2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

H_2 state-feedback control of continuous-time Markov jump Lur'e systems with partial mode information and sector bound optimization

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***Abstract*— This paper investigates the problem of H_2 state-feedback control of continuous-time Markov jump Lur'e systems with sector-bounded nonlinearity. It is provided synthesis conditions for the design of robust stabilizing controllers in which the Markov mode $\theta(t)$ is not available but is estimated by a detector $\hat{\theta}(t)$. As performance criteria, it is proposed a multi-objective optimization problem where an H_2 guaranteed cost is minimized and the amplitude of the sector is maximized. The synthesis framework is provided in terms of linear matrix inequalities, and a numerical example based on a practical system is provided to illustrate the results.**

I. INTRODUCTION

The dynamics of systems are often studied through mathematical representations, making feasible the use of theoretical tools that frequently demand the employment of numerical solutions using some optimization. Most physical phenomena exhibit nonlinearities, which are important for accurately capturing their true behavior. In addressing nonlinearities [16], the class of systems known as *Lur'e systems* [19], [20] offers a valuable framework for modeling various phenomena of interest such as saturation, dead zones, and quantizations, which are commonly encountered in practical engineering applications. A Lur'e system consists of a linear time-invariant model coupled with a nonlinear static input-output map and can be addressed by absolute stability results. The literature offers a wealth of stability analysis and synthesis tools based on the Lyapunov stability theory [21], [13] and frequency domain methods, such as Zames-Falb multipliers [29], [27], [4], [2], often formulated in terms of Linear Matrix Inequalities (LMIs) [3].

When the behavior of systems exhibits abrupt changes, single deterministic models might not be sufficiently precise to predict their trajectories and perform stability analysis. These changes, often arising from random events, are best described using stochastic processes. Their statistical characteristics, such as probabilities, expected values, and variances, become essential for providing an accurate model. One way to model such abrupt changes is by employing Markov Jump Linear Systems (MJLS). This approach decomposes the system into different operation modes governed by standard linear differential (or difference) equations. Transitions between these modes are modeled as jumps driven by randomness. An appealing feature of this class of models is the availability of convex optimization tools to perform analysis and synthesis. MJLS has been extensively researched, with [5] and [6] providing comprehensive references.

Recently, a class of systems known as Markov jump Lur'e systems has been created specially to address dynamics involving stochastic jumps and sector-bounded nonlinearities [24], [14]. In this context, it is worth of mentioning works dealing with stability analysis [14], ℓ_2-gain state-feedback control [15], $\ell_2-\ell_\infty$ observer-based control with sensors saturations [30], H_∞ filtering for stochastic time-delay systems [28], and H_2 and H_∞ filtering and dynamic output-feedback control with sector bound optimization [9], [8].

This paper investigates the problem of H_2 state-feedback control for continuous-time Markov Jump Lur'e systems. Aiming to address a more realistic scenario, where only cluster and partial observations [25] are possible, it is assumed that the controller has access only to estimations of the Markov process provided by a detector, denoted as $\hat{\theta}(t)$. In this case, the controller adapts according to the information provided by the detector, as presented in [22], [23], [11], [10]. Concerning the performance criteria, the proposed method can optimize both a bound for the H_2 norm from the disturbance input to the controlled output, and the amplitude of the sector associated with the sector-bounded nonlinearity. As technical contribution, the synthesis conditions are formulated in terms of LMIs, which are convex optimization problems well established in the control literature [3] with specialized software available [1], [17]. A numerical example based on two coupled electrical machines is presented to illustrate the achievements.

II. PRELIMINARIES

Notation: Throughout the text, $(\cdot)'$ indicates transposed matrices or vectors, $\mathrm{diag}\,(\cdot)$ a diagonal matrix, $\mathrm{tr}(\cdot)$ the trace operator, and $\star$ a block induced by symmetry in square matrices. The operator $\mathbb{E}(\cdot)$ ($\mathbb{E}(\cdot|\cdot)$) is the (conditional) expected value of a random variable, $\mathbb{R}_+$ indicates the set of positive real numbers, and $\mathrm{Her}\,(A) \triangleq A + A'$. The inequality $X > 0$ means that X is a positive definite symmetric matrix. The set of nonlinear functions belonging to a given sector is denoted by $\mathbb{F}$. The notation $[a_{ij}]$ is used to represent a matrix where a_{ij} are its entries. The 2-norm (squared) of

Supported by the Brazilian agencies National Council for Scientific and Technological Development (CNPq), Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) – Finance Code 001, and Project Stic-Amsud/CAPES NetConHybSDP, code 22-STIC-09.

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any stochastic signal $x : \mathbb{R}_+ \to \mathbb{R}^{n_x}$ is defined as

$$\|x\|_2^2 \triangleq \mathbb{E}\left(\int_0^\infty \|x(t)\|^2 \, dt\right),$$

and $\|x(t)\|$ is the Euclidean norm of vector $x(t)$.

We assume the probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, where Ω is the sample space, $\mathfrak{F}$ is the σ-field into Ω, and $\mathbb{P}$ is the probability measure.

