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Infinite-dimensional output-feedback bounded bilinear control of a parallel-flow heat exchanger

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Abstract— **In this paper, we consider the problem of output bounded controller design for a parallel-flow heat exchanger described by 2×2 coupled linear hyperbolic partial differential equations (PDEs) of balance laws. We aim to drive the internal fluid outlet temperature to track a reference trajectory by manipulating the external fluid velocity. Due to physical limitations, this manipulated variable has to be bounded to avoid a laminar regime. Consequently, the control problem becomes bounded and bilinear. Based on the set-invariance concept and the Lyapunov's stability theory, first, we design a bounded state feedback controller. Then, since only boundary measurements are available, we synthesize an output feedback controller, and demonstrate the exponential stability of the closed-loop system. Finally, simulation results are provided to illustrate the performance of the proposed control technique.**

I. INTRODUCTION

Heat exchangers (HEXs) are thermal devices that became ubiquitous in many industries where thermal energy is exchanged between two or many fluid streams. To name a few, refineries, power stations, and air conditioning are some examples where HEXs play a major role (see e.g. [?], [?]). They are employed both in heating and cooling processes and are very important in reducing energy consumption. Depending on the fluids flow directions, they can be described either as parallel-flow, counter-flow, or cross-flow HEXs. In this work, we are interested in tubular parallel-flow HEXs. They consist of a fluid circulating inside an internal tube with a constant velocity (flow rate) and a non-condensing vapour fluid flowing inside an external tube (or jacket). Both fluids move in the same direction, and heat is thus transferred through the internal tube's wall.

From a theoretical viewpoint, HEXs belong to the realm of distributed parameter systems (DPSs), and to improve their efficiency, we have to account for their spatio-temporal nature. In the literature, their dynamics can be modeled by a 2×2 hyperbolic PDE system of balance laws. These PDEs describe many physical phenomena, e.g., road traffic, hydrodynamic channel flows, and oil drilling. As a result, exhaustive research work was dedicated to the control problem of these systems. In parallel-flow HEXs, the controlled variable is the outlet temperature of the internal fluid, and the objective is to track a predefined reference trajectory. This can be achieved either by manipulating the inlet temperature of the external fluid or its velocity. Accordingly, this leads to either a boundary or a bilinear control problem.

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In the control community, much research was dedicated to the boundary control problem of hyperbolic PDEs using mainly the backstepping method e.g., [?] (and references therein) or optimal control e.g., [?]. However, many physical processes, like HEXs, chemostats, and traffic flow can be manipulated through the fluid velocity (see e.g., [1] and references therein). In practice, this is more efficient regarding energy consumption and simplicity. However, from a theoretical viewpoint, this is more challenging since the control problem becomes bilinear and constrained.

Research work on constrained control of PDEs dates back to the work of [2] and [3]. In those studies, input saturation was expressed as a bound on the state norm. Later, pointwise saturation in PDEs was handled using linear and bilinear matrix inequalities (LMIs) [4]. However, in these papers, the PDE operator was independent of the control input. Thus, these results cannot handle control problems in which the manipulated variable appears in the PDE operator itself, as in the case of parallel-flow HEXs . On the other hand, various techniques exist to tackle the problem of saturated control in finite-dimensional systems [?], [?], [?] (and references therein). However, the already mentioned methods either rely on *a priori* knowledge of the model or use algebraic solutions, and therefore, are difficult to apply on DPSs.

In this paper, we investigate the problem of bounded bilinear controller design of a 2×2 hyperbolic PDE system of balance laws. To the extent of the author's knowledge, bounded bilinear control of such models, where the input defines the PDE operator, are less investigated in the literature, especially in the late lumping framework. In our proposed control approach, as in [5], the key concept is to use the L^1-norm of the distributed tracking error as the controlled variable instead of the outlet tracking error. Also, inspired by the work of [6], we make use of the application of the set-invariance concept in control (see e.g. [?]) to develop feasibility conditions that guarantee the boundedness of the control law. The resulting controller needs the distributed state profiles, which are not available because only boundary measurements are available. Therefore, we develop a boundary state observer based on the Lyapunov approach as in [7]. Afterward, we prove the stability of the system in closed-loop and the convergence of the tracking error in the Lyapunov framework. We also validate the proposed regulator, in simulation, on a system modeling a parallel-flow HEX.

Notation. Time and space dependencies are dropped whenever it is necessary to alleviate the notation. The symbols ∂_t and ∂_x express time and space derivatives, respectively.

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$L^p(\Omega)$ ($\Omega =]0,1]$) is the Lebesgue space of order-p. $H^s(\Omega)$ is the Sobolev space of functions on Ω whose derivatives up to order s are in $L^2(\Omega)$. $\Xi(I, H^s(\Omega))$ is the space of functions $f: \Xi(I) \to H^s(\Omega)$. The notation $T_{i,e}$ signifies that the statement under investigation is valid for both T_i and T_e. Also, $T_{i,e} \in [\underline{T}_{i,e}, \bar{T}_{i,e}]$ suggests that $\underline{T}_{i,e}(x,t) \leq T_{i,e}(x,t) \leq \bar{T}_{i,e}(x,t)$ for $x \in]0,1]$ and $t \geq 0$. For readability, throughout this work we adopt the following notation $\partial_t T_{i,e}|_{T_{i,e}=T^\bullet_{i,e}} := \partial_t T_{i,e}$ when $T_{i,e} = T^\bullet_{i,e}$.

