2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

Closing the Loop in Moment Matching

Alessio Moreschini[1] and Alessandro Astolfi[2]

***Abstract*— We present a novel closed-loop interpretation of the steady-state notion of moment. In particular, to close the loop in an interconnection-based moment-matching configuration we introduce a signal generator that ensures internal stability of the interconnection while interpolating the moments of the underlying system. Unlike the traditional open-loop configuration, the closed-loop scheme allows the relaxation of the internal stability property of the underlying system. Furthermore, for a fixed signal generator, we provide conditions for any linear models to achieve moment matching in closed loop. In particular, we define the family of all (stable and unstable) admissible plants achieving moment matching through the same signal generator. These results offer a new perspective on moment matching and yield a moment matching interpretation of the dual Youla-Kučera parameterization.**

***Index Terms*— Moment matching; Closed-loop interpolation; Model reduction; Steady-state behaviour.**

I. INTRODUCTION

Large-scale systems present a significant research challenge in systems and control, as they involve a large number of interconnected subsystems described by ordinary differential equations [1]–[3]. These systems also arise from the spatial discretization of partial differential equations, see [4]–[7]. To overcome the computational complexity associated with numerical simulations and to improve controller design for these systems, model order reduction techniques have gained considerable attention in recent years, *e.g.* [8]–[13].

Moment matching is a popular model reduction technique that combines rational interpolation theory and the concept of *moment* to approximate and interpolate high-order dynamical systems. Traditionally, moments of linear systems have predominantly been associated with the coefficients of the Laurent series expansion of the transfer function at some point. However, in [14], the notion of moment was related to the solution of a certain Sylvester equation. Building upon this insight, the concept of moment was reinterpreted in [15] in terms of the steady-state output response of the underlying system driven by a signal generator. This configuration, representative of an open-loop cascade interconnection, has gained considerable attention and has been instrumental in advancing model reduction techniques, *e.g.* [16]–[21].

Yet, the open-loop characterization of the notion of moment has a *caveat*: the underlying system to be interpolated must be internally stable. Indeed, the stability property is crucial for ensuring the existence of a steady-state output response, which imposes constraints on the applicability of the open-loop interpretation of moment matching. In experimental settings, relaxing stability conditions is only possible when the initial state of the system lies on the manifold characterizing the moment. However, if the system's state is not known or fully measured, this caveat persists.

In this article, we revisit the notion of interconnection-based moment matching by introducing the concept of closed-loop interpolation. This notion relies on constructing a signal generator that takes the output of the plant to be interpolated as its input. The signal generator is designed to ensure the internal stability of the interconnection scheme, while interpolating the underlying system at desired operating points. As a result, unlike the open-loop configuration, a closed-loop interpolation scheme allows for the relaxation of any stability conditions of the underlying system. We further demonstrate the existence of a family of linear models that parameterize all linear models achieving moment matching in a closed-loop fashion. Specifically, for a fixed generalized signal generator characterizing the moment of a certain system, we provide conditions for the family of parameterized models to achieve moment matching in a closed-loop configuration.

Organization: In Section II the interconnection-based notion of moment is recalled, and a motivating example is provided. Section III introduces the concept of closed-loop interpolation. In Section IV a family of parameterized models achieving moment matching in a closed-loop interpolation scheme is presented. Section V uses the benchmark example of a NASA HiMAT aircraft model [22] to demonstrate the applicability of the proposed closed-loop scheme. Finally, Section VI offers concluding remarks and discusses future perspectives.

Notation: The field of real (complex) numbers is denoted by $\mathbb{R}$ ($\mathbb{C}$). The set of vectors having n rows with real entries is denoted by $\mathbb{R}^n$, and the set of matrices having n rows and m columns with real-valued entries is denoted by $\mathbb{R}^{n\times m}$. The spectrum of a square matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\sigma(A)$. With some abuse of notation, the time variable is omitted whenever this does not cause confusion.

*This work has been partially supported by the European Union's Horizon 2020 Research and Innovation Programme under grant agreement No. 739551 (KIOS CoE); by the Italian Ministry for Research in the framework of the 2020 Program for Research Projects of National Interest (PRIN), Grant 2020RTWES4; and by the EPSRC grants EP/W005557 and EP/X033546.