Consider the following MJLS $\mathcal{G}_{\tilde{\theta}(t)}$:

$$\begin{aligned}
\dot{x}(t) &= A_{\theta(t)}x(t) + B_{p\theta(t)}p(t) + B_{u\theta(t)}u(t) + B_{w\theta(t)}w(t),\\
q(t) &= C_{q\theta(t)}x(t),\\
z(t) &= C_{z\theta(t)}x(t) + D_{zp\theta(t)}p(t) + D_{zu\theta(t)}u(t),\\
p(t) &= -\phi\left(q(t)\right),
\end{aligned}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^{n_x}$ is the state, $z : \mathbb{R}_+ \to \mathbb{R}^{n_z}$ the performance output, $w : \mathbb{R}_+ \to \mathbb{R}^{n_w}$ the exogenous input, $u : \mathbb{R}_+ \to \mathbb{R}^{n_u}$ the control input, $p : \mathbb{R}_+ \to \mathbb{R}^{n_p}$ is an input to the linear part of the system, $q : \mathbb{R}_+ \to \mathbb{R}^{n_p}$ is the output signal, and $\phi : \mathbb{R}^{n_p} \to \mathbb{R}^{n_p}$ is a static, time-invariant and decentralized nonlinearity between $q(t)$ and $p(t)$. It is important to emphasize that $p(t)$ and $q(t)$ may not necessarily have physical meaning; instead, they could be defined mathematically to facilitate the representation of the nonlinear system within the Lur'e framework. All matrices have compatible dimensions, and the nonlinear function ϕ is characterized by $\phi(0) = 0$, ensuring that the origin $x = 0$ serves as an equilibrium point. Additionally, $\phi(\cdot)$ belongs to the sector $\left[0_{n_p \times n_p}, \boldsymbol{\kappa}\right]$, with $\boldsymbol{\kappa} \triangleq \operatorname{diag}(\kappa_1, \cdots, \kappa_{n_p})$, $\kappa_j > 0$ for $j \in \{1, 2, \cdots, n_p\}$, satisfying the following condition

$$\left(\phi\left(q(t)\right) - \boldsymbol{\kappa}q(t)\right)' \phi\left(q(t)\right) \le 0, \, \forall t \in \mathbb{R}_+. \tag{1}$$

which means that $\phi(\cdot)$ is sector-bounded.

The hidden process is denoted by the pair $\tilde{\theta}(t) = (\theta(t),\hat{\theta}(t))$, where the random variable $\theta(t)$ is a homogeneous Markov chain taking values in $\mathbb{K} \triangleq \{1,\, 2,\, \cdots,\, N\}$ with transition rate matrix $[\lambda_{ij}] \in \mathbb{R}^{N\times N}$. Besides, to cope with situations where the process $\theta(t)$ cannot be measured precisely, the random variable $\hat{\theta}(t) \in \mathbb{M} = \{1,\, \cdots,\, M\}$ is assumed to be provided by a suitable detector at all times, as presented in [22], [23] and [11]. Thus, $\tilde{\theta}(t)$ is a homogeneous Markov process

$$\begin{array}{rccc} \tilde{\theta}: & \mathbb{R}_+ & \to & \mathbb{L} \\ & t & \mapsto & (i,k) \end{array},$$

where $\mathbb{L} \triangleq \mathbb{K} \times \mathbb{M}$, and the transition probabilities $p_{(i,k)(j,\ell)}(\varepsilon) = \mathbb{P}\Big(\tilde{\theta}(t+\varepsilon) = (j,\ell) | \tilde{\theta}(t) = (i,k)\Big)$ are given by [11]

$$p_{(i,k)(j,\ell)}(\varepsilon) = \begin{cases} \tilde{\lambda}_{(i,k)(j,\ell)}\varepsilon + o(\varepsilon),\ (j,\ell) \neq (i,k), \\ 1 + \tilde{\lambda}_{(i,k)(i,k)}\varepsilon + o(\varepsilon),\ (j,\ell) = (i,k), \end{cases} \tag{2}$$

where,

$$\tilde{\lambda}_{(i,k)(j,\ell)} = \begin{cases} \alpha^k_{j\ell}\lambda_{ij},\ j \neq i,\ \ell \in \mathbb{M}, \\ \hat{\lambda}^i_{k\ell},\ \ell \neq k,\ j = i,\ i \in \mathbb{K}, \\ \lambda_{ii} + \hat{\lambda}^i_{kk},\ j = i,\ \ell = k, \end{cases} \tag{3}$$

and $\sum_{j\in\mathbb{K}} \lambda_{ij} = 0$, with $\lambda_{ii} \le 0$, $\forall i \in \mathbb{K}$, $\sum_{\ell\in\mathbb{M}} \hat{\lambda}^i_{k\ell} = 0$, with $\hat{\lambda}^i_{kk} \le 0$, $\forall k \in \mathbb{M}$ and $\sum_{\ell\in\mathbb{M}} \alpha^k_{j\ell} = 1$, $\forall k \in \mathbb{M}$. Additionally, from (2) one has $\sum_{(j,\ell)\in\mathbb{L}} \tilde{\lambda}_{(i,k)(j,\ell)} = 0$, and $\hat{\lambda}^i_{k\ell}$ represents the transition rates from the detector $\hat{\theta}(t)$. The probabilities $\alpha^k_{j\ell}$ dictate what should happen to the detector after a jump takes place in the Markov chain [23]. Thus, we obtain the matrix $\left[\tilde{\lambda}_{(i,k)(j,\ell)}\right] \in \mathbb{R}^{NM\times NM}$, that informs transition rates between joint Markov chain modes $\tilde{\theta}(t) \in \mathbb{L}$.

