2024 IEEE 63rd Conference on Decision and Control (CDC)
December 16-19, 2024. MiCo, Milan, Italy

# Linear port-Hamiltonian boundary control models and their equivalence

Arjan van der Schaft[1] and Bernhard Maschke[2]

***Abstract*— Systems of partial differential equations often admit different Hamiltonian representations, leading to different boundary variables that are either power or energy conjugate. It is shown that any linear infinite-dimensional Hamiltonian system can be transformed into one with constant symplectic matrix. Alternatively, any passive linear Hamiltonian system can be converted into one with constant energy storage matrix. The consideration of energy boundary variables points towards a new approach to control by interconnection. All this is illustrated on the example of the elastic rod.**

## I. INTRODUCTION

Standard port-Hamiltonian modeling of boundary control systems, as initiated in [15], starts from the *Stokes-Dirac structure* associated to a formally skew-adjoint matrix differential operator. This leads to the consideration of *power* conjugate boundary variables representing the interaction with the surroundings or with a controller system. Very recently in [4], see also [16], [5], it was demonstrated how this can be extended to *energy* conjugate boundary variables corresponding to a *Stokes-Lagrange structure* associated to the energy storage of the system. This extension from power port variables to energy port variables parallels recent developments in the theory of finite-dimensional port-Hamiltonian systems, where the graph of the gradient of a Hamiltonian function is replaced by a general Lagrangian subspace or submanifold [12], [13].

The present paper is continuing this line of research. Its contributions can be summarized as follows.

First, the results of [4] are specialized to a standard class of linear infinite-dimensional Hamiltonian systems, thereby obtaining more explicit expressions and providing additional insights.

Second, since the *same* partial differential equations may give rise to *different* Hamiltonian systems with *different* sets of boundary variables, this leads to the question of *equivalence* of port-Hamiltonian boundary control models, and how to transform one into the other. A main result of the present paper is that any linear infinite-dimensional Hamiltonian system can be transformed into one with a constant *symplectic* matrix; thus having only energy boundary variables. In particular, such formulations admit a standard Lagrangian description. Conversely, any passive linear infinite-dimensional Hamiltonian system can be transformed into one with energy storage described by a *constant* positive-definite matrix; thus having only power boundary variables.

[1]Arjan van der Schaft is with Bernoulli Institute and Jan C. Willems Center for Systems and Control, University of Groningen, 9700 AK Groningen, the Netherlands, a.j.van.der.schaft@rug.nl

[2]Bernhard Maschke is with LAGEPP, Université Claude Bernard Lyon-1, Villeurbanne, France, bernhard.maschke@univ-lyon1.fr

Third, the inclusion of energy boundary variables gives rise to new schemes for 'control by interconnection', as compared to the standard approach based on power variables. Some initial ideas are presented.

## II. MOTIVATING EXAMPLE

Consider a linear elastic rod, on a 1-dimensional spatial domain $[a, b]$. The dynamics is described by the partial differential equation

$$\mu \frac{\partial^2 u}{\partial t^2}(z,t) = -ku(z,t) + \kappa \frac{\partial^2 u}{\partial z^2}(z,t), \quad z \in [a,b],\, t \in \mathbb{R} \tag{1}$$

Here $u(z,t)$, $z \in [a,b]$, $t \in \mathbb{R}$, is the *displacement* of the rod, $\mu$ is the mass density, $\kappa$ the elasticity modulus, and $k$ the spring constant.

In order to provide a Hamiltonian formulation one starts with the expression for the energy at any time $t$, given as

$$\int_a^b \frac{1}{2\mu} p^2(z,t) + \frac{1}{2}\kappa \left(\frac{\partial u}{\partial z}\right)^2 (z,t) + \frac{1}{2} k u^2(z,t)\, dz, \tag{2}$$

where $\frac{\partial u}{\partial z}(z,t)$ is the *strain*, and $p(z,t) = \mu \frac{\partial u}{\partial t}(z,t)$ is the *momentum*. The first term in (2) represents the kinetic energy, the second the potential energy due to deformation of the rod (structural elastic energy), and the third term the potential energy due to stretching (elastic energy).

The dynamics (1) can be formulated as an infinite-dimensional Hamiltonian system in at least two ways.

I. Denote $\epsilon(z,t) := \frac{\partial u}{\partial z}(z,t)$, and define the Hamiltonian $\mathcal{H}_I$ as the energy expressed in the state variables $u, \epsilon, p$, all considered as functions of the spatial variable $z \in [a,b]$:

$$\mathcal{H}_I(u,\epsilon,p) := \int_a^b \frac{1}{2\mu} p^2(z) + \frac{1}{2}\kappa\epsilon^2(z) + \frac{1}{2} k u^2(z)\, dz \tag{3}$$

Then the dynamics of (1) can be written as

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ \epsilon \\ p \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & \frac{\partial}{\partial z} \\ -1 & \frac{\partial}{\partial z} & 0 \end{bmatrix}}_{\mathcal{J}_I(\frac{\partial}{\partial z})} \begin{bmatrix} \frac{\delta \mathcal{H}_\mathcal{I}}{\delta u} \\ \frac{\delta \mathcal{H}_\mathcal{I}}{\delta \epsilon} \\ \frac{\delta \mathcal{H}_\mathcal{I}}{\delta p} \end{bmatrix} \tag{4}$$