II. Problem formulation

Based on energy balance laws and some standard simplifications (see [?]), the temperature's dynamics of a parallel-flow heat exchanger can be described by the following system of hyperbolic PDEs.

$$\begin{cases} \partial_t T_i = -u_i(t)\,\partial_x T_i(x,t) + \alpha_i\,(T_e(x,t) - T_i(x,t)), \\ \partial_t T_e = -u_e(t)\,\partial_x T_e(x,t) + \alpha_e\,(T_i(x,t) - T_e(x,t)), \\ T_i(0,t) = g_i(t), \quad T_e(0,t) = g_e(t), \\ T_i(x,0) = T_i^0(x), \quad T_e(x,0) = T_e^0(x). \end{cases} \tag{1}$$

where (x,t) are space and time variables. T_i, T_e are respectively the internal and external fluid's temperatures. The internal and external fluid velocities are denoted by $u_i > 0$ and $u_e > 0$, respectively. $(\alpha_i, \alpha_e) > 0$ are thermal coefficients that depend on the fluids and tubes' characteristics. The third and fourth lines of system (1) denote the boundary and initial conditions, respectively.

Equation (1) describes heat transport in a parallel-flow HEX where the external (internal) fluid is used to either heat or cool the internal (external) fluid. Without loss of generality, in the present work, we investigate the heating process of the internal fluid using the external one. Therefore, the following assumptions are considered.

Assumption 2.1: h1- The parameters $\alpha_{i,e}$ are positive, the fluid velocities $u_{i,e} \in L^\infty(\mathbb{R}^+)$ are strictly positive, the initial conditions are such as $(T_i^0, T_e^0) \in (L^2(]0,1]))^2$ are positive, the boundary conditions are $(g_i, g_e) \in (L^2(]0,\infty]))^2$ are also positive, and the zero-compatibility condition (see [8]) is satisfied.

h2- The boundary and initial conditions are chosen such that $g_e(t) > g_i(t) > 0, \forall t \geq 0$ and $T_e^0(x) > T_i^0(x) > 0, \forall x \in [0,1]$. To ensure a laminar flow regime the fluid velocities should be lower bounded, whereas due to the pumps' physical limitations, they should be upper-bounded, thus $0 < u_{i,e_{min}} \leq u_{i,e} \leq u_{i,e_{max}} < \infty$.

According to [9] assumption 2.1-h1 ensures that system (1) admits a unique exponentially stable solution $(T_i, T_e) \in C(0,\infty, H^1(\Omega))$. Moreover this solution is positive. Whereas, assumption 2.1-h2 guarantees that $\partial_x T_e \leq 0$ and $\partial_x T_i \geq 0$ (this can be directly inferred from the steady-state solution of equation (1)).

In the present work, we consider that $\alpha_{i,e}$ and $0 < u_{i_{min}} < u_i < u_{i_{max}}$ are known constants, and the external fluid velocity u_e is the manipulated variable (control input). We aim to control the outlet internal temperature $T_i(1,t)$ to track a reference trajectory. The measurements are located at the tubes' inlets and outlets as follows

$$y(t) = [T_i(0,t) \quad T_e(0,t) \quad T_i(1,t) \quad T_e(1,t)]^T \tag{2}$$

III. Bounded bilinear control design

Instead of using the tracking error as the controlled variable, to simplify the controller design, we select the function γ which is the L^1-norm of the external and internal tracking errors as the controlled variable, i.e.,

$$\gamma(t) = \int_0^1 |T_e^*(x,t) - T_e(x,t)|\,dx + \int_0^1 |T_i^*(x,t) - T_i(x,t)|\,dx. \tag{3}$$

where T_e^* and T_i^* denote the target external and internal temperature profiles. Note that since we are interested in solutions in $C(0,\infty,H^1(\Omega))$, then from the embeddedness of $H^1(\Omega)$ in $C(\Omega)$, the convergence of the norm $\gamma(t)$ implies the point-wise convergence of the distributed tracking error, and thus the convergence of the internal fluid's outlet tracking error.

Let us consider the following control input

$$u_e(t) = \begin{cases} u_{e_{min}} & \text{if} \quad \rho(t) \leq u_{e_{min}}, \\ \rho(t) & \text{if} \quad u_{e_{min}} < \rho(t) < u_{e_{max}}, \\ u_{e_{max}} & \text{if} \quad \rho(t) \geq u_{e_{max}}, \end{cases} \tag{4}$$

where

$$\begin{cases} \rho(t) = \phi(t;T_e)^{-1}\Big[+ \int_0^1 sgn(\delta T_e)\,(-\partial_t T_e^* + \alpha_e\,(T_i - T_e))\,dx \\ + \int_0^1 sgn(\delta T_i)\,(-\partial_t T_i^* + \partial_t T_i)\,dx - k_p \gamma(t)\Big], \\ \phi(t;T_e) = \int_0^1 sgn(\delta T_e)\,\partial_x T_e\,dx, \end{cases} \tag{5}$$