Remark 2.1: As presented in [11], we can retrieve some interesting cases presented in the literature, such as

1) Complete observation: $\hat{\theta}(t) \equiv \theta(t)$, i.e., $M = N$ $(\mathbb{K} = \mathbb{M})$, which implies $\hat{\lambda}^i_{k\ell} = 0$, $\alpha^k_{jj} = 1$ and $\alpha^k_{j\ell} = 0$ for $j \neq \ell$;
2) Cluster observation: In this scenario, the Markov modes may be written as the union of $M \le N$ disjoint sets (clusters) $\mathbb{K}_i$ so that $\mathbb{K} = \bigcup_{i\in\mathbb{M}} \mathbb{K}_i$. We consider the function $h : \mathbb{K} \to \mathbb{M}$ such that $h(i) = j$, that is, the function that represents the cluster where the Markov mode belongs, and thus the controller has access to $h(i)$. This is equivalent to $\hat{\lambda}^i_{k\ell} = 0$ and $\alpha^k_{ih(i)} = 1$, so that whenever $\theta(t)$ jumps to i, $\hat{\theta}(t)$ jumps simultaneously to $h(i)$.

Note that the partial observation $(M < N)$ is a particular case of cluster observation, item (2).

The following subsections define stochastic stability and the H_2 norm for system $\mathcal{G}_{\tilde{\theta}(t)}$.

A. Stochastic Stability and H_2 Norm

The following definitions are necessary for obtaining analysis and synthesis conditions for the problem under investigation. The reader can find more details in [6], [3].

Definition 1: (Infinitesimal Generator). Let a stochastic process defined by $\{X(t),\, t \in \mathbb{R}_+\}$. A function of this process is represented by $f(X(t))$. The infinitesimal generator $\varphi(\cdot)$ applied to f is given by

$$\varphi(f) \triangleq \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\frac{f(X(t+\varepsilon)) - f(X(t))}{\varepsilon}\right). \tag{4}$$

Definition 2: (Stochastic Stability (SS) and Mean-Square Stability (MSS)). The system $\mathcal{G}_{\tilde{\theta}(t)}$ with null input is said

1) SS if there exists $G = G' > 0$, such that

$$\mathbb{E}\left(\int_0^\infty \|x(t)\|^2 dt \middle| \tilde{\theta}_0\right) \le x_0' G x_0.$$

2) MSS if

$$\lim_{t\to\infty} \mathbb{E}(\|x(t)\|^2) = 0,$$

for initial conditions $x_0 \in \mathbb{R}^{n_x}$ and $\tilde{\theta}_0 \in \mathbb{L}$.

Definition 3: (H_2 norm). Considering that $\mathcal{G}_{\tilde{\theta}(t)}$ is MSS and $x(0) = 0$, the H_2 norm of $\mathcal{G}_{\tilde{\theta}(t)}$ is defined as

$$\|\mathcal{G}_{\tilde{\theta}(t)}\|_2^2 = \sum_{j=1}^{n_w} \mathbb{E}\left(\int_0^\infty \|z_j(t)\|^2 dt\right) = \sum_{j=1}^{n_w} \|z_j\|_2^2 = \|z\|_2^2,$$

where $z_j(t)$ is the system response to impulsive input $w(t) = e_j\delta(t)$, e_j is a vector with all null entries except the j^{th}, which is 1, and $\delta(t)$ is the Dirac's impulse.

The next lemma is useful for the derivation of the main achievements of the paper, presented in the next section.

Lemma 1: [12] For $P = P' > 0$, the inequality $G'P^{-1}G \geq \mathrm{Her}(G) - P$ holds for any square matrix G of compatible dimensions.

B. Problem Statement

The problem to be investigated is the design of the stabilizing detector-based state-feedback control law $u(t) = K_{\hat{\theta}(t)}x(t)$, where $K_{\hat{\theta}(t)}$ are gains to be computed, yielding the following closed-loop system $\mathcal{G}_{K_{\tilde{\theta}(t)}}$

$$\begin{aligned} \dot{x}(t) &= A_{\tilde{\theta}(t)}x(t) + B_{p\theta(t)}p(t) + B_{w\theta(t)}w(t), \\ q(t) &= C_{q\theta(t)}x(t), \\ p(t) &= -\phi\left(q(t)\right), \\ z(t) &= C_{z\tilde{\theta}(t)}x(t) + D_{zp\theta(t)}p(t), \end{aligned} \tag{5}$$

with

$$\begin{aligned} A_{\tilde{\theta}(t)} &= A_{\theta(t)} + B_{u\theta(t)}K_{\hat{\theta}(t)}, \\ C_{z\tilde{\theta}(t)} &= C_{z\theta(t)} + D_{zu\theta(t)}K_{\hat{\theta}(t)}, \end{aligned} \tag{6}$$

As control objectives, an H_2 guaranteed cost is minimized, and the sector amplitudes $\boldsymbol{\kappa}$ are maximized. To compact the notation regarding the indexation of the matrices depending on $\theta(t)$ and $\hat{\theta}(t)$, from this point we write $X_{\theta(t)} = X_i$, $i \in \mathbb{K}$, $X_{\hat{\theta}(t)} = X_k$, $k \in \mathbb{M}$, and $X_{\theta(t)\hat{\theta}(t)} = X_{ik}$, $(i,k) \in \mathbb{L}$ where X can be any matrix of the system.

III. Main Results

The next theorem presents the first result of the paper, which is an LMI-based approach to address the problem under investigation, assuming that the state-feedback gains are given. The result is an extension of [8] to deal with the detector-based feedback gains.