Here the last vector is the *variational derivative* of $\mathcal{H}_\mathcal{I}$, which is the vector-valued function defined by the requirement that for all functions $\nu(z) = \begin{bmatrix} \nu_u(z) & \nu_\epsilon(z) & \nu_p(z) \end{bmatrix}$ vanishing at the boundary points $a, b$

$$\mathcal{H}_I(u + \varepsilon\nu) = \mathcal{H}_I(u) + \varepsilon \int_a^b \nu(z) \begin{bmatrix} \frac{\delta \mathcal{H}_I}{\delta u}(z) \\ \frac{\delta \mathcal{H}_I}{\delta \epsilon}(z) \\ \frac{\delta \mathcal{H}_I}{\delta p}(z) \end{bmatrix} dz \; + \; O(\varepsilon^2) \tag{5}$$

for small $\varepsilon$. The variational derivative of $\mathcal{H}_I$ is simply given as the vector of partial derivatives of the *Hamiltonian density*

$$H_I(u,\epsilon,p)(z) := \frac{1}{2\mu}p^2(z) + \frac{1}{2}\kappa\epsilon^2(z) + \frac{1}{2}ku^2(z), \quad (6)$$

that is

$$\begin{bmatrix} \frac{\delta\mathcal{H}_\mathcal{I}}{\delta u} \\ \frac{\delta\mathcal{H}_\mathcal{I}}{\delta \epsilon} \\ \frac{\delta\mathcal{H}_\mathcal{I}}{\delta p} \end{bmatrix} = \underbrace{\begin{bmatrix} k & 0 & 0 \\ 0 & \kappa & 0 \\ 0 & 0 & \frac{1}{\mu} \end{bmatrix}}_{Q_I} \begin{bmatrix} u \\ \epsilon \\ p \end{bmatrix} \quad (7)$$

On the other hand, *another* Hamiltonian formulation of (1) is obtained by considering the energy (2) as a function of $u, p$, *together* with their spatial derivatives up to arbitrary order (in this case only the first-order spatial derivative $\frac{\partial u}{\partial z}$). This leads to the alternative Hamiltonian

$$\mathcal{H}_{II}(u,p) := \int_a^b \frac{1}{2\mu}p^2(z) + \frac{1}{2}\kappa\left(\frac{\partial u}{\partial z}\right)^2(z) + \frac{1}{2}ku^2(z)\; dz \quad (8)$$

The computation of the variational derivative of $\mathcal{H}_{II}$ is more involved. In fact, the variational derivative of the middle term

$$\int_a^b \frac{1}{2}\kappa\left(\frac{\partial u}{\partial z}\right)^2(z)dz \quad (9)$$

is computed by noting that for any $\nu_u$

$$\int_a^b \frac{1}{2}\kappa\bigg(\frac{\partial(u+\varepsilon\nu_u)}{\partial z}\bigg)^2 - \frac{1}{2}\kappa\bigg(\frac{\partial u}{\partial z}\bigg)^2 dz = \varepsilon\int_a^b \kappa\frac{\partial\nu_u}{\partial z}\frac{\partial u}{\partial z}dz \quad (10)$$

up to terms $O(\varepsilon^2)$. Then integration by parts of the last integral, since $\nu_u$ is zero at the boundary points $a, b$, yields

$$\int_a^b -\kappa\nu_u(z)\frac{\partial^2 u}{\partial z^2}(z)\; dz \quad (11)$$

Therefore the variational derivative of $\mathcal{H}_{II}(u,p)$ is given as

$$\begin{bmatrix} \frac{\delta\mathcal{H}_{II}}{\delta u} \\ \frac{\delta\mathcal{H}_{II}}{\delta p} \end{bmatrix} = \begin{bmatrix} k-\kappa\frac{\partial^2}{\partial z^2} & 0 \\ 0 & \frac{1}{\mu} \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} \quad (12)$$

This results in the following alternative Hamiltonian formulation of the elastic rod:

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ p \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\mathcal{J}_{II}} \underbrace{\begin{bmatrix} k-\kappa\frac{\partial^2}{\partial z^2} & 0 \\ 0 & \frac{1}{\mu} \end{bmatrix}}_{Q_{II}(\frac{\partial}{\partial z})} \begin{bmatrix} u \\ p \end{bmatrix} \quad (13)$$

Here we recognize $\mathcal{J}_{II}$ as the (inverse of the) standard *symplectic matrix*. This second Hamiltonian formulation is also referred to as the *symplectic* formulation of (1).

*Remark 2.1:* The symplectic formulation of the elastic rod has the special feature that it admits as well a *Lagrangian* representation, and thus can be derived from a *variational principle*. Using shorthand notation $\frac{\partial u}{\partial z} =: u_z$ and $\frac{\partial u}{\partial t} =: u_t$, one defines the *Lagrangian density*

$$L(u,u_z,u_t)(z,t) := \frac{1}{2}\mu u_t^2(z,t) - \frac{1}{2}ku^2(z,t) - \frac{1}{2}\kappa u_z^2(z,t) \quad (14)$$

Then the dynamics (1) also arises as the *Euler-Lagrange equation* of the variational problem (Hamilton's principle)

$$0 = \delta\int_{t_0}^{t_f}\int_a^b L(u,u_z,u_t)(z,t)dzdt, \quad (15)$$

where the variations are taken over all (smooth) functions $\delta u : [a,b]\times[t_0,t_f] \to \mathbb{R}$, which are fixed on the boundary of the rectangle $[a,b]\times[t_0,t_f]$. In fact, the Euler-Lagrange equation for (15) is given by the partial differential equation

$$\frac{\partial L}{\partial u}(u,u_z,u_t) - \frac{d}{dt}\frac{\partial L}{\partial u_t}(u,u_z,u_t) - \frac{d}{dz}\frac{\partial L}{\partial u_z}(u,u_z,u_t) = 0, \quad (16)$$

which is seen to reduce to (1).

Both formulations (4) + (7) and (13) of the elastic rod are *Hamiltonian systems*; defined for the purpose of this paper as follows.

