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LMI Feasibility Analysis in Observer Design for Some Families of Nonlinear Systems

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Abstract— **This note deals with observer design for nonlinear systems via Linear Matrix Inequalities (LMIs). The main goal consists of showing that for some families of nonlinear systems, the LMI-based observer design techniques always provide exponential convergent observer. Indeed, until now, this advantageous feature is unique to some types of observers/estimators, such as the high-gain observer, the sliding mode observer, and the moving horizon estimator, under certain conditions of detectability or observability. More specifically, the LMI conditions we propose in this paper always provide solutions to both systems in companion form and feedforward structure. An extension to a general class of nonlinear triangular systems without linear components is provided, which renders the applicability of LMI-based methods possible for a wide class of nonlinear systems without the need for nonlinear diffeomorphism-based transformations.**

Index Terms— **Observers design, Lipschitz systems, LMIs.**

I. INTRODUCTION

Observer design for nonlinear systems has become a vital and crucial step in modern control design issues due to the integration of novel technologies and the development of new and sophisticated sensors. Such technologies involve numerous variables or states from different natures, such as cyber-physical states, for instance [1]. The observer design step is motivated by the often very high cost of sensors, and sometimes because of the unavailability of sensors at any cost.

Tremendous research activities have been paid to nonlinear observer design and various methods have been proposed in the literature. Among these methods, apart from the optimization/minimization of cost functions-based techniques, like the extended Kalman filter, and the moving horizon estimator, we can mention the famous high-gain observer design methodology [2], the sliding mode observer approach [3], and the LMI-based techniques [4]. While the first two techniques guarantee the existence of the observer design under only some assumptions on the nonlinearity of the system, however, the last one provides only sufficient conditions expressed in terms of LMIs for which feasibility is not always ensured, which is the main drawback of LMI-based observer design approach. In this paper, we will address this problem and we will analyze the feasibility of LMIs for some specific families of nonlinear systems, namely systems in companion form, and systems having feedforward structure.

Several LMI-based techniques have been developed in the literature, where each technique attempts to reduce the conservatism of the LMI design conditions ensuring exponential convergence of the observer (2). Among these methods, there are the old techniques, which are conservative [5], [6], [7], [8], and the recent approaches [9], which provide feasible LMI conditions for a wider class of nonlinear systems. Feasibility of the LMI conditions depends on the Lipschitz constant and the structure of the nonlinearity of the system. To overcome these limitations, the recent LMI approaches use some mathematical tools in convenient ways to dominate the Lipschitz constant and to compensate for the structure of the nonlinearity due to additional decision variables. Despite the considerable efforts made to propose enhanced LMI conditions, this approach suffers from a major drawback, which is the absence of a guarantee of feasibility for any Lipschitz constant. This weakens the LMI techniques and sometimes makes them useless. Recently in [10], instead of guaranteeing the feasibility of LMI conditions, the authors proposed new results on guaranteeing *infeasibility* of the LMIs for systems where all the system components or all the output functions are non-monotonic. In spite of this result, the problem of guaranteeing feasibility is the most important and still remains open. It would therefore be interesting to work on the analysis and guarantee of the feasibility of LMIs for at least some particular families of nonlinear systems as it is the case with some famous nonlinear observers, namely high-gain observers and sliding mode observers. This goal is the motivation of this paper.

II. PROBLEM STATEMENT AND PRELIMINARY RESULTS

A. Problem statement

Consider the class of systems described by the following equations:

$$\begin{cases} \dot{x} = Ax + f(x) \\ y = Cx + h(x) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state and $y \in \mathbb{R}^p$ is the output measurement vector. Without loss of generality, and for the sake of brevity, we consider the system (1) without control input. We assume that the functions $f(.)$ and $h(.)$ are respectively γ_f-Lipschitz and γ_h-Lipschitz with respect to their arguments. Without loss of generality, the Lipschitz constraint is assumed to be global. Otherwise, we need to apply the *Hilbert* projection theorem [11], [12] or the

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Kirszbraun–Valentine extension theorem [13], [14] to extend $f(.)$ and $h(.)$ to global Lipschitz functions. We only need the system (1) to admit a positively invariant compact set on which $f(.)$ and $h(.)$ are Lipschitz. The reader can also find details on this extension in [2].

As usual in the LMI context, which is the objective of the paper, we consider the following Luenberger observer structure corresponding to (1):

$$\dot{\hat{x}} = A\hat{x} + f(\hat{x}) + L\big(y - C\hat{x} - h(\hat{x})\big) \tag{2}$$

where $\hat{x} \in \mathbb{R}^n$ is the estimate of x and the matrix $L \in \mathbb{R}^{n\times p}$ is the observer gain to be determined such that the estimation error $\epsilon \triangleq x - \hat{x}$ converges exponentially towards zero.