Theorem 3.1: Let K_k, $k \in \mathbb{M}$ be given stabilizing state-feedback gains. If there exist symmetric positive definite matrices P_{ik}, $(i,k) \in \mathbb{L}$, a diagonal matrix $\boldsymbol{\kappa} > 0$, and positive scalars γ_2 and ξ satisfying the LMIs

$$\gamma_2 > \sum_{(i,k)\in\mathbb{L}} \pi_{ik}\mathrm{tr}(B'_{w_i}P_{ik}B_{w_i}), \tag{7}$$

$$\mathrm{tr}(\boldsymbol{\kappa}) > \xi, \tag{8}$$

$$\begin{bmatrix} \mathcal{T}_{ik}(\mathbf{P}) & \star & \star \\ -B'_{p_i}P_{ik} + \boldsymbol{\kappa}C_{q_i} & -2I_{n_p} & \star \\ C_{z_{ik}} & -D_{zp_i} & -I_{n_z} \end{bmatrix} < 0, \tag{9}$$

with the operator

$$\mathcal{T}_{ik}(\mathbf{P}) \triangleq \mathrm{Her}\left(\left(A_i + B_{u_i}K_k\right)' P_{ik}\right) + \sum_{(j,\ell)\in\mathbb{L}} \tilde{\lambda}_{(i,k)(j,\ell)}P_{j\ell},$$

and $\pi_{ik} \triangleq \mathbb{P}\left(\tilde{\theta}_0 = (i,k)\right)$, then system $\mathcal{G}_{K_{\tilde{\theta}(t)}}$ is MSS for any nonlinearity $\phi(\cdot)$ belonging to sector $[0_{n_p\times n_p}, \boldsymbol{\kappa}]$, and

$$\max_{\mathbb{F}} \|z\|_2 < \sqrt{\gamma_2},\ \mathbb{F} \triangleq \left\{\phi(q(t)) \in \left[0_{n_p\times n_p}, \boldsymbol{\kappa}\right]\right\} \tag{10}$$

Proof: We start defining the Lyapunov stochastic function $v_{x(t),\tilde{\theta}(t)} \triangleq x(t)'P_{\tilde{\theta}(t)}x(t)$ and its infinitesimal generator

$$\varphi(v_{x(t),\tilde{\theta}(t)}) = \dot{x}'P_{\tilde{\theta}(t)}x + x'P_{\tilde{\theta}(t)}\dot{x} + x'\varphi(P_{\tilde{\theta}(t)})x, \tag{11}$$

where $\varphi(P_{\tilde{\theta}(t)})$ is given by,

$$\begin{aligned} &= \lim_{\varepsilon\to 0^+} \mathbb{E}\left(\frac{P_{\tilde{\theta}(t+\varepsilon)} - P_{\tilde{\theta}(t)}}{\varepsilon} \middle| \tilde{\theta}(t) = (i,k) \right) \\ &= \lim_{\varepsilon\to 0^+} \frac{\sum_{(j,\ell)\in\mathbb{H}}^{NM} \tilde{\lambda}_{(i,k)(j,\ell)}P_{j\ell}\varepsilon + (1 + \tilde{\lambda}_{(i,k)(i,k)}\varepsilon)P_{ik} - P_{ik}}{\varepsilon} \\ &= \sum_{(j,\ell)\in\mathbb{L}} \tilde{\lambda}_{(i,k)(j,\ell)}P_{j\ell}, \end{aligned}$$

$$\mathbb{H} \triangleq \mathbb{L} \setminus (i,k) = \{(j,\ell) \in \mathbb{L} : (j,\ell) \neq (i,k)\}\,. \tag{12}$$

Multiplying (9) on the left by $\begin{bmatrix} x' & \phi(q)' & 0' \end{bmatrix}$ and on the right by the transpose, yields

$$x'\mathcal{T}_{ik}(\mathbf{P})x - 2\phi(q)'\phi(q) + \mathrm{Her}(\phi(q)'(-B'_{p_i}P_{ik} + \boldsymbol{\kappa}C_{q_i})x) < 0, \tag{13}$$

which can be rewritten, using (11), as follows

$$\varphi(v_{x(t),\tilde{\theta}(t)}) < \mathrm{Her}\Big((\phi(q) - \boldsymbol{\kappa}q)'\phi(q)\Big) \leq 0, \tag{14}$$

assuring the stochastic stability of the system. Next, rewrite (9) as follows

$$\begin{bmatrix} \mathcal{T}_{ik}(\mathbf{P}) & \star \\ -B'_{p_i}P_{ik} + \boldsymbol{\kappa}C_{q_i} & -2I_{n_p} \end{bmatrix} + \begin{bmatrix} C'_{z_{ik}} \\ -D'_{zp_i} \end{bmatrix} \begin{bmatrix} C'_{z_{ik}} \\ -D'_{zp_i} \end{bmatrix}' < 0,$$

and multiply on the left by $\begin{bmatrix} x' & \phi(q)' \end{bmatrix}$ and on the right by the transpose to obtain

$$x'(\mathcal{T}_{ik}(\mathbf{P}) + C'_{z_{ik}}C_{z_{ik}})x + \phi(q)'(-2I_{n_p} + D'_{zp_i}D_{zp_i})\phi(q) + \mathrm{Her}\big(\phi(q)'(-(B'_{p_i}P_{ik} + D'_{zp_i}C_{z_{ik}}) + \boldsymbol{\kappa}C_{q_i})x\big) < 0,$$

which can be rewritten, using (11), as follows

$$\varphi(v_{x(t),\tilde{\theta}(t)}) + \|z(t)\|^2 < \mathrm{Her}((\phi(q) - \boldsymbol{\kappa}q)'\,\phi(q)) \leq 0. \tag{15}$$