*Definition 2.2 (Infinite-dimensional Hamiltonian system):* A linear *infinite-dimensional Hamiltonian system* is a system of partial differential equations of the form

$$\frac{\partial x}{\partial t}(z,t) = \mathcal{J}(\frac{\partial}{\partial z})\,Q(\frac{\partial}{\partial z})x(z,t),\; z\in[a,b],\; x(z,t)\in\mathbb{R}^n, \quad (17)$$

where the $n\times n$ matrix differential operator

$$\mathcal{J}(\frac{\partial}{\partial z}) = -\mathcal{J}^\top(-\frac{\partial}{\partial z}) \quad (18)$$

is *formally skew-adjoint* (or constant skew-symmetric), and

$$Q(\frac{\partial}{\partial z}) = Q^\top(-\frac{\partial}{\partial z}) \quad (19)$$

is an $n\times n$ *formally self-adjoint* matrix differential operator (or constant symmetric matrix).

Although both (4) + (7) and (13) are Hamiltonian formulations of the *same* dynamics (1), and their Hamiltonians $\mathcal{H}_I$ and $\mathcal{H}_{II}$ express the *same* energy (2), we will see that they lead to *different* sets of *boundary variables*. In fact, as detailed in Section IV, the boundary variables are obtained by 'integration by parts' applied to the matrix differential operators $\mathcal{J}(\frac{\partial}{\partial z})$ (formally skew-adjoint) and $Q(\frac{\partial}{\partial z})$ (formally self-adjoint). (As such, this is different from a finite-dimensional model, where the input and output equations are specified a priori.) It turns out that the boundary variables arising from $\mathcal{J}$ are *power* boundary variables (they come in pairs, with their product having dimension of power), while the boundary variables arising from $Q$ are *energy* boundary variables (again appearing in pairs, but now with product having dimension of energy).

## III. TWO-VARIABLE POLYNOMIAL CALCULUS

An efficient tool to obtain the boundary variables based on $\mathcal{J}(\frac{\partial}{\partial z})$ and $Q(\frac{\partial}{\partial z})$ is *two-variable polynomial matrix calculus*. Based on [18], [14] we first collect relevant results on one-variable and two-variable polynomial matrices and the matrix differential operators associated with them. First recall that a *matrix differential operator* is given by an expression

$$A(\frac{d}{dz}) = \sum_{k=0}^N A_k\frac{d^k}{dz^k} \quad (20)$$

for $p \times q$ matrices $A_k$, where $z \in [a, b]$ is a scalar variable. Associated with $A(\frac{d}{dz})$ is the $p \times q$ polynomial matrix $A(s) = \sum_{k=0}^{N} A_k s^k$ in the indeterminate $s$.

Next we consider *two-variable* polynomial matrices and their corresponding bilinear differential operators.

*Definition 3.1:* A $p \times q$ *two-variable polynomial matrix* $\Phi(\zeta, \eta)$ in the two indeterminates $\zeta$ and $\eta$ is given by an expression of the form

$$\Phi(\zeta, \eta) := \sum_{k,l=0}^{M} \Phi_{k,l} \zeta^k \eta^l, \tag{21}$$

for some nonnegative integer $M$ and $p \times q$ matrices $\Phi_{k,l} \in \mathbb{R}^{p \times q}$. It is equivalently defined by its *coefficient matrix* $\widetilde{\Phi}$ which is the $Mp \times Mq$ matrix whose $(k, l)$-th block is the matrix $\Phi_{k,l}$, $k, l = 0, \ldots, M$,.

The *bilinear differential operator* $D_\Phi$ associated with the two-variable polynomial matrix $\Phi(\zeta, \eta)$ in (21) is

$$D_\Phi(v, w)(z) = \sum_{k,l=0}^{M} \left[ \frac{d^k}{dz^k} v(z) \right]^\top \Phi_{k,l} \frac{d^l}{dz^l} w(z), \tag{22}$$

acting on $M$-differentiable functions $v : \mathbb{R} \to \mathbb{R}^p, w : \mathbb{R} \to \mathbb{R}^q$. This operator defines, in turn, the following *bilinear functional* on the functions on the domain $[a,\, b]$ :

$$\mathcal{D}_\Phi(v, w) := \int_a^b D_\Phi(v, w)(z)\, dz \tag{23}$$

A $p \times p$ two-variable polynomial matrix $\Phi(\zeta, \eta)$ is called *symmetric* if $\Phi(\zeta, \eta) = \Phi^\top(\eta, \zeta)$, or equivalently its coefficient matrix $\widetilde{\Phi}$ is symmetric. A symmetric two-variable polynomial matrix $\Phi(\zeta, \eta)$ defines the *quadratic differential operator* $D_\Phi(v, v)(z)$. Furthermore, the expression

$$\mathfrak{D}_\Phi(v) := \frac{1}{2} \int_a^b D_\Phi(v, v)(z) dz \tag{24}$$

is the corresponding *quadratic functional*. Repeated integration by parts for functions with support included in $(a, b)$ yields

$$\mathfrak{D}_\Phi(v + \varepsilon \delta v) - \mathfrak{D}_\Phi(v) = \int_a^b \delta v^\top(z)\, Q(\frac{d}{dz})(v)(z) dz \tag{25}$$

up to $O(\varepsilon^2)$, where $Q(s) := \Phi(-s, s)$. Thus the variational derivative of $\mathfrak{D}_\Phi(v)$ is the function $Q(\frac{d}{dz})v$. Since $\Phi(\zeta, \eta) = \Phi^\top(\eta, \zeta)$ one has $Q(s) = Q^\top(-s)$, i.e., the matrix differential operator $Q(\frac{d}{dz})$ is formally self-adjoint.

Two-variable polynomial matrices can be used to express *integration by parts* in a compact way [14, Section 2.2].