[1]Alessio Moreschini is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K. a.moreschini@imperial.ac.uk

[2]Alessandro Astolfi is with the Department of Electrical and Electronic Engineering, Imperial College London, SW7 2AZ London, U.K. and with the Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico 1, 00133 Roma, Italy a.astolfi@imperial.ac.uk

II. Preliminaries and Problem Statement

Consider a linear, time-invariant, system described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \\ y(t) = Cx(t), \tag{1}$$

in which $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the input, $y(t) \in \mathbb{R}^p$ is the output, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$ are constant matrices, and the associated transfer function

$$W(s) := C(sI - A)^{-1}B.$$

Throughout the article, we assume that system (1) is minimal, *i.e.* (A, B) is controllable, and (A, C) is observable.

Definition 1. *Let $s^\star \in \mathbb{C} \setminus \sigma(A)$ and $k \in \mathbb{N} \setminus 0$. The 0-moment of system (1) at $s^\star$ is given by the complex matrix $\eta_0(s^\star) = W(s^\star)$. For $k \geq 1$, the k-moment of the system (1) at $s^\star$ is given by the complex matrix*

$$\eta_k(s^\star) = \frac{(-1)^k}{k!}\frac{d^kW}{ds^k}(s^\star).$$

The moments of the system (1) (in the sense of Definition 1) have been equivalently characterized in terms of a certain Sylvester equation in [14] and [15].

Lemma 1. *Consider the system (1) with $s^\star \in \mathbb{C}\setminus\sigma(A)$. Then the moments $\eta_0(s^\star), \ldots, \eta_k(s^\star)$ are in one-to-one relation with the matrix $C\Pi$, where $\Pi \in \mathbb{R}^{n\times\nu}$ is the (unique) solution of the Sylvester equation*

$$A\Pi + BL = \Pi S, \tag{2}$$

$S \in \mathbb{R}^{\nu\times\nu}$ is any non-derogatory[1] real matrix with characteristic polynomial given by $\det(sI-S) = \prod_{i=1}^{N}(s-s^\star)^{k+1}$, and $L \in \mathbb{R}^{m\times\nu}$ is such that the pair (S, L) is observable.

Definition 2. *Consider the system (1) and suppose that there exists a unique $\Pi \in \mathbb{R}^{n\times\nu}$ such that (2) holds. The moment of the system (1) at (S, L) is defined as $C\Pi$.*

The one-to-one characterization of Lemma 1 combined with the algebraic notion of moment of Definition 2 allow for an interpretation of the notion of moment in the time domain [15]. This interpretation relies upon the steady-state output response [23] of the interconnection (open-loop) of the system (1) with a signal generator described by the equations

$$\dot{\omega}(t) = S\omega(t), \quad \omega(0) = \omega_0, \tag{3a}$$

$$u(t) = L\omega(t), \tag{3b}$$

with $\omega(t) \in \mathbb{R}^\nu$ and $u(t) \in \mathbb{R}^m$. The open-loop interconnection yields the augmented system

$$\dot{\omega}(t) = S\omega(t), \tag{4a}$$

$$\dot{x}(t) = Ax(t) + BL\omega(t), \tag{4b}$$

$$y(t) = Cx(t), \tag{4c}$$

[1] A matrix is non-derogatory if its characteristic and minimal polynomials coincide.

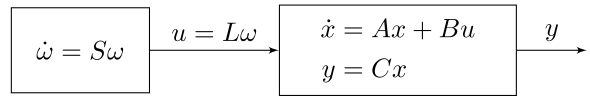

Fig. 1: The open-loop notion of moment.

which possesses a well-defined invariant set

$$\mathcal{M} := \{(x, \omega) \in \mathbb{R}^n \times \mathbb{R}^\nu \mid x = \Pi\omega\}.$$

The interconnection-based interpretation (4) of the notion of moment enables the problem of interpolation of points to be interpreted as a problem of interpolation of signals.

Definition 3. *The linear system described by the equations*

$$\dot{\xi}(t) = \overline{A}\xi(t) + \overline{B}u(t), \quad \xi(0) = \xi_0, \tag{5a}$$

$$\overline{y}(t) = \overline{C}\xi(t), \tag{5b}$$

with $\xi(t) \in \mathbb{R}^\rho$ and $\overline{y}(t) \in \mathbb{R}^p$ is a model at (S, L) of the system (1) if the system (5) has the same moment at (S, L) as (1). In this case, the system (5) is said to achieve moment matching at (S, L). Furthermore, if $\rho < n$ then the system (5) is a reduced order model of the system (1).