From inequality (15), we have

$$\|z(t)\|^2 < -\varphi(v_{x(t),\tilde{\theta}(t)}). \tag{16}$$

Integrating (16) and applying the mathematical expectation on both sides leads to

$$\begin{aligned} &\mathbb{E}\left(\int_0^\infty \|z(t)\|^2 dt\right) < \\ &-\mathbb{E}\left(\int_0^\infty \varphi\left(v_{x(t),\tilde{\theta}(t)}\right) dt \middle| x_0 = B_{w_i}e_j,\, \tilde{\theta}_0 = (i,k)\right), \end{aligned} \tag{17}$$

implying

$$\max_{\mathbb{F}} \|z\|_2^2 < \mathbb{E}\left(x'_0P_{ik}x_0 \middle| x_0 = B_{w_i}e_j\right), \tag{18}$$

where $\mathbb{F}$ is the set of possible nonlinearities. To get (18) from (17), we used the fact that $\lim_{t\to\infty}\mathbb{E}\left(v_{x(t),\tilde{\theta}(t)}\right) = 0$, since the system is stochastically stable. Moreover, note that the initial condition $x_0 \triangleq B_{w_i}e_j$ is equivalent to the impulsive input discussed in Definition 3. Thus, we have that

$$\begin{aligned} \max_{\mathbb{F}}\|z\|_2^2 &< \mathbb{E}\left(\sum_{j=1}^{n_w}\left(e'_jB'_{w_i}P_{ik}B_{w_i}e_j\right)\right) \\ &= \mathbb{E}\left(\mathrm{tr}\left(B'_{w_i}P_{ik}B_{w_i}\right)\right) \\ &= \sum_{(i,k)\in\mathbb{L}} \pi_{ik}\mathrm{tr}\left(B'_{w_i}P_{ik}B_{w_i}\right), \end{aligned} \tag{19}$$

and using (7), that

$$\max_{\mathbb{F}} \|z\|_2^2 < \sum_{(i,k)\in\mathbb{L}} \pi_{ik} \mathrm{tr}\left(B_{w_i}' P_{ik} B_{w_i}\right) < \gamma_2, \tag{20}$$

Finally, note that in inequality (8) the scalar $\xi > 0$ is a lower bound for $\mathrm{tr}(\boldsymbol{\kappa})$, which refers to the sum of all $\kappa_j > 0, \forall j \in \{1, \cdots, n_p\}$, concluding the proof. ■

To determine the minimum H_2 guaranteed cost (γ_2) and the maximum lower bound for the sum of all sector amplitudes (ξ), we propose the following multi-objective convex optimization problem

$$\inf_{P_{ik}>0,\ \boldsymbol{\kappa}>0} \{g(\gamma_2, \xi) : \text{ subject to } (7)-(9)\}, \tag{21}$$

where $g(\gamma_2, \xi) \triangleq -b\xi + a\gamma_2$ is the objective function to be optimized, with weights $a > 0$ and $b > 0$ chosen by the designer *a priori*. This optimization problem simultaneously minimizes the H_2 guaranteed cost with weight a and $-\xi$ with weight b, subject to constraints (7)-(9). In other words, it maximizes the amplitudes of the sectors while minimizing the H_2 guaranteed cost.

The following theorem presents the primary contribution of the paper: a synthesis condition for designing H_2 detector-based state-feedback controllers for continuous-time Markov jump Lur'e systems. Moreover, these results hold even when $\phi_i(q(t))$ is different for each mode $i \in \mathbb{K}$, as discussed in [7].

Theorem 3.2: Let $\zeta_{ik} > 0, \forall (i,k), (j,\ell) \in \mathbb{L}$ be given scalars. If there exist positive definite symmetric matrices X_{ik}, W_{ik} and $Z_{(i,k)(j,\ell)}$, matrices Y_k, Q_k and R_{ik}, for all $(i,k),(j,\ell) \in \mathbb{L}$, $k \in \mathbb{M}$, a diagonal matrix $\boldsymbol{\tau} > 0$ and positive scalars ν and γ_2 such that the following LMIs

$$\sum_{(i,k)\in\mathbb{L}} \pi_{ik} \mathrm{tr}(W_{ik}) < \gamma_2, \quad \mathrm{tr}(\boldsymbol{\tau}) < \nu, \tag{22}$$

$$\begin{bmatrix} W_{ik} & \star \\ B_{w_i} & X_{ik} \end{bmatrix} > 0, \quad \begin{bmatrix} Z_{(i,k)(j,\ell)} & \star \\ R_{ik} & X_{j\ell} \end{bmatrix} > 0, \tag{23}$$

$$\mathcal{W}_{ik} + \mathrm{Her}(\mathcal{U}_{ik}\mathcal{V}_{ik}) < 0, \tag{24}$$

hold with the matrices $\mathcal{W}_{ik}$, $\mathcal{U}_{ik}$ and $\mathcal{V}_{ik}$ respectively given by

$$\begin{bmatrix} \tilde{\lambda}_{(i,k)(i,k)} X_{ik} & \star & \star & \star & \star \\ X_{ik} & 0_{n_x\times n_x} & \star & \star & \star \\ -\boldsymbol{\tau} B_{p_i}' & 0_{n_p\times n_x} & -4\boldsymbol{\tau} + 2I_{n_p} & \star & \star \\ 0_{n_z\times n_x} & 0_{n_z\times n_x} & -D_{zp_i}\boldsymbol{\tau} & -I_{n_z} & \star \\ X_{ik} & 0_{n_x\times n_x} & 0_{n_x\times n_p} & 0_{n_x\times n_z} & \Pi_{ik} \end{bmatrix},$$