*Lemma 3.2:* The derivative of the bilinear differential operator $D_\Psi$ associated with the $p \times q$ two-variable polynomial matrix $\Psi(\zeta, \eta)$ is the bilinear differential operator

$$D_\Phi(v, w)(z) = \frac{d}{dz} \left( D_\Psi(v, w) \right)(z)\,, \tag{26}$$

associated with the two-variable polynomial matrix

$$\Phi(\zeta, \eta) = (\zeta + \eta) \Psi(\zeta, \eta) \tag{27}$$

Conversely, a bilinear differential operator $D_\Phi(v, w)(z)$ is the derivative of another bilinear differential operator if and only if the two-variable polynomial matrix $\Phi(\zeta, \eta)$ is divisible by $(\zeta + \eta)$, i.e., (27) holds for some $\Psi(\zeta, \eta)$).
In integral form, one derives from (26) the expression

$$\mathcal{D}_\Phi(v, w) = D_\Psi(v, w)(b) - D_\Psi(v, w)(a) =: [D_\Psi(v, w)(z)]_a^b \tag{28}$$

Finally, we will make use of the following observation concerning *factorization* of two-variable polynomial matrices into one-variable polynomial matrices [18]. Consider the $p \times q$ two-variable polynomial matrix $\Phi(\zeta, \eta)$ with $Mp \times Mq$ coefficient matrix $\widetilde{\Phi}$. Consider any factorization $\widetilde{\Phi} = \widetilde{X}^\top \widetilde{Y}$ of $\widetilde{\Phi}$, where $\widetilde{X}$ is a $k \times Mp$ matrix and $\widetilde{Y}$ is a $k \times Mq$ matrix for some nonnegative integer $k$

$$\widetilde{X} = \begin{bmatrix} X_0 & X_1 & \cdots & X_M \end{bmatrix},\ \widetilde{Y} = \begin{bmatrix} Y_0 & Y_1 & \cdots & Y_M \end{bmatrix} \tag{29}$$

Define the $k \times p$, respectively $k \times q$, one-variable polynomial matrices

$$\begin{aligned} X(s) &:= X_0 + X_1 s + \cdots + X_M s^M \\ Y(s) &:= Y_0 + Y_1 s + \cdots + Y_M s^M \end{aligned} \tag{30}$$

Then it is immediately verified that $\Phi(\zeta, \eta) = X^\top(\zeta) Y(\eta)$.

## IV. PORT HAMILTONIAN BOUNDARY CONTROL SYSTEMS

Consider a Hamiltonian system (17), with $\mathcal{J}(\frac{\partial}{\partial z})$ a formally *skew-adjoint*, and $Q(\frac{\partial}{\partial z})$ a formally *self-adjoint* matrix differential operator. Equivalently, the corresponding polynomial matrices satisfy

$$\mathcal{J}(s) = -\mathcal{J}^\top(-s), \quad Q(s) = Q^\top(-s) \tag{31}$$

*Boundary variables* are now introduced as follows. First, let us start with $\mathcal{J}(s)$. Since $\mathcal{J}(s) = -\mathcal{J}^\top(-s)$ the two-variable polynomial expression $\mathcal{J}^\top(\zeta) + \mathcal{J}(\eta)$ is zero for $\zeta + \eta = 0$. Therefore

$$\mathcal{J}^\top(\zeta) + \mathcal{J}(\eta) = (\zeta + \eta)\, \Pi(\zeta, \eta) \tag{32}$$

for some two-variable symmetric polynomial matrix $\Pi(\zeta, \eta)$. Using factorization of two-variable polynomial matrices (see the previous section) it follows that

$$\Pi(\zeta, \eta) = Z^\top(\zeta)\, \Sigma\, Z(\eta) \tag{33}$$

for some polynomial matrix $Z(s)$ and invertible symmetric matrix $\Sigma$. In most cases of interest, see e.g. [15], $\Sigma$ has as many positive as negative eigenvalues[1], in which case we can take $\Sigma$ to be in the canonical form

$$\Sigma = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \tag{34}$$

Partition accordingly $Z(s) = \begin{bmatrix} Z_f(s) \\ Z_e(s) \end{bmatrix}$. Then the *boundary variables* corresponding to $\mathcal{J}(\frac{\partial}{\partial z})$ are defined as

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} := \begin{bmatrix} Z_f(\frac{\partial}{\partial z}) \\ Z_e(\frac{\partial}{\partial z}) \end{bmatrix} Q(\frac{\partial}{\partial z}) x \tag{35}$$

[1]If this is *not* the case we have to utilize the more general theory developed in [3].

evaluated at the boundary points $a, b$ of the spatial domain $[a, b]$. We obtain the following power balance.

*Proposition 4.1:* Consider the Hamiltonian system (17), where $Q(\frac{\partial}{\partial z})$ is a *constant* matrix $Q$, defining the Hamiltonian

$$\mathcal{H}(x) := \frac{1}{2}\int_a^b x^\top(z) Q x(z) dz \tag{36}$$

Then (17) with boundary variables (35) satisfies

$$\frac{d}{dt}\mathcal{H} = e_\partial(b)^\top f_\partial(b) - e_\partial(a)^\top f_\partial(a) \tag{37}$$

*Proof:* First substitute $\frac{\partial x}{\partial t} = \mathcal{J}(\frac{\partial}{\partial z}) Q(\frac{\partial}{\partial z}) x$ into

$$\frac{d}{dt}\mathcal{H} = \frac{1}{2}\int_a^b \left(\frac{\partial x}{\partial t}(z)\right)^\top Q x(z) + x^\top(z) Q \frac{\partial x}{\partial t}(z) dz \tag{38}$$

Second, notice that (32) expresses the following integration by parts identity for $q(z) := Qx(z)$

$$\begin{aligned} &\textstyle\int_a^b \left(\mathcal{J}(\frac{\partial}{\partial z}(z) q(z)\right)^\top q(z) + q^\top(z)\left(\mathcal{J}(\frac{\partial}{\partial z} q(z)\right) dz \\ &= \textstyle\int_a^b \frac{\partial}{\partial z}\left[(Z(\frac{\partial}{\partial z}) q(z))^\top \Sigma\, Z(\frac{\partial}{\partial z}) q(z))\right] dz \\ &= 2\left[e_\partial(z)^\top f_\partial(z)\right]_a^b, \end{aligned} \tag{39}$$

where for the last equality we substituted (35). Taken together this results in (37). ∎

Since $\mathcal{H}$ has dimension of energy, the product of the vectors $f_\partial, e_\partial$ has dimension of *power*. Therefore the elements of $f_\partial, e_\partial$ define *power boundary variables*.