$$\text{and} \quad \begin{cases} \delta T_i(x,t) = T_i^*(x,t) - T_i(x,t), \\ \delta T_e(x,t) = T_e^*(x,t) - T_e(x,t), \end{cases} \tag{6}$$

while the gain $k_p > 0$. Note that if the control input u_e was unbounded, then by selecting $u_e(t) = \rho(t)$, we can ensure, using straightforward computations, that $\dot{\gamma}(t) \leq -k_p \gamma(t)$. Therefore, the tracking error converges exponentially to zero. However, since u_e is bounded then the direct method to handle input constraints is to use the saturation operator as in equations (4)-(5). On the other hand, input saturation hinders the nominal stabilization properties of the controller, and therefore we need to specify the region of attraction under which the system's closed-loop desired behavior is guaranteed. In the following assumption, we state feasibility conditions on the choice of the target temperature profiles $T^*_{i,e}$.

Assumption 3.1: Desired temperature profiles $T^*_{i,e}$ are chosen such as

$$\begin{cases} u_{e_{min}} \leq \dfrac{1}{(T_e^*(1,t) - T_e^*(0,t))}\Big[\int_0^1 \Big(-\partial_t T_e^* + \alpha_e\,(T_i^* - T_e^*)\Big)\,dx \\ + \int_0^1 \Big(-\partial_t T_i^* - \alpha_i\,(T_i^* - T_e^*) - u_i\,\partial_x T_i^*\Big)\,dx\Big] \leq u_{e_{max}}. \end{cases} \tag{7}$$

Assumption 3.1 expresses that the desired profile T_e^* (and thus T_i^*) must be chosen such that they satisfy equation (1) and the control input generating them satisfies the input constraints.

By a continuity argument we can ensure that since the pair (T_e^*, T_i^*) is continuous, then there exist temperature intervals $[\underline{T}_{i,e}, \bar{T}_{i,e})]$ such that $\underline{T}_{i,e} < T^*_{i,e} < \bar{T}_{i,e}$, $(\underline{T}_{i,e}, \bar{T}_{i,e}) \in (C(0,\infty,H^1(\Omega)))^2$ and $(\underline{T}_{i,e}, \bar{T}_{i,e})$ satisfy equation (1). Moreover, the following inequality holds for all $T_e \in [\underline{T}_e, \bar{T}_e]$ and $T_i \in [\underline{T}_i, \bar{T}_i]$

$$\begin{cases} u_{e_{min}} \leq \phi(t;T_e)^{-1}\Big[\int_0^1 sgn(\delta T_e)\,(-\partial_t T_e^* + \alpha_e\,(T_i - T_e))\,dx \\ \quad + \int_0^1 sgn(\delta T_i)\,(-\partial_t T_i^* + \partial_t T_i)\,dx - k_p\gamma(t)\Big] \leq u_{e_{max}}, \\ \phi(t;T_e) = \int_0^1 sgn(\delta T_e)\,\partial_x T_e\,dx. \end{cases} \tag{8}$$

Inequality (8) states that, from a continuity argument, we can always find a region around the desired profiles $T^*_{i,e}$ for which the unbounded control law $\rho(t)$ satisfies the control constraints. This region– described by inequality (8)– defines the region of attraction.

Remark III.1 From equation (1), in a heating process of the internal fluid using the external one, for a fixed u_i, when the external fluid velocity u_e approaches its maximum value $u_{e_{max}}$ then it generates a temperature profile T_i that is close to its maximum profile (and T_e tends towards its minimum profile), and vice versa when it reaches its minimum value. This statement can be verified by analyzing the steady-state solution of equation (1) (for space limitation, we do not investigate it further).

Remark III.1 is important especially in analyzing the stabilizing properties of the control input (4)-(5) in the following proposition.

Proposition 3.2: Consider system (1). Provided that assumptions 2.1 and 3.1 are fulfilled then

1- The domain $]0,\bar{T}_e]\times]0,\bar{T}_i]$ is positively invariant, where $\bar{T}_{i,e}$ satisfy assumption 3.1.

2- The tracking error of the closed-loop system using control law (4)-(5) is exponentially stable (relatively to the domain $]0,\bar{T}_e]\times]0,\bar{T}_i]$).

Proof: First, note that $\partial_t T_{i,e}|_{T_{i,e}=0} = 0$. Based on assumption 2.1 we have $\forall t \geq 0 : g_{i,e}(t) > 0$ and $\forall x \in [0,1] : T^0_{i,e}(x) > 0$, then $T_{i,e}(x,t) > 0,\ \forall t \geq 0, \forall x \in [0,1]$.