$$\begin{bmatrix} \bar{A}_{ik}' & -Q_k' & (C_{q_i}Q_k)' & \bar{C}_{z_{ik}}' & (0_{n_x\times n_x})' \end{bmatrix}',$$

$$\begin{bmatrix} \zeta_{ik} I_{n_x} & I_{n_x} & 0_{n_x\times n_p} & 0_{n_x\times n_z} & 0_{n_x\times n_x} \end{bmatrix},$$

and

$$\begin{aligned} &\bar{A}_{ik} = A_i Q_k + B_{u_i} Y_k,\ \bar{C}_{z_{ik}} = C_{z_i} Q_k + D_{zu_i} Y_k, \\ &\Pi_{ik} \triangleq -\mathrm{Her}(R_{ik}) + \sum_{(j,\ell)\in\mathbb{H}} \tilde{\lambda}_{(i,k)(j,\ell)} Z_{(i,k)(j,\ell)}, \end{aligned} \tag{25}$$

with $\mathbb{H}$ given in (12), then the state-feedback gains

$$K_k = Y_k Q_k^{-1},\ k \in \mathbb{M} \tag{26}$$

associated with the sector amplitudes

$$\boldsymbol{\kappa} = \boldsymbol{\tau}^{-1}, \tag{27}$$

assure that the closed-loop system $\mathcal{G}_{K_{\hat{\theta}(t)}}$ is MSS for any nonlinearity $\phi(\cdot)$ belonging to sector $[0_{n_p\times n_p}, \boldsymbol{\kappa}]$ and (10).

Proof: Suppose that (22)-(24) hold. Applying the Schur complement in second inequality of (23) provides $Z_{(i,k)(j,\ell)} > R_{ik}'\left(X_{j\ell}\right)^{-1} R_{ik}$, which multiplied by $\tilde{\lambda}_{(i,k)(j,\ell)}$ and summed up for all $(j,\ell) \in \mathbb{H}$, yields

$$\sum_{(j,\ell)\in\mathbb{H}} \tilde{\lambda}_{(i,k)(j,\ell)} Z_{(i,k)(j,\ell)} > R_{ik}' \Delta_{ik} R_{ik} \geq \mathrm{Her}\left(R_{ik}\right) - \Delta_{ik}^{-1},$$

where $\Delta_{ik} = \sum_{(j,\ell)\in\mathbb{H}} \tilde{\lambda}_{(i,k)(j,\ell)} \left(X_{j\ell}\right)^{-1}$. Rewriting the previous inequality, yields

$$-\mathrm{Her}\left(R_{ik}\right) + \sum_{(j,\ell)\in\mathbb{H}} \tilde{\lambda}_{(i,k)(j,\ell)} Z_{(i,k)(j,\ell)} \geq -\Delta_{ik}^{-1}.$$

Using Lemma 1 on block (3×3) of (24) gives

$$-4\boldsymbol{\tau} + 2I_{n_p} = -2(\boldsymbol{\tau}' + \boldsymbol{\tau} - I_{n_p}) \geq -2\boldsymbol{\tau}' I_{n_p} \boldsymbol{\tau} = -2\boldsymbol{\tau}^2.$$

Thus, (24) still holds with $\mathcal{W}_{ik}$ rewritten as

$$\mathscr{W}_{ik} = \begin{bmatrix} \tilde{\lambda}_{(i,k)(i,k)} X_{ik} & \star & \star & \star & \star \\ X_{ik} & 0_{n_x\times n_x} & \star & \star & \star \\ -\boldsymbol{\tau} B_{p_i}' & 0_{n_p\times n_x} & -2\boldsymbol{\tau}^2 & \star & \star \\ 0_{n_z\times n_x} & 0_{n_z\times n_x} & -D_{zp_i}\boldsymbol{\tau} & -I_{n_z} & \star \\ X_{ik} & 0_{n_x\times n_x} & 0_{n_x\times n_p} & 0_{n_x\times n_z} & -\Delta_{ik}^{-1} \end{bmatrix}.$$

Moreover, (24) can be factorized as follows

$$\mathscr{W}_{ik} + \mathrm{Her}(\mathscr{U}_{ik} Q_k \mathcal{V}_{ik}) < 0, \tag{28}$$

with $\mathcal{V}_{ik}$ as previously defined and $\mathscr{U}_{ik}$ given by

$$\begin{bmatrix} A_{ik}' & -I_{n_x} & C_{q_i}' & C_{z_{ik}}' & 0_{n_x\times n_x}' \end{bmatrix}'.$$