In case $Q(\frac{\partial}{\partial z})$ is a true matrix differential operator (not equal to a constant matrix), other boundary variables arise. Since $Q(s) = Q^\top(-s)$ the two-variable polynomial expression $Q(\eta) - Q^\top(\zeta)$ is zero for $\zeta + \eta = 0$. Therefore

$$Q(\eta) - Q^\top(\zeta) = (\zeta + \eta)\,\Gamma(\zeta, \eta) \tag{40}$$

for some two-variable polynomial matrix $\Gamma(\zeta, \eta)$, which is skew-symmetric by skew-symmetry of $Q(\eta) - Q^\top(\zeta)$. Hence, one factorizes

$$\Gamma(\zeta, \eta) = G^\top(\zeta)\,\Delta\, G(\eta) \tag{41}$$

for some polynomial matrix $G(s)$ and invertible *skew-symmetric* matrix $\Delta$, which we can take to be in the canonical form

$$\Delta = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{42}$$

Partitioning accordingly $G(s) = \begin{bmatrix} G_\chi(s) \\ G_\epsilon(s) \end{bmatrix}$, one defines the boundary variables

$$\begin{bmatrix} \chi_\partial \\ \epsilon_\partial \end{bmatrix} := \begin{bmatrix} G_\chi(\frac{\partial}{\partial z}) \\ G_\epsilon(\frac{\partial}{\partial z}) \end{bmatrix} x, \tag{43}$$

to be evaluated at the boundary points $a, b$. These boundary variables are of a *different* nature than the ones defined in (35), and lead to a *different* form of power balance. First, one needs to extend the definition of the Hamiltonian $\mathcal{H}$ as compared with the expression (36). In fact, $\mathcal{H}$ is defined as the quadratic functional, cf. (24), corresponding to the two-variable symmetric polynomial matrix

$$\begin{aligned} \mathfrak{H}(\zeta, \eta) &:= \tfrac{1}{2}\left(Q(\eta) + Q^\top(\zeta)\right) \\ &+ \tfrac{1}{2}(\zeta + \eta)\left(G_\chi^\top(\zeta) G_\epsilon(\eta) + G_\epsilon^\top(\zeta) G_\chi(\eta)\right) \end{aligned} \tag{44}$$

Explicitly,

$$\mathcal{H}(x) = \frac{1}{2}\int_a^b x^\top Q(\frac{\partial}{\partial z}) x\, dz + \frac{1}{2}\left[G_\chi^\top(\frac{\partial}{\partial z}) x \cdot G_\epsilon(\frac{\partial}{\partial z}) x\right]_a^b \tag{45}$$

Note that since $\mathfrak{H}(-s, s) = Q(s)$, the variational derivative of $\mathcal{H}(x)$ is the function $Q(\frac{\partial}{\partial z})x$, as required. One obtains the following power balance, extending (37).

*Theorem 4.2:* Consider the Hamiltonian system (17) with Hamiltonian $\mathcal{H}$ defined by (45). Then

$$\frac{d}{dt}\mathcal{H} = \left[e_\partial^\top f_\partial + \epsilon_\partial^\top \frac{\partial}{\partial t}\chi_\partial\right]_a^b, \tag{46}$$

where $\chi_\partial, \epsilon_\partial$ are the boundary variables defined in (43), and $f_\partial, e_\partial$ are the boundary variables as defined in (35).

*Proof:* By the definition of $\mathcal{H}$ in (45)

$$\begin{aligned} &\tfrac{d}{dt}\mathcal{H} = \tfrac{1}{2}\textstyle\int_a^b \left(x^\top Q(\frac{\partial}{\partial z})\frac{\partial x}{\partial t} + \frac{\partial x}{\partial t}^\top Q(\frac{\partial}{\partial z}) x\right) dz + \\ &\tfrac{1}{2}\left[\left(G_\chi(\tfrac{\partial}{\partial z}) x\right)^\top \cdot G_\epsilon(\tfrac{\partial}{\partial z})\tfrac{\partial x}{\partial t} + \left(G_\epsilon(\tfrac{\partial}{\partial z}) x\right)^\top \cdot G_\chi(\tfrac{\partial}{\partial z})\tfrac{\partial x}{\partial t}\right]_a^b \end{aligned} \tag{47}$$

Furthermore, using the 'integration by parts' properties of the matrix differential operator $Q(\frac{\partial}{\partial z})$, cf. (40), (41), (42), one obtains

$$\begin{aligned} &x^\top Q(\tfrac{\partial}{\partial z})\tfrac{\partial x}{\partial t} = \tfrac{\partial x}{\partial t}^\top Q(\tfrac{\partial}{\partial z}) x + \\ &\tfrac{\partial}{\partial z}\left((G_\chi(\tfrac{\partial}{\partial z}) x)^\top \cdot G_\epsilon(\tfrac{\partial}{\partial z})\tfrac{\partial x}{\partial t} - (G_\epsilon(\tfrac{\partial}{\partial z}) x)^\top \cdot G_\chi(\tfrac{\partial}{\partial z})\tfrac{\partial x}{\partial t}\right) \end{aligned} \tag{48}$$