When $T_i = \bar{T}_i$ then $u_e = \min(u_{e_{max}},\rho) > 0$, $T_e = \underline{T}_e$, and from equation (1) we get

$$\begin{aligned} \int_0^1 \partial_t T_e|_{T_e=\underline{T}_e}\,dx &= -\min(u_{e_{max}},\rho)\int_0^1 \partial_x \underline{T}_e\,dx + \alpha_e\int_0^1 (\bar{T}_i - \underline{T}_e)\,dx \\ &= \min(u_{e_{max}},\rho)\underbrace{[\underline{T}_e(0,t) - \underline{T}_e(1,t)]}_{>0} + \alpha_e\int_0^1 (\bar{T}_i - \underline{T}_e)\,dx. \end{aligned} \tag{9}$$

Also we have $sgn(\delta T_e)\big|_{T_e=\underline{T}_e} = 1$ and $sgn(\delta T_i)\big|_{T_i=\bar{T}_i} = -1$. From equation (8), by replacing T_e by $\underline{T}_e$ and consequently T_i by $\bar{T}_i$, after straightforward simplifications, we know that

$$\begin{aligned} u_{e_{max}} \geq \quad & \frac{1}{[\underline{T}_e(1,t) - \underline{T}_e(0,t)]}\Big[\int_0^1\Big(-\partial_t T_e^* - \alpha_e\,(\bar{T}_i - \underline{T}_e)\Big)\,dx \\ & + \int_0^1\Big(\partial_t T_i^* - \partial_t \bar{T}_i\Big)\,dx - k_p\gamma(t)\Big]. \end{aligned} \tag{10}$$

By replacing inequality (10) in equation (9) we get

$$\begin{aligned} \int_0^1 \partial_t T_e|_{T_e=\underline{T}_e}\,dx \geq & -\Big[\int_0^1\Big(-\partial_t T_e^* - \alpha_e\,(\bar{T}_i - \underline{T}_e)\Big)\,dx \\ & + \int_0^1\Big(\partial_t T_i^* - \partial_t \bar{T}_i\Big)\,dx - k_p\gamma(t)\Big] + \alpha_e\int_0^1 (\underline{T}_i - \bar{T}_e)\,dx. \end{aligned} \tag{11}$$

By rearranging the terms in the previous inequality (11), we obtain

$$\int_0^1 (\partial_t \underline{T}_e - \partial_t T_e^*)\,dx + \int_0^1 (\partial_t T_i^* - \partial_t \bar{T}_i)\,dx \geq k_p\gamma(t), \tag{12}$$

However, when $T_i = \bar{T}_i$ and $T_e = \underline{T}_e$, then

$$\gamma(t) = \int_0^1 (T_e^*(x,t) - \underline{T}_e(x,t))\,dx + \int_0^1 (\bar{T}_i(x,t) - T_i^*(x,t))\,dx$$

and thus inequality (12) becomes

$$\dot{\gamma}(t) \leq -k_p\,\gamma(t).$$

Also, if $u_e = \rho(t)$ then direct calculations lead to $\dot{\gamma}(t) = -k_p\,\gamma(t)$. Accordingly, we conclude that when the temperature profile T_i reaches $\bar{T}_i$ then $T_{i,e}$ converges exponentially towards $T^*_{i,e}$.

On the other hand, when T_i approaches its lower bound $\underline{T}_i$, then T_e becomes close to $\bar{T}_e$ and $u_e = \max(u_{e_{min}},\rho) > 0$. In this case, $sgn(\delta T_e)\big|_{T_e=\bar{T}_e} = -1$ and $sgn(\delta T_i)\big|_{T_i=\underline{T}_i} = 1$. Moreover,

$$\begin{aligned} \int_0^1 \partial_t T_e|_{T_e=\bar{T}_e}\,dx &= -\max(u_{e_{min}},\rho)\int_0^1 \partial_x \bar{T}_e\,dx + \alpha_e\int_0^1 (\underline{T}_i - \bar{T}_e)\,dx \\ &= \max(u_{e_{min}},\rho)\underbrace{[\bar{T}_e(0,t) - \bar{T}_e(1,t)]}_{>0} + \alpha_e\int_0^1 (\underline{T}_i - \bar{T}_e)\,dx. \end{aligned} \tag{13}$$

By replacing T_e by $\bar{T}_e$ and T_i by $\underline{T}_i$, from equation (8) we infer that

$$\begin{aligned} u_{e_{min}} \leq \quad & \frac{1}{[\bar{T}_e(0,t) - \bar{T}_e(1,t)]}\Big[\int_0^1 -\Big(-\partial_t T_e^* + \alpha_e\,(\underline{T}_i - \bar{T}_e)\Big)\,dx \\ & + \int_0^1\Big(-\partial_t T_i^* + \partial_t \underline{T}_i\Big)\,dx - k_p\gamma(t)\Big]. \end{aligned} \tag{14}$$

Therefore, if $u_e = u_{e_{min}}$ by replacing inequality (14) in equation (13) we infer that

$$\begin{aligned} \int_0^1 \partial_t T_e|_{T_e=\underline{T}_e}\,dx \leq & \Big[\int_0^1 -\Big(-\partial_t T_e^* + \alpha_e\,(\underline{T}_i - \bar{T}_e)\Big)\,dx \\ & + \int_0^1\Big(-\partial_t T_i^* + \partial_t \underline{T}_i\Big)\,dx - k_p\gamma(t)\Big] + \alpha_e\int_0^1 (\underline{T}_i - \bar{T}_e)\,dx. \end{aligned} \tag{15}$$