While the notion of moment in Definition 1 holds for every linear-time invariant system, the interconnection-based notion of moment (4) requires the invariant set $\mathcal{M}$ to be attractive. When the system (1) in unstable, the interconnection-based concept of moment can be made meaningful by using a stabilizer that renders the set $\mathcal{M}$ attractive. Yet, the trajectories of the closed-loop system for arbitrary stabilizers may converge towards a set different than $\mathcal{M}$, and hence the moment of the system might be modified. This issue is revealed by a simple example.

Example 1. Consider a two-dimensional system of the form (1) with matrices

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

in which $a_0 \in \mathbb{R}\setminus 0$ and $a_1 \in \mathbb{R}$. Consider a first-order signal generator of the form (3) with $S = 0$ and $L = 1$.

The Sylvester equation (2) yields a unique solution $\Pi^\top = \begin{bmatrix} a_0^{-1} & 0 \end{bmatrix}$ and the moment of the system at (S, L) is $C\Pi = a_0^{-1}$.[2] Note that for $a_0 > 0$ and $a_1 > 0$ the system is exponentially stable, the invariant set $\mathcal{M}$ is attractive, and the (open-loop) interconnection (4) has a well-defined steady-state output response, $y \to C\Pi\omega$. On the contrary, for any other a_0 and a_1 the origin of the system is not asymptotically stable, and $\mathcal{M}$ is not attractive. Hence, to render $\mathcal{M}$ attractive we consider a state feedback of the form

$$u = -Kx + L\omega,$$

where $K = \begin{bmatrix} k_0 & k_1 \end{bmatrix}$. Note that if $k_0 > -a_0$ and $k_1 > -a_1$ the closed-loop system is exponentially stable, *i.e.* $\sigma(A -$

[2] The moment $C\Pi = a_0^{-1}$ is the DC gain of the transfer function $W(s)$, *i.e.* $W(0) = -CA^{-1}B = a_0^{-1}$. Hence the condition $a_0 \neq 0$ is required to have a well-defined moment.

$BK) \subset \mathbb{C}^-$. Note, however, that the Sylvester equation associated with the closed-loop system is given by

$$(A - BK)\Pi + BL = \Pi S,$$

which yields a unique $\Pi^\top = \begin{bmatrix}(a_0 + k_0)^{-1} & 0\end{bmatrix}$, and thus the moment of the closed-loop system at (S, L) is $C\Pi = (a_0 + k_0)^{-1}$. It can be readily seen that the moment at (S, L) of the closed-loop system is now parameterized by k_0. Specifically, for every $k_0 \neq 0$ the feedback modifies the moment of the underlying system, while for $k_0 = 0$ the feedback leaves the moment of the underlying system untouched. Therefore, we may conclude that whenever $k_0 = 0$ and $a_0 > 0$ the (closed-loop) interconnection

$$\begin{aligned}\dot{\omega}(t) &= S\omega(t),\\ \dot{x}(t) &= (A - BK)x(t) + BL\omega(t),\\ y(t) &= Cx(t),\end{aligned}$$

possesses a well-defined invariant set $\mathcal{M}$ which is attractive for every $k_1 > -a_1$ and the moment of the closed-loop system remains untouched. On the contrary, for any $a_0 < 0$, the state feedback controller is incapable of achieving simultaneous stabilization of the closed-loop system and preservation of the moment of the underlying system. Note that, it is still possible to achieve stabilization of the closed-loop system without altering the moment of the system through the use of a suitable dynamic output feedback.

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Motivated by the previous example, in the remainder of this article we provide a closed-loop interpolation scheme that aims at relaxing the stability properties of the underlying plant in the interconnection-based notion of moment.

III. Closed-Loop Moment Matching

In this section we introduce the concept of closed-loop interpolation by moment matching. To begin with, we define an enhancement of the signal generator (3), called *generalized signal generator*.

A generalized signal generator is a linear time-invariant system modeled by the equations

$$\dot{\omega}(t) = S\omega(t), \qquad \omega(0) = \omega_0, \tag{6a}$$

$$\dot{z}(t) = Fz(t) - K\left(C\Pi\omega(t) - y(t)\right), \qquad z(0) = z_0, \tag{6b}$$

$$u(t) = Hz(t) + L\omega(t), \tag{6c}$$

with states $\omega(t) \in \mathbb{R}^\nu$ and $z(t) \in \mathbb{R}^r$, output $u(t) \in \mathbb{R}^m$, and input $y(t) \in \mathbb{R}^p$. The matrices characterizing the generalized signal generator (6) have the following properties.