Substituting into (47) then yields

$$\frac{d}{dt}\mathcal{H} = \int_a^b \frac{\partial x}{\partial t}^\top Q(\frac{\partial}{\partial z}) x\, dz + \left[G_\epsilon^\top(\frac{\partial}{\partial z}) x \cdot G_\chi(\frac{\partial}{\partial z})\frac{\partial x}{\partial t}\right]_a^b \tag{49}$$

Permuting in the last term spatial and time derivatives, and using the properties of the matrix differential operator $\mathcal{J}(\frac{\partial}{\partial z})$ leading to the power boundary variables $f_\partial, e_\partial$ (see the computation resulting in (37)), one finally obtains (46). ∎

Since $\frac{d}{dt}\mathcal{H}$ has dimension of power, the product of the variables $\chi_\partial$ and $\epsilon_\partial$ has dimension of *energy*. Thus $\chi_\partial, \epsilon_\partial$ are called *energy* boundary variables.

All this leads to the following definition.

*Definition 4.3 (pH boundary control system):* The linear *port-Hamiltonian boundary control system* corresponding to (17) is obtained by supplementing (17) with the *power* boundary variables (35) resulting from $\mathcal{J}(\frac{\partial}{\partial z})$, and the *energy* boundary variables (43) corresponding to $Q(\frac{\partial}{\partial z})$, all evaluated at the boundary points $a, b$. The Hamiltonian $\mathcal{H}$ is given by (45), and satisfies the power balance (46).

*Example 4.4:* Let us illustrate the developed theory on the two Hamiltonian formulations of the linear elastic rod discussed before in Section II. Consider $\mathcal{J}_I$ as given in the first Hamiltonian formulation (4) of the elastic rod. Then

$$\mathcal{J}_I^\top(\zeta)+\mathcal{J}_I(\eta) = (\zeta+\eta)\begin{bmatrix}0 & 0\\ 0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}}_{Z} \quad (50)$$

Hence the power boundary variables are given as

$$\begin{bmatrix}f_\partial\\ e_\partial\end{bmatrix} = \begin{bmatrix}0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\underbrace{\begin{bmatrix}k & 0 & 0\\ 0 & \kappa & 0\\ 0 & 0 & \frac{1}{\mu}\end{bmatrix}}_{Q}\begin{bmatrix}u\\ \varepsilon\\ p\end{bmatrix} = \begin{bmatrix}\frac{p}{\mu}\\ \kappa\varepsilon\end{bmatrix},$$

which are the *velocity* and the *stress*. On the other hand, in the formulation (13), $Q_{II}(s) = \begin{bmatrix}k-\kappa s^2 & 0\\ 0 & \frac{1}{\mu}\end{bmatrix}$. Hence

$$\begin{aligned} Q_{II}(\eta) - Q_{II}^\top(\zeta) &= \begin{bmatrix}\kappa(\zeta^2-\eta^2) & 0\\ 0 & 0\end{bmatrix}\\ &= (\zeta+\eta)\begin{bmatrix}\kappa(\zeta-\eta) & 0\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}1 & \kappa\zeta\\ 0 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ \kappa\eta & 0\end{bmatrix} \end{aligned} \quad (51)$$

Thus

$$\begin{bmatrix}\chi_\partial\\ \epsilon_\partial\end{bmatrix} = \begin{bmatrix}1 & 0\\ \kappa\frac{\partial}{\partial z} & 0\end{bmatrix}\begin{bmatrix}u\\ p\end{bmatrix} = \begin{bmatrix}u\\ \kappa\frac{\partial u}{\partial z}\end{bmatrix} \quad (52)$$

Note that the second boundary variable $\epsilon_\partial$ is equal to the *stress* $\kappa\frac{\partial u}{\partial z}$, and thus happens to be equal to the boundary variable $e_\partial$ in the first Hamiltonian formulation. On the other hand, the boundary variable $\chi_\partial = u$ is the *displacement*, in contrast with the boundary variable $f_\partial = \frac{p}{\mu}$ (*velocity*).

The resulting Hamiltonian $\mathcal{H}$ defined in (45) is computed by considering the two-variable polynomial matrix, cf. (44),

$$\begin{aligned} \mathfrak{H}(\zeta,\eta) &= \tfrac{1}{2}\begin{bmatrix}k-\kappa\eta^2 & 0\\ 0 & \frac{1}{\mu}\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}k-\kappa\zeta^2 & 0\\ 0 & \frac{1}{\mu}\end{bmatrix}\\ &+\tfrac{1}{2}(\zeta+\eta)(\begin{bmatrix}1\\ 0\end{bmatrix}\begin{bmatrix}\kappa\eta & 0\end{bmatrix} + \begin{bmatrix}\kappa\zeta\\ 0\end{bmatrix}\begin{bmatrix}1 & 0\end{bmatrix})\\ &= \tfrac{1}{2}\begin{bmatrix}2k-\kappa(\zeta^2+\eta^2) & 0\\ 0 & \frac{2}{\mu}\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}\kappa(\zeta+\eta)^2 & 0\\ 0 & 0\end{bmatrix}\\ &= \begin{bmatrix}k+\kappa\zeta\eta & 0\\ 0 & \frac{1}{\mu}\end{bmatrix} \end{aligned} \quad (53)$$

This yields the Hamiltonian $\mathcal{H}_{II}$ already given in (8).

## V. EQUIVALENCE OF HAMILTONIAN FORMULATIONS

The example of the linear elastic rod shows that there may be *different* Hamiltonian formulations of the same dynamics (17), leading to *different* sets of boundary variables, and thus to *different* port-Hamiltonian boundary control systems. This raises the question of *equivalence*, and the possibilities of *transforming* one Hamiltonian formulation into another.