Multiplying (28) on the right by $\mathscr{U}_{ik}^{\perp}$ and on the left by the transpose, where $\mathscr{U}_{ik}^{\perp}$ is a basis for the null space of $\mathscr{U}_{ik}$, (i.e., $\mathscr{U}_{ik}\mathscr{U}_{ik}^{\perp} = 0$), with

$$\mathscr{U}_{ik}^{\perp} = \begin{bmatrix} I_{n_x} & 0_{n_x\times n_p} & 0_{n_x\times n_z} & 0_{n_x\times n_x} \\ A_{ik}' & C_{q_i}' & C_{z_{ik}}' & 0_{n_x\times n_x} \\ 0_{n_p\times n_x} & I_{n_p} & 0_{n_p\times n_z} & 0_{n_p\times n_x} \\ 0_{n_z\times n_x} & 0_{n_z\times n_p} & I_{n_z} & 0_{n_z\times n_x} \\ 0_{n_x\times n_x} & 0_{n_x\times n_p} & 0_{n_x\times n_z} & I_{n_x} \end{bmatrix}.$$

provides

$$\begin{bmatrix} \mathcal{H}_{ik}(\mathbf{X}) & \star & \star & \star \\ -\boldsymbol{\tau} B_{p_i}' + C_{q_i} X_{ik} & -2\boldsymbol{\tau}^2 & \star & \star \\ C_{z_{ik}} X_{ik} & -D_{zp_i}\boldsymbol{\tau} & -I_{n_z} & \star \\ X_{ik} & 0_{n_x\times n_p} & 0_{n_x\times n_z} & -\Delta_{ik}^{-1} \end{bmatrix} < 0, \tag{29}$$

where, $\mathcal{H}_{ik}(\mathbf{X}) \triangleq \mathrm{Her}(A_{ik}X_{ik}) + \tilde{\lambda}_{(i,k)(i,k)} X_{ik}$. Applying the Schur complement in (29) and a congruence transformation in the result with $\mathrm{diag}(X_{ik}^{-1}, \boldsymbol{\tau}^{-1}, I_{n_z})$, gives

$$\begin{bmatrix} \mathcal{T}_{ik}(\mathbf{P}) & \star & \star \\ -B_{p_i}' P_{ik} + \boldsymbol{\kappa} C_{q_i} & -2I_{n_p} & \star \\ C_{z_{ik}} & -D_{zp_i} & -I_{n_z} \end{bmatrix} < 0, \tag{30}$$

with $X_{ik}^{-1} = P_{ik}$ and $\boldsymbol{\tau} \triangleq \left(1/\boldsymbol{\kappa}_1, \cdots, 1/\boldsymbol{\kappa}_{n_p}\right)$, which is (9).

The application of the Schur complement in the first inequality of (23) provides $W_{ik} > B'_{w_i} X_{ik}^{-1} B_{w_i}$, which multiplied by π_{ik} and summed up for all $(i,k) \in \mathbb{L}$, and taking into account (22), gives

$$\gamma_2 > \sum_{(i,k)\in\mathbb{L}} \pi_{ik}\mathrm{tr}(W_{ik}) > \sum_{(i,k)\in\mathbb{L}} \pi_{ik}\mathrm{tr}(B'_{w_i} P_{ik} B_{w_i}),$$

as in (20), thereby concluding the proof. ■

To obtain a feasible controller, we can solve the following multi-objective optimization problem, akin to (21):

$$\inf \{g(\gamma_2,\, \nu) : \text{ subject to (22) – (24)}\}, \tag{31}$$

with $g(\gamma_2,\, \nu) \triangleq b\nu + a\gamma_2$.

IV. Example

The synthesis conditions are implemented within Matlab (version 2015A) using a DELL computer equipped with an Intel Core i7 processor, 16GB of RAM. The LMIs are programmed using the parser Yalmip [17] and solved using the Mosek toolbox [1]. Regarding the scalars ζ_{ik} necessary to test the proposed synthesis conditions, we fixed $\zeta_{ik} = \zeta$ and performed a linear search on ζ considering 40 values logarithmically spaced in the interval $[10^{-5},\, 10^2]$.

This experiment is devoted to a practical example involving two coupled electrical machines, borrowed from [18], [26], [8]. The goal is to design a controller for this nonlinear model using the same parameters adopted in [8]. Furthermore, two scenarios are investigated: (1) complete observation and (2) partial observation. The state transition rates matrices $[\lambda_{ij}]$, $\left[\hat{\lambda}^i_{j\ell}\right]$ and probability matrix $\left[\alpha^k_{j\ell}\right]$ are given by

$$[\lambda_{ij}] = \begin{bmatrix} -3.0 & 0.5 & 2.5 & 0.0 \\ 0.5 & -3.0 & 0.0 & 2.5 \\ 2.5 & 0.0 & -3.0 & 0.5 \\ 0.0 & 2.5 & 0.5 & -3.0 \end{bmatrix},$$

$$(1) : \hat{\lambda}^i_{j\ell} = 0,\ \forall i \in \mathbb{K},\ \alpha^k_{jj} = 1,\ \forall k \in \mathbb{M},$$

$$(2) : \left[\hat{\lambda}^i_{j\ell}\right] = \tau_i \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix},\ \left[\alpha^1_{j\ell}\right] = \begin{bmatrix} 3/4 & 1/4 \\ 2/5 & 3/5 \\ 5/8 & 3/8 \\ 1/3 & 2/3 \end{bmatrix},\ \left[\alpha^2_{j\ell}\right] = \begin{bmatrix} I_2 \\ I_2 \end{bmatrix},$$

where $\tau_i = \{2,\, 1,\, 0.5,\, 0.1\}$. Consider also the sets $\mathbb{K} = \{1,\, 2,\, 3,\, 4\}$, $\mathbb{M} = \{1,\, 2,\, 3,\, 4\}$ for scenario (1) and $\mathbb{M} = \{1,\, 2\}$ for scenario (2). The nonlinear dynamic system is given by[1]

$$\ddot{\varphi}_1(t) = -\varphi_1(t) - \mu_i^{(1)} \sin(\varphi_1(t) - \varphi_2(t))$$
$$\ddot{\varphi}_2(t) = -100\varphi_2(t) - 0.1\dot{\varphi}_2(t) + \mu_i^{(2)} \sin(\varphi_1(t) - \varphi_2(t)),$$