Hence

$$\int_0^1\Big(\partial_t \underline{T}_e - \partial_t T_e^*\Big)\,dx + \int_0^1\Big(\partial_t T_i^* - \partial_t \underline{T}_i\Big)\,dx \leq -k_p\gamma(t) \tag{16}$$

and

$$\dot{\gamma}(t) \leq -k_p\,\gamma(t).$$

If $u_e = \rho$ then straightforward computations lead to $\dot{\gamma}(t) \leq -k_p\,\gamma(t)$. In conclusion, we have proved that the temperatures $T_{i,e}$ are trapped in the domain $]0,\bar{T}_e]\times]0,\bar{T}_i]$, i.e., this domain is positively invariant, and that the L^1- norm of the tracking error is exponentially stable. From the embeddedness of $H^1(\Omega)$ in $C(\Omega)$, we conclude that the pointwise tracking error is also exponentially converging to zero (relatively to the domain $]0,\bar{T}_i]\times]0,\bar{T}_e]$). ■

IV. STATE ESTIMATION

The control law presented in (4)-(5) requires the exact knowledge of the distributed profiles of both the internal and external temperatures T_i and T_e. However, only boundary measurements are available. Therefore, in this section, we develop an estimator able to reconstruct the distributed profiles in finite time. The observability of system (1)-(2) was established in the work of [**?**].

Let $\hat{T}_{i,e}$ denote the estimated temperature profiles of $T_{i,e}$, respectively. We consider the estimator given by the following system of equations:

$$\begin{cases} \partial_t \hat{T}_i = -u_i(t)\,\partial_x \hat{T}_i(x,t) + \alpha_i\,(\hat{T}_e(x,t) - \hat{T}_i(x,t)), \\ \partial_t \hat{T}_e = -u_e(t)\,\partial_x \hat{T}_e(x,t) + \alpha_e\,(\hat{T}_i(x,t) - \hat{T}_e(x,t)), \\ \hat{T}_i(0,t) = g_i(t) - \kappa_i \Delta T_i(1,t), \\ \hat{T}_e(0,t) = g_e(t) - \kappa_e \Delta T_e(1,t), \\ \hat{T}_i(x,0) = \hat{T}_i^0(x),\ \hat{T}_e(x,0) = \hat{T}_e^0(x). \end{cases} \tag{17}$$

where $\kappa_{i,e}$ are the tuning parameters and $\Delta T_{i,e} = \hat{T}_{i,e} - T_{i,e}$ are the state estimation errors. Their dynamics are computed using equations (1) and (17) and are given by

$$\begin{cases} \partial_t \Delta T_i = -u_i(t)\,\partial_x \Delta T_i(x,t) + \alpha_i\,(\Delta T_e - \Delta T_i), \\ \partial_t \Delta T_e = -u_e(t)\,\partial_x \Delta T_e + \alpha_e\,(\Delta T_i - \Delta T_e), \\ \Delta T_i(0,t) = -\kappa_i \Delta T_i(1,t), \\ \Delta T_e(0,t) = -\kappa_e \Delta T_e(1,t), \\ \Delta T_i(x,0) = \hat{T}_i^0(x) - T_i^0(x),\ \Delta T_e(x,0) = \hat{T}_e^0(x) - T_e^0(x). \end{cases} \tag{18}$$

Proposition 4.1: Consider system (1) under assumptions 2.1 and 3.1. Provided that the fluid u_i and u_e are positive and bounded, then if the tuning parameters κ_i and κ_e are selected such that $0 < \kappa_{i,e} < 1$ then the L^2-norm of the state estimation errors $\Delta T_{i,e}$ is globally exponentially stable.

Proof: The proof is based on the Lyapunov's stability theory and inspired from the work of [7]. Let $V_1(t)$ be a Lyapunov candidate function defined by

$$V_1(t) = \int_0^1 e^{-\sigma x}\left(\frac{1}{\alpha_i}(\Delta T_i)^2 + \frac{1}{\alpha_e}(\Delta T_e)^2\right)dx. \tag{19}$$

Using (18) and straightforward simplifications the derivative of (19) is computed as follows

$$\begin{aligned} \dot{V}_1 &= 2\int_0^1 e^{-\sigma x}\left(\frac{-u_i}{\alpha_i}\Delta T_i\,\partial_x \Delta T_i - \frac{u_e}{\alpha_e}\Delta T_e\,\partial_x \Delta T_e - (\Delta T_e - \Delta T_i)^2\right)dx \\ &\le -\frac{\sigma u_i}{\alpha_i}\int_0^1 e^{-\sigma x}(\Delta T_i)^2\,dx - \frac{\sigma u_e}{\alpha_e}\int_0^1 e^{-\sigma x}(\Delta T_e)^2\,dx \\ &\quad - \frac{u_i}{\alpha_i}(e^{-\sigma} - \kappa_i)(\Delta T_i(1,t))^2 - \frac{u_e}{\alpha_e}(e^{-\sigma} - \kappa_e)(\Delta T_e(1,t))^2, \end{aligned} \tag{20}$$