H1. The matrix $S \in \mathbb{R}^{\nu\times\nu}$ is non-derogatory and verifies the spectrum conditions

$$\sigma(S) \cap \sigma(A) = \emptyset, \;\; \sigma(S) \cap \sigma(F) = \emptyset.$$

The matrix $L \in \mathbb{R}^{m\times\nu}$ is such that the pair (S, L) is observable.

H2. The matrices $F \in \mathbb{R}^{r\times r}$, $K \in \mathbb{R}^{r\times\nu}$, and $H \in \mathbb{R}^{m\times r}$ are such that $\sigma(J) \subset \mathbb{C}^-$ where

$$J := \begin{bmatrix} F & KC \\ BH & A \end{bmatrix}.$$

In addition, K is such that $\operatorname{rank} K = \nu$.

H3. The matrix $\Pi \in \mathbb{R}^{n\times\nu}$ is the (unique) solution of the Sylvester equation (2).

The notion of closed-loop interpolation relies on the steady-state output response of the interconnection of the system (1) and the generalized signal generator (6), which yields the (closed-loop) interconnected system

$$\dot{\omega}(t) = S\omega(t), \tag{7a}$$

$$\dot{z}(t) = Fz(t) + KC\left(x(t) - \Pi\omega(t)\right), \tag{7b}$$

$$\dot{x}(t) = BHz(t) + Ax(t) + BL\omega(t), \tag{7c}$$

$$y(t) = Cx(t). \tag{7d}$$

Lemma 2. *Consider the system* (1) *and the generalized signal generator* (6)*. Then, for every* $(z(0), x(0), \omega(0)) \in \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^\nu$ *the trajectories of* (7) *converge to the invariant set* $\mathcal{M}_{zx\omega} := \{(z, x, \omega) \in \mathbb{R}^r \times \mathbb{R}^n \times \mathbb{R}^\nu \mid (z, x) = (0, \Pi\omega)\}$.

Proof. By **H1** and the center manifold theorem [24], the closed-loop interconnected system (7) has a globally well-defined invariant manifold (which is a hyperplane) given by $\mathcal{M}_{zx\omega}$. In particular, assume $(x, z) = (\Pi\omega, \Psi\omega)$ for some $\Pi \in \mathbb{R}^{n\times\nu}$ and $\Psi \in \mathbb{R}^{r\times\nu}$, then

$$\dot{z} = \Psi\dot{\omega} \quad\Longrightarrow\quad F\Psi = \Psi S, \tag{8a}$$

$$\dot{x} = \Pi\dot{\omega} \quad\Longrightarrow\quad A\Pi + B(H\Psi + L) = \Pi S. \tag{8b}$$

The spectrum condition in **H1** (*i.e.* $\sigma(S) \cap \sigma(F) = \emptyset$) ensures that (8a) is satisfied only by $\Psi = 0$, whereas (8b) reads as $A\Pi + BL = \Pi S$ which is (2), whereby a unique solution exists since $\sigma(S) \cap \sigma(A) = \emptyset$ by **H1**. With this in mind we have that (7b) and (7c) yield the following equations

$$\begin{aligned}\dot{z} &= Fz + KC\left(x - \Pi\omega\right),\\ \dot{x} - \Pi\dot{\omega} &= BHz + A\left(x - \Pi\omega\right).\end{aligned}$$

Finally, by virtue of **H2** we have that for every initial condition $(z(0), x(0), \omega(0))$, $\lim_{t\to\infty} z(t) = 0$ and $\lim_{t\to\infty} x(t) - \Pi\omega(t) = 0$.

□

The generalized signal generator (6) yields a closed-loop interconnection scheme (as in Figure 2) in which $\mathcal{M}_{zx\omega}$ is an invariant and attractive set for every $(x(0), z(0), \omega(0))$ regardless of the eigenvalues of A. The restriction of the system (7) to the set $\mathcal{M}_{zx\omega}$ yields a copy of the system (3a). Every signal generator of the form (3) can be enhanced with the generalized version (6), which is obtained by combining the dynamics (6a), containing the interpolation points, with the dynamics (6b).

