. The results are shown in Fig. 3. Both LMIs with $\mathbf{\Pi}_{m,\mathbb{DHD}}^{[1]}$ and $\mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]}$ turned out to be feasible for (a, b) in the green region, whereas only the LMI with $\mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]}$ turned out to be feasible for (a, b) in the magenta region. By employing the novel set of multipliers $\mathbf{\Pi}_{m,\mathbb{RHD}}^{[1]}$, we can confirm that the region with stability certificate has been enlarged.

Next, assuming $\Phi \in \mathbf{\Phi}_{0,1}^m \cap \mathbf{\Phi}_{\rm id} \cap \mathbf{\Phi}_{\rm odd}$ that includes symmetric saturation nonlinearities as typical cases, we tested the LMI (5) with the set of known OZF multipliers $\mathbf{\Pi}_{m,\mathbb{DD}}^{[1]} \subset \mathbf{\Pi}^\star$

given by (11) and newly proposed $\mathbf{\Pi}^{[1]}_{m,\mathbb{RD}} \subset \mathbf{\Pi}^\star$ given by (22), where $\mathbf{\Pi}^{[1]}_{m,\mathbb{DD}} \subsetneq \mathbf{\Pi}^{[1]}_{m,\mathbb{RD}}$ holds. The results are shown in Fig. 4. Similarly to Fig. 3, both LMIs with $\mathbf{\Pi}^{[1]}_{m,\mathbb{DD}}$ and $\mathbf{\Pi}^{[1]}_{m,\mathbb{RD}}$ turned out to be feasible for (a, b) in the green region, whereas only the LMI with $\mathbf{\Pi}^{[1]}_{m,\mathbb{RD}}$ turned out to be feasible for (a, b) in the magenta region. By employing the novel set of multipliers $\mathbf{\Pi}^{[1]}_{m,\mathbb{RD}}$, we can confirm that the region with stability certificate has been drastically enlarged.

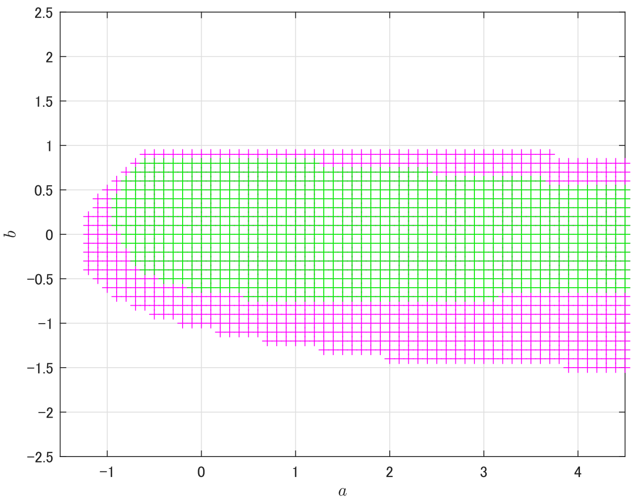

Fig. 3. Regions certified to be stable for $\Phi \in \mathbf{\Phi}_{0,1} \cap \mathbf{\Phi}_{\rm id}$.

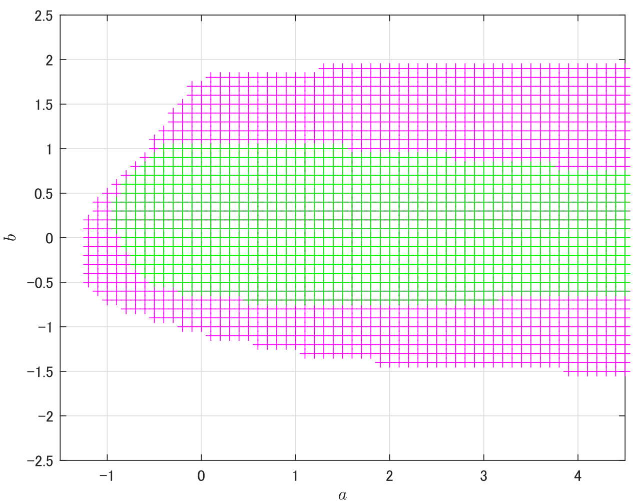

Fig. 4. Regions certified to be stable for $\Phi \in \mathbf{\Phi}_{0,1} \cap \mathbf{\Phi}_{\rm id} \cap \mathbf{\Phi}_{\rm odd}$.

VI. Conclusion

We investigated static OZF multipliers for slope-restricted and idempotent nonlinearities. By actively using the idempotence property, we first showed that we can enlarge the set of the standard OZF multipliers for slope-restricted nonlinearities. In particular, for slope-restricted and idempotent nonlinearities with slope $[0, 1]$, we provided a clear understanding on how the set of the standard OZF multipliers is enlarged. We finally illustrated the effectiveness of the newly proposed multipliers by numerical examples.

In this paper, we dealt with static OZF multipliers that are valid in dealing with both continuous- and discrete-time nonlinear feedback systems. Future topics include the extension of the current results to dynamic OZF multipliers. In such extension, existing studies suggest that it might be necessary to distinguish the treatments for continuous- and discrete-time nonlinear feedback systems. This topic is currently under investigation.

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