Let us start from the skew-symmetric polynomial matrix $\mathcal{J}(s)$. Using a similar line of reasoning as in [17], but for the skew-symmetric instead of the symmetric case, one associates to $\mathcal{J}(s)$ the two-variable polynomial matrix

$$\Phi(\zeta,\eta) := \frac{1}{2}(J(\eta) - J^\top(\zeta)) \quad (54)$$

Since $\mathcal{J}(s) = -\mathcal{J}^\top(-s)$, it follows that $\Phi(-s,s) = J(s)$ and, furthermore, that $\Phi(\zeta,\eta)$ is skew-symmetric. This means that the coefficient matrix $\tilde{\Phi}$ is skew-symmetric, and hence for some polynomial matrix $W(s)$

$$\mathcal{J}(s) = \Phi(-s,s) = W(s)\widetilde{\mathcal{J}}W^\top(-s),\ \widetilde{\mathcal{J}} := \begin{bmatrix}0 & I_k\\ -I_k & 0\end{bmatrix}, \quad (55)$$

where $\widetilde{\mathcal{J}}$ is the standard symplectic matrix. It follows that the subspace of functions of the spatial variable $z$ that are in the image of the differential operator $W(\frac{\partial}{\partial z})$ is *invariant* under the dynamics. The dynamics *restricted* to this invariant subspace of functions $x = W(\frac{\partial}{\partial z})\tilde{x}$ is generated by

$$\dot{\tilde{x}} = \widetilde{\mathcal{J}}\,\widetilde{Q}(\frac{\partial}{\partial z})\tilde{x}, \quad (56)$$

with $\widetilde{Q}(s) := W^\top(-s)Q(s)W(s)$. Since $\widetilde{\mathcal{J}}$ is a constant symplectic matrix this is a *symplectic* Hamiltonian system as in the symplectic formulation (13) of the elastic rod.

Conversely, start from the symmetric polynomial matrix $Q(s)$, where we make the additional assumption (motivated by impedance passivity) that the Hermitian matrices $Q(i\omega)$ satisfy $Q(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. Then it is well-known from spectral factorization theory, see e.g. [7], that there exists a polynomial matrix $V(s)$ such that

$$Q(s) = V^\top(-s)V(s) \quad (57)$$

Now define the new state vector $\tilde{x} := V(\frac{\partial}{\partial z})x$. Then by substituting $Q(\frac{\partial}{\partial z}) = V^\top(-\frac{\partial}{\partial z})V(\frac{\partial}{\partial z})$ it follows that

$$\dot{\tilde{x}} = V(\frac{\partial}{\partial z})\mathcal{J}(\frac{\partial}{\partial z})V^\top(-\frac{\partial}{\partial z})\,V(\frac{\partial}{\partial z})x = \widetilde{\mathcal{J}}(\frac{\partial}{\partial z})\tilde{x}, \quad (58)$$

where $\widetilde{\mathcal{J}}(s) := V(s)\mathcal{J}(s)V^\top(-s)$ is the new $\mathcal{J}$-matrix. Summarizing we obtain

*Proposition 5.1:* Consider a Hamiltonian system (17). It can be always transformed into a Hamiltonian system with $\mathcal{J}$ equal to a symplectic matrix. In particular, any Hamiltonian system (17) can be derived from a variational principle (15). Conversely, if $Q(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, then the Hamiltonian system (17) can be transformed into a Hamiltonian system with $Q = I$.

*Example 5.2:* Consider the elastic rod (1) with its Hamiltonian formulations (4) and (13). Then

$$W(s) = \begin{bmatrix}1 & 0\\ s & 0\\ 0 & 1\end{bmatrix},\ V(s) = \begin{bmatrix}\sqrt{k}+\sqrt{\kappa}s & 0\\ 0 & \frac{1}{\sqrt{\mu}}\end{bmatrix} \quad (59)$$

Note that $\tilde{x} := V(\frac{\partial}{\partial z})x$ transforms (13) into a system with *two*-dimensional state function, with first component given by $\sqrt{k}u+\sqrt{\kappa}\frac{\partial u}{\partial z}$ (weighted combination of displacement and strain). Alternatively, the Hamiltonian formulation (4), with $Q$ scaled to the identity matrix, is obtained by considering

$$V'(s) = \begin{bmatrix} \sqrt{k} & 0 \\ \sqrt{\kappa} s & 0 \\ 0 & \frac{1}{\sqrt{\mu}} \end{bmatrix}, \tag{60}$$

still satisfying $V'^{\top}(-s)V'(s) = Q_{II}(s)$.

## VI. CONTROL BY INTERCONNECTION VIA ENERGY PORTS

Although control by interconnection of physical systems is typically performed through the *power ports* of the system, see e.g. [6], it can be done through the *energy ports* as well. In fact, traces of this can be already found in [9]. In the context of infinite-dimensional Hamiltonian systems (17) it can be illustrated as follows. Consider a Hamiltonian system (17) with energy boundary variables $\chi_\partial, \epsilon_\partial$ as defined in (43). For simplicity, denote $\chi_a := \chi_\partial(a), \chi_b := \chi_\partial(b)$ and $\epsilon_a := \epsilon_\partial(a), \epsilon_b := \epsilon_\partial(b)$. Now consider the feedback

$$\epsilon_b = -\frac{\partial P}{\partial \chi_b}(\chi_b) \tag{61}$$

for some function $P(\chi_b)$. Then the closed-loop port-Hamiltonian boundary control system can be seen to satisfy

$$\frac{d}{dt}\left(\mathcal{H}(x) + P(G_\chi(\frac{\partial}{\partial z})x(b))\right) = \left[e_\partial^\top f_\partial\right]_a^b - \epsilon_a^\top \frac{d}{dt}\chi_a \tag{62}$$

Thus we have effectively *shaped* the Hamiltonian $\mathcal{H}$ of the system by an additional (boundary) energy term $P(\chi_b)$.