Of all the existing methods, the less conservative one is the LPV/LMI approach which is based on transforming the dynamics of the estimation error into a polytopic system, and then the application of the convexity principle leads to solving a finite number of LMI conditions without using strong upper bounds to dominate the nonlinearity of the system. For this reason, in this paper, we will exploit this method and we will show that the LMIs are always feasible for some families of nonlinear systems. Hence, we will first recall the LPV/LMI technique.

By applying [15, Lemma 7], there exist functions $\psi_{ij}(.,.)$ and $\phi_{ij}(.,.)$ such that the dynamics of the estimation error is given as

$$\begin{aligned}\dot{\epsilon} &= \Big(A - LC\Big)\epsilon + \big[f(x) - f(\hat{x})\big] + \big[h(x) - h(\hat{x})\big] \\ &= \Big(\mathcal{A}(\psi) - L\mathcal{C}(\phi)\Big)\epsilon \end{aligned}\tag{3}$$

where

$$\mathcal{A}(\psi) \triangleq A + \sum_{i,j=1}^{n,n} \psi_{ij}\mathcal{H}_{ij}^{n,n} \tag{4}$$

$$\mathcal{C}(\phi) \triangleq C + \sum_{i,j=1}^{p,n} \phi_{ij}\mathcal{H}_{ij}^{p,n} \tag{5}$$

$$-\gamma_{f_i} \le \underline{\gamma}_{\psi_{ij}} \le \psi_{ij} \le \bar{\gamma}_{\psi_{ij}} \le \gamma_{f_i} \tag{6}$$

$$-\gamma_{h_i} \le \underline{\gamma}_{\phi_{ij}} \le \phi_{ij} \le \bar{\gamma}_{\phi_{ij}} \le \gamma_{h_i} \tag{7}$$

with

$$\psi_{ij} \triangleq \psi_{ij}\Big(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big), \quad \phi_{ij} \triangleq \phi_{ij}\Big(x^{\hat{x}_{j-1}}, x^{\hat{x}_j}\Big).$$

It is clear from (6) and (7) that the parameters ψ and ϕ belong to the bounded convex sets

$$\mathcal{S}_f = \Big\{\varphi \in \mathbb{R}^{n\times n} : \underline{\gamma}_{\psi_{ij}} \le \varphi_{ij} \le \bar{\gamma}_{\psi_{ij}}\Big\}, \tag{8}$$

$$\mathcal{S}_h = \Big\{\varphi \in \mathbb{R}^{p\times n} : \underline{\gamma}_{\phi_{ij}} \le \varphi_{ij} \le \bar{\gamma}_{\phi_{ij}}\Big\} \tag{9}$$

for which the sets of vertices are respectively given by

$$\mathcal{V}_f = \Big\{\varphi \in \mathbb{R}^{n\times n} : \varphi_{ij} \in \{\underline{\gamma}_{\psi_{ij}}, \bar{\gamma}_{\psi_{ij}}\}\Big\} \tag{10}$$

and

$$\mathcal{V}_h = \Big\{\varphi \in \mathbb{R}^{p\times n} : \varphi_{ij} \in \{\underline{\gamma}_{\phi_{ij}}, \bar{\gamma}_{\phi_{ij}}\}\Big\}. \tag{11}$$

Hence, by using the quadratic Lyapunov function

$$\vartheta(\epsilon) \triangleq \epsilon^\top \mathbb{P}\epsilon$$

and developing its derivative along the trajectories of (3), we obtain following theorem.

Theorem 1: The estimation error ϵ satisfying (3) converges exponentially towards zero if there exists a positive definite matrix $\mathbb{P} = \mathbb{P}^\top$, a matrix $\mathcal{X} \in \mathbb{R}^{n\times p}$, and a scalar $\lambda > 0$ such that the following LMIs are feasible:

$$\mathcal{A}(\psi)^\top \mathbb{P} + \mathbb{P}\mathcal{A}(\psi) - \mathcal{C}(\phi)^\top \mathcal{X}^\top - \mathcal{X}\mathcal{C}(\phi) + \lambda\mathbb{I}_n < 0 \quad \forall\psi \in \mathcal{V}_f,\ \forall\phi \in \mathcal{V}_h. \tag{12}$$

Moreover, the observer gain is computed by $L = \mathbb{P}^{-1}\mathcal{X}$.