Theorem 1. *Consider the system* (1) *and the generalized signal generator* (6)*. Then, the steady-state output response*

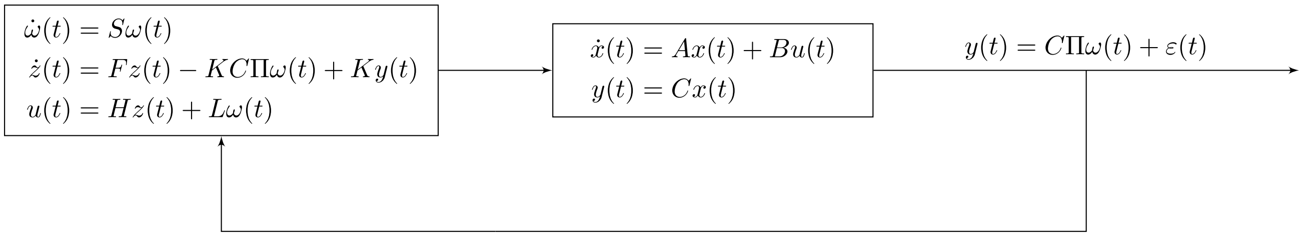

Fig. 2: Diagrammatic illustration of the closed-loop interpolation scheme for linear systems with decay term $\varepsilon(t)$.

of the (closed-loop) interconnected system (7) *is in one-to-one relation with the moment at* (S, L) *of the system* (1).

Proof. By the steady-state response of (7) discussed in Lemma 2, since for t sufficiently large $z(t) \to 0$ and $x(t) \to \Pi\omega(t)$, it follows that the output response (7d) yields

$$\begin{aligned}&\lim_{t\to\infty} y(t) - C\Pi\omega(t)\\ &= Cx(t) - C\Pi\omega(t) = 0,\end{aligned}$$

which, by Definition 2, implies that the moment of (1) at (S, L) through the generalized signal generator (6) is $C\Pi$. $\square$

The dynamics described in (6b) introduce degrees of freedom in constructing the signal generator. Additionally, the parameters of the dynamics (6b) can be adjusted to enhance the attractiveness of the invariant set characterizing the moment. As a result, for any matrix A, the closed-loop interpolation scheme maintains the one-to-one relation with the k-moment, *i.e.* $\eta_0(s^\star), \ldots, \eta_k(s^\star)$, of system (1) at $s^\star \in \mathbb{C} \setminus \sigma(A)$.

IV. Model Reduction by Closed-Loop Moment Matching

Given a generalized signal generator (6) defining the moment at (S, L) of the system (1), the focus of this section is to characterize linear models achieving moment matching at (S, L) by using the generalized signal generator (6).

Consider a family of multi-input, multi-output, linear systems $\overline{\Sigma}$ of dimension $\rho \geq \nu$ described by the equations

$$\dot{\xi}_a = \overline{A}_{11}\xi_a + \overline{A}_{12}\xi_b + \overline{B}_1 u, \qquad \xi_a(0) = \xi_{1_0}, \tag{9a}$$
$$\dot{\xi}_b = \overline{A}_{21}\xi_a(t) + \overline{A}_{22}\xi_b + \overline{B}_2 u, \qquad \xi_b(0) = \xi_{2_0}, \tag{9b}$$
$$\overline{y} = \overline{C}_1\xi_a + \overline{C}_2\xi_b, \tag{9c}$$

with states $\xi_a(t) \in \mathbb{R}^\nu$ and $\xi_b(t) \in \mathbb{R}^{\rho-\nu}$, input $u(t) \in \mathbb{R}^m$, output $\overline{y}(t) \in \mathbb{R}^p$, and matrices $\overline{A}_{11}$, $\overline{A}_{12}$, $\overline{A}_{21}$, $\overline{A}_{22}$, $\overline{B}_1$, $\overline{B}_2$, $\overline{C}_1$, and $\overline{C}_2$ of appropriate dimension. As per system (1), we let the state-space realization of system (9), with transfer function $\overline{W}(s)$, be minimal, *i.e.* (9) is controllable and observable.

To study the moment of the family of systems described by equations (9) we consider the closed-loop interpolation scheme given by the interconnection of (9) and the generalized signal generator (6) which yields

$$\dot{\omega} = S\omega, \tag{10a}$$
$$\dot{z} = Fz + K\overline{C}_1\xi_a + K\overline{C}_2\xi_b - KC\Pi\omega, \tag{10b}$$
$$\dot{\xi}_a = \overline{A}_{11}\xi_a + \overline{A}_{12}\xi_b + \overline{B}_1 Hz + \overline{B}_1 L\omega, \tag{10c}$$
$$\dot{\xi}_b = \overline{A}_{21}\xi_a + \overline{A}_{22}\xi_b + \overline{B}_2 Hz + \overline{B}_2 L\omega, \tag{10d}$$
$$\overline{y} = \overline{C}_1\xi_a + \overline{C}_2\xi_b. \tag{10e}$$