where the inequality in (20) is achieved using integration by parts, and by replacing the boundary conditions given in (18) while assuming that assumption 2.1 is fulfilled, and $u_{i,e}$ are positive. Since $0 < \kappa_{i,e} < 1$ then there exists always a σ such that $(e^{-\sigma} - \kappa_{i,e}) > 0$. Accordingly, inequality (20) becomes

$$\begin{aligned} \dot{V}_1 &\le -\frac{\sigma u_i}{\alpha_i}\int_0^1 e^{-\sigma x}(\Delta T_i)^2\,dx - \frac{\sigma u_e}{\alpha_e}\int_0^1 e^{-\sigma x}(\Delta T_e)^2\,dx, \\ &\le -\sigma \min(u_i, u_e)\,V_1(t). \end{aligned} \tag{21}$$

Therefore, we conclude that the L^2-norm of the state estimation errors are globally exponentially stable. ■

V. OUTPUT-FEEDBACK CONTROL

In this section, we demonstrate that the output-feedback controller obtained by substituting $T_{i,e}$ by its estimate $\hat{T}_{i,e}$ in (4)-(5) ensures the stability of the closed-loop system and guarantees the asymptotic convergence of the tracking error. Let the bounded output-control law be formulated as follows.

$$u_e(t) = \begin{cases} u_{e_{min}} & \text{if } \hat{\rho}(t) \le u_{e_{min}}, \\ \hat{\rho}(t) & \text{if } u_{e_{min}} < \hat{\rho}(t) < u_{e_{max}}, \\ u_{e_{max}} & \text{if } \hat{\rho}(t) \ge u_{e_{max}}, \end{cases} \tag{22}$$

where

$$\begin{cases} \hat{\rho}(t) = \phi(t;\hat{T}_e)^{-1}\Big[\int_0^1 sgn(\delta\hat{T}_e)\Big(-\partial_t T_e^* + \alpha_e\,(\hat{T}_i - \hat{T}_e)\Big)dx \\ \quad + \int_0^1 sgn(\delta\hat{T}_i)\Big(-\partial_t T_i^* - \alpha_i\,(\hat{T}_i - \hat{T}_e) - u_i\,\partial_x \hat{T}_i\Big)dx - k_p\hat{\gamma}(t)\Big], \\ \phi(t;\hat{T}_e) = \int_0^1 sgn(\delta\hat{T}_e)\,\partial_x \hat{T}_e\,dx, \end{cases} \tag{23}$$

and

$$\begin{cases} \delta\hat{T}_{i,e}(x,t) = T_{i,e}^*(x,t) - \hat{T}_{i,e}(x,t), \\ \hat{\gamma}(t) = \int_0^1 \Big(|\delta\hat{T}_i(x,t)| + |\delta\hat{T}_e(x,t)|\Big)\,dx. \end{cases} \tag{24}$$

A relationship between $\gamma(t)$ and $\hat{\gamma}(t)$ can be formulated as follows.

$$\begin{aligned} \gamma(t) &= \int_0^1 |T_i^* - T_i|\,dx + \int_0^1 |T_e^* - T_e|\,dx, \\ &= \int_0^1 |T_i^* - \hat{T}_i + \hat{T}_i - T_i|\,dx + \int_0^1 |T_e^* - \hat{T}_e + \hat{T}_e - T_e|\,dx \\ &= \int_0^1 |\delta\hat{T}_i + \Delta T_i|\,dx + \int_0^1 |\delta\hat{T}_e + \Delta T_e|\,dx \end{aligned} \tag{25}$$

where $\Delta T_{i,e}$ are the state estimation errors defined by equation (18).

Considering that we can always find a constant $0 < w_{i,e} < \infty$ such that

$$\int_0^1 |\Delta T_{i,e}|\,dx \le w_{i,e}\int_0^1 (\Delta T_{i,e})^2\,dx,$$

and using the triangle inequality, equation (25) becomes

$$\begin{aligned} \gamma(t) &\le \int_0^1 |\delta\hat{T}_i|\,dx + \int_0^1 |\Delta T_i|\,dx + \int_0^1 |\delta\hat{T}_e|\,dx + \int_0^1 |\Delta T_e|\,dx, \\ &\le \hat{\gamma}(t) + w_i\int_0^1 (\Delta T_i)^2\,dx + w_e\int_0^1 (\Delta T_e)^2\,dx, \\ &\le \hat{\gamma}^2(t) + \bar{w}_i\int_0^1 e^{-\sigma x}(\Delta T_i)^2\,dx + \bar{w}_e\int_0^1 e^{-\sigma x}(\Delta T_e)^2\,dx. \end{aligned} \tag{26}$$

For the third inequality in (26), from the boundedness of the space interval $[0,1]$, we can always find $\bar{w}_{i,e}$ that satisfies it.

Inequality (26) states that the tracking error $\gamma(t) > 0$ is bounded by $\hat{\gamma}(t)$ and $\Delta T_{i,e}$. Accordingly, if we prove that the right hand side of inequality (26) is asymptotically stable then we can guarantee that the tracking error $\gamma(t)$ is also asymptotically stable. Note that since we have already proved, in the previous section, that provided that the control input $u_e(t)$ is positive, the state estimation error is exponentially stable, it remains to analyze convergence properties of $\hat{\gamma}(t)$ in closed-loop.