Proof: The proof is straightforward from the LPV/LMI technique in [15]. The term $\lambda\mathbb{I}_n$ is added to get exponential convergence instead of asymptotic convergence. ■

Although (12) are the less restrictive LMI conditions that can exist in the literature, they are still strongly dependent on the Lipschitz constants of the nonlinearities, namely the set of vertices $\mathcal{V}_f$ and $\mathcal{V}_h$. They are not always feasible for all values of the bounds $\underline{\gamma}_{\phi_{ij}}, \bar{\gamma}_{\phi_{ij}}, \underline{\gamma}_{\psi_{ij}}$, and $\bar{\gamma}_{\psi_{ij}}$. To improve the feasibility, some guidelines have been given in [16]. Therefore, this note is a continuation of the work in [16]. We will not only improve the feasibility of LMI conditions as in [16], but we will show that LMIs (12) are still always feasible for some classes of nonlinear systems independently from the value of the Lipschitz constant of the nonlinearity of the system.

III. Feasible LMIs for Particular Families of Systems

For some classes of nonlinear systems, we can always guarantee feasibility of the LMIs for any bounds $\underline{\gamma}_{\phi_{ij}}, \bar{\gamma}_{\phi_{ij}}, \underline{\gamma}_{\psi_{ij}}$, and $\bar{\gamma}_{\psi_{ij}}$. This is the objective of this section.

A. Systems in canonical form

Here we will study the case where system (1) can be transformed into the following triangular form through a diffeomorphism $z = \Phi(x)$:

$$\begin{cases} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ f_z(z) \end{bmatrix} \\ y = z_1 \end{cases} \tag{13}$$

which can be written under the following compact form (14):

$$\begin{cases} \dot{z} = A_z z + B_z f_z(z) \\ y = C_z z \end{cases} \tag{14}$$

where A_z, C_z, and B_z have the companion structure as in [17]. Note that a more general class of systems with a nonlinearity $f_i(z_1,\ldots,z_i)$ in each component of the system can be considered, without loss of generality. However, for the sake of brevity, we investigate (13) with only a single nonlinear function in the last component of the system.

Now introduce the linear transformation

$$\zeta = \mathbb{T}_\tau z, \text{ where } \mathbb{T}_\tau \triangleq \text{diag}\left(\frac{1}{\tau}, \ldots, \frac{1}{\tau^n}\right) \quad (15)$$

which transforms (13) into

$$\dot{\zeta} = \tau A_z \zeta + \frac{1}{\tau^n} f_z(\mathbb{T}_\tau^{-1}\zeta). \quad (16)$$

Let us consider the following state observer corresponding to (16):

$$\dot{\hat{\zeta}} = \tau A_z \hat{\zeta} + \frac{1}{\tau^n} f_z\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big) + L\left(y - C_z \mathbb{T}_\tau^{-1}\hat{\zeta}\right) \quad (17)$$

where L, independent from τ, is the constant observer gain to be determined. Then the dynamics of the estimation error $e_\zeta = \zeta - \hat{\zeta}$ is expressed as

$$\dot{e}_\zeta = \tau\Big(A_z - LC_z\Big)e_\zeta + B_z \Delta f_z \quad (18)$$

where

$$\Delta f_z \triangleq \frac{1}{\tau^n}\left[f_z(\mathbb{T}_\tau^{-1}\zeta) - f_z\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big)\right]. \quad (19)$$

Applying [15, Lemma 7], there exist functions

$$\psi_j \;:\; \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$$

and constants $\underline{\gamma}_j \leq 0$ and $\bar{\gamma}_j \geq 0$, such that

$$\Delta f_z = \Big[\sum_{j=1}^{j=n} \frac{\psi_j}{\tau^{n-j}} \mathbf{e}_n^\top(j)\Big] e_\zeta \quad (20)$$

and

$$\underline{\gamma}_j \leq \gamma_j \leq \bar{\gamma}_j, \quad (21)$$

where $\mathbf{e}_n^\top(j)$ is the j^{th} element of the canonical basis of $\mathbb{R}^n$.

Similarly to (4), we introduce the affine matrix $\mathcal{A}(\tau, \Psi)$ defined as

$$\mathcal{A}\Big(\tau, \Psi\Big) = A_z + \sum_{j=1}^{n}\Big[\frac{1}{\tau^{1+(n-j)}}\psi_j \mathbf{e}_n^\top(j)\Big] \quad (22)$$

where $\Psi = \big[\psi_1, \ldots, \psi_n\big]^\top$. Then, the parameter Ψ belongs to a bounded convex set for which the set of vertices is given by

$$\mathcal{V}_{f_z} \triangleq \Big\{v \in \mathbb{R}^n : v_j \in \{\underline{\gamma}_j, \bar{\gamma}_j\}\Big\}. \quad (23)$$

From (18), (20), and (22), it follows that the dynamics of the estimation error becomes

$$\dot{e}_\zeta = \tau\Big[\mathcal{A}\big(\tau, \Psi\big) - LC_z\Big]e_\zeta. \quad (24)$$

Consequently, we can state the following corollary as a particular case of Theorem 1.