Lemma 3. *For all* $(z(0), \xi_a(0), \xi_b(0), \omega(0)) \in \mathbb{R}^r \times \mathbb{R}^\nu \times \mathbb{R}^{\rho-\nu} \times \mathbb{R}^\nu$ *the trajectories of the interconnected system* (10) *converge to the invariant set*

$$\mathcal{M}_{z\xi\omega} := \{(z, \xi_a, \xi_b, \omega) \in \mathbb{R}^r \times \mathbb{R}^\nu \times \mathbb{R}^{\rho-\nu} \times \mathbb{R}^\nu \mid (z, \xi_a, \xi_b) = (0, \omega, 0)\}$$

if and only if

$$\overline{A}_{11} = S - \overline{B}_1 L, \tag{11a}$$
$$\overline{A}_{21} = -\overline{B}_2 L, \tag{11b}$$
$$\overline{C}_1 = C\Pi, \tag{11c}$$

and $\overline{A}_{12}$, $\overline{A}_{22}$, $\overline{B}_1$, $\overline{B}_2$, *and* $\overline{C}_2$ *are such that* $\sigma(S) \cap \sigma(S - \overline{B}_1 L) = \emptyset$, $\sigma(S) \cap \sigma(\overline{A}_{22}) = \emptyset$, $\sigma(S) \cap \sigma(F) = \emptyset$, *and* $\sigma(\overline{J}_0) \subset \mathbb{C}^-$, *where*

$$\overline{J}_0 = \begin{bmatrix} F & K\overline{C}_1 & K\overline{C}_2 \\ \overline{B}_1 H & S - \overline{B}_1 L & \overline{A}_{12} \\ \overline{B}_2 H & -\overline{B}_2 L & \overline{A}_{22} \end{bmatrix}.$$

Proof. Following the same rationale as per Lemma 2, assume $(\xi_a, \xi_b, z) = (\overline{\Pi}_1\omega, \overline{\Pi}_2\omega, \Psi\omega)$ for some $\overline{\Pi}_1 \in \mathbb{R}^{\nu\times\nu}$, $\overline{\Pi}_2 \in \mathbb{R}^{(\rho-\nu)\times\nu}$, and $\Psi \in \mathbb{R}^{r\times\nu}$, then

$$\dot{z} = \Psi\dot{\omega} \implies$$
$$F\Psi + K\overline{C}_1\overline{\Pi}_1 + K\overline{C}_2\overline{\Pi}_2 - KC\Pi = \Psi S, \tag{12a}$$
$$\dot{\xi}_a = \overline{\Pi}_1\dot{\omega} \implies$$
$$\overline{A}_{11}\overline{\Pi}_1 + \overline{A}_{12}\overline{\Pi}_2 + \overline{B}_1 H\Psi + \overline{B}_1 L = \overline{\Pi}_1 S, \tag{12b}$$
$$\dot{\xi}_b = \overline{\Pi}_2\dot{\omega} \implies$$
$$\overline{A}_{21}\overline{\Pi}_1 + \overline{A}_{22}\overline{\Pi}_2 + \overline{B}_2 H\Psi + \overline{B}_2 L = \overline{\Pi}_2 S. \tag{12c}$$

Pick $(\overline{\Pi}_1\omega, \overline{\Pi}_2\omega, \Psi\omega) = (\omega, 0, 0)$. A direct computation yields

$$\text{(12b)} \implies \overline{A}_{11} + \overline{B}_1 L - S = 0 \quad \implies \text{(11a)},$$
$$\text{(12c)} \implies \overline{A}_{21} + \overline{B}_2 L = 0 \quad \implies \text{(11b)},$$

and (12a) $\implies$ $0 = K(\overline{C}_1 - C\Pi)$. This last implication emphasizes that, if (11c) holds then (12a) is satisfied. Note that by virtue of **H2** we have that $K \in \mathbb{R}^{r\times\nu}$ is full column rank and thus the condition (11c) is also necessary as $\overline{C}_1 - C\Pi = 0$ is the unique solution of (12a). Moreover, by the conditions $\sigma(S) \cap \sigma(S - \overline{B}_1 L) = \emptyset$, $\sigma(S) \cap \sigma(\overline{A}_{22}) = \emptyset$, $\sigma(S) \cap \sigma(F) = \emptyset$ the matrices $\overline{\Pi}_1 = I$, $\overline{\Pi}_2 = 0$, and $\Psi = 0$ are the unique solution of (12a)-(12b)-(12c). Finally, by the center manifold theory [24] the attractiveness of the invariant set $\mathcal{M}_{z\xi\omega}$ is guaranteed by the asymptotic stability of the zero equilibrium of the system (10c)-(10d) for $\omega = 0$, that is $\sigma(\overline{J}_0) \subset \mathbb{C}^-$. □