Lemma 5.1: Provided that assumptions 2.1 and 3.1 are satisfied, the control input $u_e(t)$ defined by equation (22)-(23) ensures both the positive invariance of the domain $]0,\bar{T}_i]\times]0,\bar{T}_e]$ and the exponential convergence of the tracking error $\hat{\gamma}(t)$ introduced in system (24) (relatively to the domain $]0,\bar{T}_e]\times]0,\bar{T}_i]$) .

Proof: Due to space limitation, the proof of lemma 5.1 is sketched briefly.

First, note that, except for the boundary conditions, the state observer in (17) is identical to the original model given in system (1), and that $\hat{\rho}(t)$ is obtained by replacing $T_{i,e}$ by $\hat{T}_{i,e}$ in equation (5). Therefore, if assumptions 2.1 and 3.1 are fulfilled, then for extreme cases when $u_e(t) = u_{e_{min}}$ or $u_e(t) = u_{e_{max}}$ the proof of lemma 5.1 is similar to the one established for proposition 3.2. It remains to investigate the case when $u_e(t) = \hat{\rho}(t)$. Computing $\dot{\hat{\gamma}}(t)$ and replacing $u_e(t)$ by $\hat{\rho}(t)$ results in $\dot{\hat{\gamma}}(t) = -k_p\hat{\gamma}(t)$. Thus, $\hat{\gamma}(t)$ is exponentially stable, and this concludes the proof of lemma 5.1. ∎

At this stage we can formulate the following theorem.

Theorem 5.2: Consider system (1)-(2), the observer (17) and assume that assumptions 2.1 and 3.1 are satisfied. If the control gain k_p is positive then the observer-based controller described by equations (22)-(23) ensures that

1- The domain $]0,\bar{T}_i]\times]0,\bar{T}_e]$ is positively invariant.

2- The tracking error $\gamma(t)$ and the L^2-norm of the state estimation errors $\Delta T_{i,e}$ converge uniformly asymptotically to zero relatively to $]0,\bar{T}_i]\times]0,\bar{T}_e]$.

Proof: From lemma 5.1, we know that if assumptions 2.1 and 3.1 are satisfied then the domain $]0,\bar{T}_i]\times]0,\bar{T}_e]$ is positively invariant and that $\hat{\gamma}(t)$ is exponentially converging.

Let us introduce the following Lyapunov candidate function

$$V(t) = \hat{\gamma}^2(t) + \bar{w}_i\int_0^1 e^{-\sigma x}(\Delta T_i)^2\,dx + \bar{w}_e\int_0^1 e^{-\sigma x}(\Delta T_e)^2\,dx. \qquad (27)$$

Its derivative can be written as

$$\begin{aligned}\dot{V} &\leq -2k_p\hat{\gamma}^2(t) + \min(\bar{w}_i,\bar{w}_e)\dot{V}_1,\\ &\leq -2k_p\hat{\gamma}^2(t) - \sigma, \min(\bar{w}_i\,u_i,\,\bar{w}_e\,u_e)V_1(t), \qquad (28)\end{aligned}$$

where $V_1(t)$ is defined in (19) and we have employed inequality (21). From this last inequality (28) we conclude that

$$\dot{V}(t) \leq -\min(2k_p, \frac{\bar{w}_i\,\sigma\,u_i}{\alpha_i}, \frac{\bar{w}_e\,\sigma\,u_e}{\alpha_e})V(t). \qquad (29)$$

Therefore the control law (22)-(23) guarantee the exponential stability of both $\hat{\gamma}$ and $\|\Delta T_{i,e}\|_2$. As a result, from inequality (26) we conclude that γ also is asymptotically stable (relatively to the domain $]0,\bar{T}_i]\times]0,\bar{T}_e]$).

∎

VI. Simulation Results

To illustrate the performance of the proposed control strategy in (22)-(23), the dynamic model of a parallel-flow heat exchanger as in (1) is considered with $\alpha_e = \alpha_i = 1$, $u_i = 1$ and where the time and space variable are normalized. The input u_e bounds are , $u_{e_{max}} = 20$, $u_{e_{min}} = 0.5$. All the parameters are

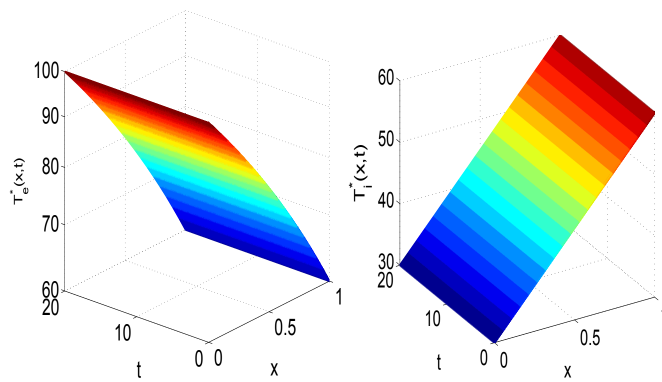

Fig. 1: External T_e^* and internal T_i^* distributed target temperature profiles