[1]References [18] and [6] are recommended for the reader interested in more details about these parameters.

which can be presented as in (5) with the matrices $\forall i \in \mathbb{K}$,

$$A_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -100 & 0 & -0.1 \end{bmatrix},\ B_{p_i} = \begin{bmatrix} 0 \\ 0 \\ \mu_i^{(1)} \\ -\mu_i^{(2)} \end{bmatrix},$$

$$B'_{u_i} = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix},\ B_{w_i} = I_4,\ C_{q_i} = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix},$$

$$C_{z_i} = \begin{bmatrix} I_4 \\ 0_{1\times 4} \end{bmatrix},\ D_{zu_i} = \begin{bmatrix} 0_{4\times 1} \\ 1 \end{bmatrix},\ D_{zp_i} = 0_{5\times 1}.$$

with, $\mu^{(1)} = \mu^{(2)} = \{-0.2,\, 0,\, 0,\, 0.2\}$, and state vector $x(t)' = [\varphi_1(t)\ \varphi_2(t)\ \dot{\varphi}_1(t)\ \dot{\varphi}_2(t)]$. As discussed in [8], this system is unstable in open-loop. Next, the optimization problem (31) is employed to design stabilizing controllers for both scenarios.

Case 1: Adopting $\zeta = 50$, $a = 5$ and $b = 12$, provides $\sqrt{\gamma_2} = 9.9135$, $\kappa = 1.0693$ and the gains

$$\begin{bmatrix} K_1 \\ \hline K_2 \\ \hline K_3 \\ \hline K_4 \end{bmatrix} = \begin{bmatrix} -0.6245 & -1.4227 & -1.5203 & -0.9050 \\ \hline -0.5769 & -1.3828 & -1.4916 & -0.9063 \\ \hline -0.6008 & -1.3978 & -1.5093 & -0.9058 \\ \hline -0.5558 & -1.3993 & -1.4902 & -0.9070 \end{bmatrix}.$$

To illustrate this result, we conducted a Monte Carlo simulation with 2500 sample paths, each lasting 15 seconds, and the initial condition $x(0)' = [1\ 0\ 0\ 0]$. The closed-loop system norm is $\|z\|_2 = 1.3899$, and time simulations are only presented for **Case 2** (to save space).

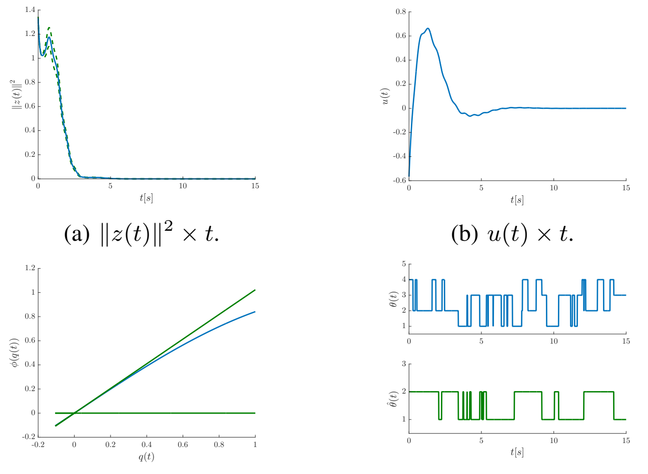
Case 2 Adopting $\zeta = 14$, $a = 5$ and $b = 10$, the optimization problem (31) gives $\sqrt{\gamma_2} = 9.9100$, $\kappa = 1.0230$ and

$$\begin{bmatrix} K_1 \\ \hline K_2 \end{bmatrix} = \begin{bmatrix} -0.5881 & -1.3963 & -1.5005 & -0.9057 \\ \hline -0.5664 & -1.3893 & -1.4897 & -0.9063 \end{bmatrix}.$$

Fig. 1a presents the mean-square performance output along with its standard deviation, indicating the stochastic stability of the electrical machines with random coupling ($\|z\|_2 = 1.3895$), utilizing the same parameters as those specified in **Case 1**. Fig. 1b depicts the behavior of the control input in the last run among all 2500 rounds. As for Fig. 1c, note that the nonlinear function belongs to the sector $[0,\, \kappa]$, and Fig. 1d presents the last round of the Markov and detector processes.

V. Conclusion

This paper presented a multi-objective optimization approach for robustly stabilizing continuous-time Markov jump Lur'e systems. Considering a more realistic scenario where only an estimate of the Markov mode is available (provided by a detector), synthesis conditions incorporating both H_2 performance and the amplitudes of the sectors associated with nonlinearities are formulated in terms of LMIs. A numerical example (based on a practical system) demonstrating the results under the partial observation of the Markov mode was presented to illustrate the effectiveness of the proposed approach.



(a) $\|z(t)\|^2 \times t$. (b) $u(t) \times t$. (c) $\phi(q(t)) \times q(t)$. (d) $\theta(t) \times t$ and $\hat{\theta}(t) \times t$.

Fig. 1: 1a Mean and standard deviation of $\|z(t)\|^2$ evaluated at all times t for **Case 2**. 1b Latter run of $u(t)$ evaluated at all times t. 1c Sector condition: in green the sector $[0, \kappa]$, and in light blue the latter mapping $q(t) \mapsto \phi(q(t))$. 1d Latter run of Markov process $\theta(t)$ and the detector's process $\hat{\theta}(t)$.

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