Instead of the static feedback (61), we could also consider a controller system that is a finite-dimensional input-output Hamiltonian system with dissipation, as defined in [10], [11],

$$\begin{aligned} \dot{x}_c &= [J(x_c) - R(x_c)]\left(\frac{\partial H_c}{\partial x_c}(x_c) - \frac{\partial C^\top}{\partial x_c}(x_c)u_c\right), \\ y_c &= C(x_c), \end{aligned} \tag{63}$$

with controller state $x_c$, where the matrices $J(x_c), R(x_c)$ satisfy $J(x_c) = -J^\top(x_c), R(x_c) = R^\top(x_c) \geq 0$. Interconnection to the Hamiltonian system (17) via the energy port at boundary point $b$, using the interconnection

$$\epsilon_b = y_c, \quad u_c = \chi_b, \tag{64}$$

results in a closed-loop system with total Hamiltonian

$$\mathcal{H}(x) + H_c(x_c) - C(x_c)^\top G_\chi(\frac{\partial}{\partial z})x(b), \tag{65}$$

where $\mathcal{H}(x)$ is defined in (45).

Further research is needed to explore the scope of control by interconnection using energy ports; see [10] for initial results in the *finite-dimensional* case.

## VII. CONCLUDING REMARKS

Although we confined ourselves in this paper to the example of the elastic rod, it is obvious that the existence of different Hamiltonian representations occurs in other physical domains as well. For instance, the Hamiltonian formulations of the transmission line, as detailed in [11], are completely analogous to those of the elastic rod for $k = 0$. Furthermore, the derivation of different sets of power and energy port variables has already been explored in [1] for the formulation of beam models such as the Timoshenko and Euler-Bernoulli model. In this reference also use is made of the notion of *Stokes-Lagrange structures* as introduced in [4], see also [16]. (Stokes-Lagrange structures generalize formally self-adjoint matrix differential operators, just like Lagrangian subspaces generalize symmetric linear mappings.) Further research is needed for the (non-trivial) extension of the developed theory from one-dimensional spatial domains to *higher-dimensional* spatial domains, for example in case of Maxwell's equations on a bounded domain in $\mathbb{R}^3$. Finally, an important extension concerns the generalization from quadratic Hamiltonians as treated in the present paper to *general* Hamiltonians depending on the state variables and their spatial derivatives; see [8] for previous related work.

## REFERENCES


[1] A. Bendimerad-Hohl, D. Matignon, G. Haine, L. Lefèvre. "On implicit and explicit representations for 1D distributed port-Hamiltonian systems", Proc. 26th Int. Symposium on Mathematical Theory of Networks and Systems (MTNS2024), Cambridge, UK, 2024.

[2] D. Jeltsema and A.J. van der Schaft. "Lagrangian and Hamiltonian formulation of transmission line systems with boundary energy flow". *Rep. Mathematical Physics*, 63(1), 55–74, 2009.

[3] Y. Le Gorrec, H. Zwart, and B.M. Maschke. "Dirac structures and boundary control systems associated with skew-symmetric differential operators". *SIAM J. Cont. Optim.*, 44(5):1864–1892, 2005.

[4] B. Maschke and A. van der Schaft. "Linear boundary port-Hamiltonian systems with implicitly defined energy". ArXiv:2305.13772, submitted for publication.

[5] B. Maschke and A.J. van der Schaft. "Linear boundary port Hamiltonian systems defined on Lagrangian submanifolds". *IFAC-PapersOnLine*, 53(2):7734–7739, 2020.

[6] R. Ortega, A. van der Schaft, B. Maschke, G. Escobar. "Interconnection and damping-assignment passivity-based control of port-controlled Hamiltonian systems". *Automatica*, 38:585–596, 2002.

[7] A. C. M. Ran and L. Rodman. "Factorization of matrix polynomials with symmetries". *IMA Preprint Series*, no. 993, 1992.

[8] M. Schöberl and A. Siuka. "Jet bundle formulation of infinite-dimensional port-Hamiltonian systems using differential operators". *Automatica*, 50(2):607–613, 2014.

[9] A.J. van der Schaft. "Controlled invariance for Hamiltonian systems". *Math. Systems Theory*, 18:257–291, 1985.

[10] A.J. van der Schaft, "Interconnections of input-output Hamiltonian systems with dissipation", Proc. 55th IEEE Conf. Decision and Control, pp. 4686–4691, 2016.

[11] A. van der Schaft and D. Jeltsema. "Port-Hamiltonian systems theory: An introductory overview". *Foundations and Trends in Systems and Control*, 1(2-3):173–378, 2014.

[12] A. van der Schaft and B. Maschke. "Generalized port-Hamiltonian DAE systems". *Systems & Control Letters*, 121:31–37, 2018.

[13] A.J. van der Schaft and B. Maschke. "Dirac and Lagrange algebraic constraints in nonlinear port-Hamiltonian systems". *Vietnam Journal of Mathematics*, 48(4):929–939, 2020.

[14] A. van der Schaft and P. Rapisarda. "State maps from integration by parts". *SIAM J. Cont. Optim.*, 49(6):2415–2439, 2011.

[15] A.J. van der Schaft and B.M. Maschke. "Hamiltonian formulation of distributed parameter systems with boundary energy flow". *J. of Geometry and Physics*, 42:166–174, 2002.

[16] A.J. van der Schaft and B.M. Maschke. "Differential operator Dirac structures". *IFAC-PapersOnLine*, 54(19):198–203, 2021. 7th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control (LHMNC 2021).

[17] H.L. Trentelman and P. Rapisarda, "New algorithms for polynomial $J$-spectral factorization", *Math. Cont., Signals Systems*, 12:24–61, 1999.

[18] J. C. Willems and H. L. Trentelman. "On quadratic differential forms". *SIAM J. Cont. and Optim.*, 36(5):1703–1749, 1998.