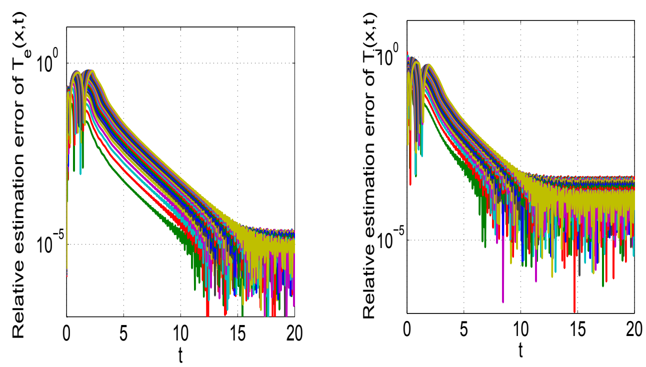

Fig. 2: External $\Delta T_e(x,t)(\%)$ and internal $\Delta T_i(x,t)(\%)$ state estimation relative errors

dimensionless. Based on these constraints and on assumption 3.1's requirements, the target temperature profiles T_i^* and T_e^* are selected and depicted in figure 1. Implementation of the system model in (1), the observer in (17) and the conroller in (22)-(23) is carried out in Matlab/Simulink® using the quadratic B-splines Galerkin method (see e.g. [10], [11]). The observer tuning gains are $\kappa_1 = 0.8$, $\kappa_2 = 0.5$ and the controller gain $k_p = 100$. Note that since the control law design was not based on Galerkin truncation, there is no risk of spillover instabilities.

Figure 2 shows the distributed external $\Delta T_e(x,t)$ and internal $\Delta T_i(x,t)$ state estimation errors. Since these errors do not exceed 1%, at steady-state, then we can confirm that the observer in equation (17) performs well and that the errors converge in finite time.

To illustrate the performance of the controller, we plot the outlet temperature $T_i(1,t)$ and its relative tracking error in figure 3. These plots show that the tracking error converges rapidly and in finite-time. The error's maximum amplitudes is below 1% (at steady-state).

The control input $u_e(t)$ is presented in figure 4 (left panel) and is clearly contained by the input upper and lower bounds, i.e., $u_{e_{min}} < u_e < u_{e_{max}}$. In addition, from figure 4 (right panel),

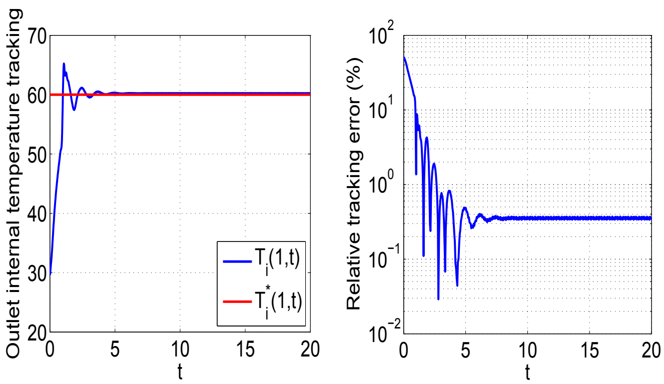

Fig. 3: Outlet internal tracking

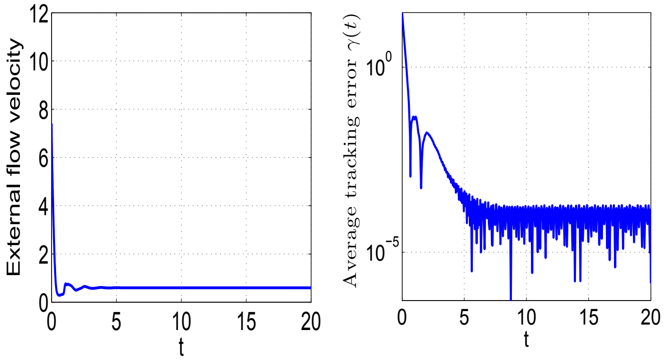

Fig. 4: Profiles of the control input $u(t)$ and the average tracking error $\gamma(t)$

the controlled variable $\gamma(t)$ is rapidly converging and in finite time.

VII. Conclusion

This work dealt with the problem of output-feedback bounded bilinear control of 2×2 hyperbolic PDE systems of balance laws. First, we selected the L^1-norm of the tracking error as the controlled variable that simplified the bilinear control problem. Then using the set invariance concept, the proposed control strategy is guaranteed to respect the input constraints while achieving the control objectives. Since only boundary measurements are available, we proposed a boundary observer to reconstruct the distributed profiles. After that, we demonstrated the stability of the closed-loop system and the convergence of the tracking errors. The results in the simulation section show the ability of the proposed control technique to guarantee satisfactory performances. Note that this work is not limited to parallel-flow hyperbolic PDE systems but can be straightforwardly applied to hetero-directional hyperbolic PDEs (as in cross-flow heat exchangers). In future works, we intend to address robustness issues related to unknown (or partially known) parameters and external disturbances.

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