Theorem 2. *Consider the family of systems* (9) *and the generalized signal generator* (6)*. Let* $C\Pi$ *be the moment at* (S, L) *of the system* (1) *obtained from* (6)*. Suppose that the solution of* (10) *converges to the set* $\mathcal{M}_{z\xi\omega}$ *for every* $(z(0), \xi_a(0), \xi_b(0), \omega(0)) \in \mathbb{R}^r \times \mathbb{R}^\nu \times \mathbb{R}^{\rho-\nu} \times \mathbb{R}^\nu$*. Then* (9) *achieves moment matching at* (S, L)*.*

Proof. The steady-state output response of (10) is obtained from Lemma 3 and this implies that

$$\begin{aligned}\lim_{t\to\infty} \overline{y}(t) - C\Pi\omega(t) &= (\overline{C}_1\xi_a(t) + \overline{C}_2\xi_b(t) - C\Pi)\omega(t) \\ &= \overline{C}_1\omega(t) - C\Pi\omega(t) = 0.\end{aligned}$$

□

For a given generalized signal generator, Theorem 2 guarantees that this family, under the conditions of Lemma 3, achieves moment matching in a feedback control scheme. We observe in Corollary 2.1 below that any system verifying the necessary and sufficient conditions in Lemma 3 is a model parameterized in $\overline{A}_{12}$, $\overline{A}_{22}$, $\overline{B}_1$, $\overline{B}_2$, and $\overline{C}_2$ which achieves moment matching at (S, L).

Corollary 2.1. *Consider the generalized signal generator* (6) *and suppose that the family of systems* (9) *satisfies the conditions in Lemma 3. Let* $\widetilde{\Sigma}$ *be any model of* (1) *of dimension* $\rho \geq \nu$ *achieving moment matching at* (S, L)*. Suppose that the solution of* $\widetilde{\Sigma}$ *converges to the (well-defined) invariant set*

$$\begin{aligned}\mathcal{M}_{z\widetilde{\xi}\omega} := \{&(z, \widetilde{\xi}_a, \widetilde{\xi}_b, \omega) \in \mathbb{R}^r \times \mathbb{R}^\nu \times \mathbb{R}^{\rho-\nu} \times \mathbb{R}^\nu \\ &\mid (z, \widetilde{\xi}_a, \widetilde{\xi}_b) = (0, \widetilde{\Pi}_1\omega, \widetilde{\Pi}_2\omega)\},\end{aligned}$$

for every $(\widetilde{z}(0), \widetilde{\xi}_1(0), \widetilde{\xi}_2(0), \omega(0)) \in \mathbb{R}^r \times \mathbb{R}^\nu \times \mathbb{R}^{\rho-\nu} \times \mathbb{R}^\nu$*. Then the family of systems* (9) *is a parameterization of* $\widetilde{\Sigma}$*.*

Proof. Let $\widetilde{X} \in \mathbb{R}^{(\rho-\nu)\times(\rho-\nu)}$ be any invertible matrix. Then, without loss of generality, for every $\widetilde{\Pi}_2 \in \mathbb{R}^{(\rho-\nu)\times\nu}$ and every invertible $\widetilde{\Pi}_1 \in \mathbb{R}^{\nu\times\nu}$ we can define the change of coordinates

$$\widetilde{\xi}_1 = \widetilde{\Pi}_1\xi_a, \quad \widetilde{\xi}_2 = \widetilde{\Pi}_2\xi_a + \widetilde{X}\xi_b. \tag{13}$$

Hence, avoiding tedious computations, the result is directly proved by substituting (13) in (10) yielding that any system $\widetilde{\Sigma}$ associated with the attractive set $\mathcal{M}_{z\widetilde{\xi}\omega}$ makes the set $\mathcal{M}_{z\xi\omega}$ invariant and attractive, by the necessary and sufficient conditions of Lemma 3. □

It is worth mentioning that if $\omega(t) = 0$ for every $t \in \mathbb{R}$, the proposed generalized signal generator acts as a stabilizer of dimension r for any system of dimension ρ within the family of systems satisfying the conditions in Lemma 3. Consequently, the family of systems described in (9) that meet the conditions in Lemma 3 also defines the set of all admissible plants stabilized by a given stabilizing controller. This leads to a moment matching interpretation of the well-known dual Youla-Kučera parameterization [25], [26], which defines the set of all admissible plants stabilized by a given stabilizing controller.

V. Numerical Validation

We consider the benchmark example of a two-input, two-output NASA HiMAT aircraft model as in [22], [27]. The dynamics of the aircraft are described by the equations (1) with $n = 6$ and matrices A, B, and C as in [27]. The zero equilibrium of the system is unstable, and the steady-state behavior is not well defined. Hence, to construct a moment matching interpolation scheme, we construct a generalized signal generator of the form (6) with matrices

$$S = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$H^\top = \begin{bmatrix} \frac{227}{8814} & -\frac{109}{10437} \\ -\frac{374}{499} & \frac{337}{3750} \\ -\frac{367}{1845} & -\frac{13}{3962} \\ -\frac{648}{2051} & \frac{116}{6675} \end{bmatrix}, \quad K = \begin{bmatrix} \frac{24847}{171} & \frac{1617}{463} \\ -\frac{16391}{15} & -\frac{35873}{133} \\ -\frac{156367}{1348} & \frac{12345}{176} \\ \frac{30811}{26} & \frac{17792}{67} \end{bmatrix},$$

$$F = \begin{bmatrix} -\frac{7121}{49} & -\frac{1721}{47} & -\frac{6784}{303} & -\frac{3209}{100} \\ \frac{16391}{15} & -\frac{1269}{668} & \frac{16513}{61} & -\frac{38}{52369} \\ \frac{5956}{51} & -\frac{2336}{217} & -\frac{6693}{85} & -\frac{1469}{155} \\ -\frac{20151}{17} & \frac{337}{125} & -\frac{16673}{63} & \frac{232}{445} \end{bmatrix},$$

satisfying the conditions **H1**-**H2**-**H3**. Moreover, we design an unstable interpolant of the form (5) of dimension $\rho = 2$ with matrices

$$\overline{A} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}, \ \overline{B} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \ \overline{C} = \begin{bmatrix} \frac{1657}{583} & \frac{285}{12158} \\ \frac{4426}{1525} & -\frac{1127}{401} \end{bmatrix},$$

satisfying the conditions of Lemma 3 and Theorem 2. The interpolant, a reduced order model of dimension $\rho = 2$, achieves moment matching at (S, L), as shown in Fig. 3.

VI. Conclusion

In this paper we have revisited the concept of interconnection-based moment matching and introduced the notion of closed-loop interpolation. While open-loop moment matching is limited by a strong stability requirements on the underlying system, closed-loop interpolation overcomes this limitation by employing a signal generator that ensures internal stability. This approach allows relaxing stability conditions, even when the system's state is not

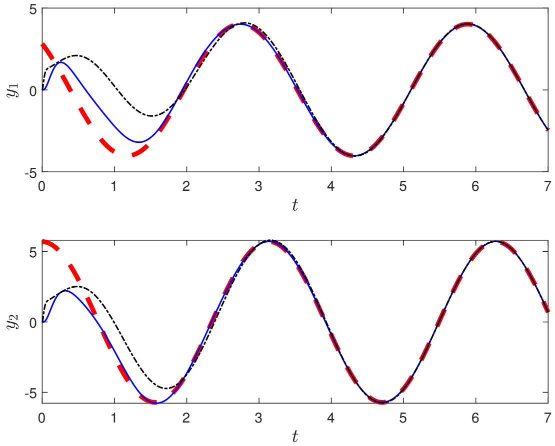

Fig. 3: Time histories of the moment (dashed red) and the outputs (solid blue) of the NASA HiMAT aircraft model, and the outputs (dotted black) of the reduced order model (of dimension 2).

fully known or measured. We have also demonstrated the existence of a family of models that parameterize all linear models achieving moment matching in a closed-loop fashion. These parameterized models offer flexibility and adaptability in achieving moment matching at desired operating points. Finally, we have discussed a benchmark example involving a two-input, two-output, NASA HiMAT aircraft model, illustrating the effectiveness of closed-loop interpolation in achieving moment matching.

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