Corollary 2: Let $\mathcal{P} = \mathcal{P}^\top > 0$ and $\mathcal{X}$ be matrices of appropriate dimensions, and $\tau > 0$ is a scalar, such that the following LMI conditions hold:

$$\mathcal{A}\big(\tau, w\big)^\top \mathcal{P} + \mathcal{P}\mathcal{A}\big(\tau, w\big) - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0, \quad \forall w \in \mathcal{V}_{f_z}. \quad (25)$$

Then the observer (17) corresponding to (16), with $L = \mathcal{P}^{-1}\mathcal{X}^\top$, converges exponentially towards zero. Moreover, the estimated state $\hat{x} = \Phi^{-1}\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big)$ converges exponentially to the state x of the original system (1).

Proof: The proof is omitted. ∎

Corollary 2 is an intermediate gateway that leads straightforwardly to the next important result from the LMI point of view. Such a result is given in the following proposition.

Proposition 1: For any fixed values of the bounds $\underline{\gamma}_j$ and $\bar{\gamma}_j$, $j = 1, \ldots, n$, there exists $\tau^* > 0$ such that the LMIs (25) are feasible for any $\tau \geq \tau^*$.

Proof: Since (A_z, C_z) is observable, then there always exists a matrix $\mathcal{P} = \mathcal{P}^\top > 0$ and a matrix $\mathcal{X}$ such that

$$A_z^\top \mathcal{P} + \mathcal{P}A_z - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0.$$

On the other hand, from the definition of $\mathcal{A}\big(\tau, \Psi\big)$ in (22), we have

$$\lim_{\tau \to +\infty} \big(\mathcal{A}(\tau, w)\big) = A_z, \; \forall w \in \mathcal{V}_{f_z}.$$

Then from continuity of $\mathcal{A}\big(\tau, w\big)$ with respect to τ, there exists $\tau^* > 0$ large enough such that the LMI (25) holds for any $\tau \geq \tau^*$. ∎

Proposition 1 means that the LMIs (25) are always feasible for any global Lipschitz nonlinear function $f_z(.)$ independently from the value of its Lipschitz constant. This result is important in the LMI context since it always guarantees the design of an LMI-based exponential observer for any Lipschitz constant of the system. This is not the case in general for the arbitrary structure of the system where the feasibility of the LMIs depends strongly on the value of the Lipschitz constant of the system.

Remark 1: High-gain observer design is a particular case of the proposed methodology. Indeed, as can be seen in the proof of Proposition 1, a sufficiently large value of τ guarantees exponential stability of the estimation error. Such a sufficiently large value of τ leads to high values of the observer gain.

Remark 2: The result of this section remains valid for the more general class of systems (26) with multi-nonlinearities, described by the following equations:

$$\left\{\begin{array}{l} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_n \\ 0 \end{bmatrix} + \begin{bmatrix} f_1(z_1) \\ f_2(z_1, z_2) \\ \vdots \\ f_{n-1}(z_1, z_2, \ldots, z_{n-1}) \\ f_n(z) \end{bmatrix} \\ y = z_1. \end{array}\right. \quad (26)$$

The generalization is straightforward, therefore it is not necessary to provide the developments. On the other hand, we avoid repetition since in the next section, we will consider multi-nonlinearities in the system description.

B. Systems having feedforward structure

Consider the class of feedforward systems described by the equations below, which can be obtained by transforming (1) through the diffeomorphism $z = \Phi(x)$:

$$\begin{cases} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-2} \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} z_2 + f_1(z_3, \ldots, z_n) \\ z_3 + f_2(z_4, \ldots, z_n) \\ \vdots \\ z_{n-1} + f_{n-2}(z_n) \\ z_n \\ u(t) \end{bmatrix} \\ \quad = A_z z + f^{\text{feed}}(z) + Bu(t) \\ y = z_1 \end{cases} \tag{27}$$

where $u(t)$ is any known signal. In general, when we consider system (1) with the presence of control input, namely $f(x,u)$ and $h(x,u)$ instead of $f(x)$ and $h(x)$, we get the structure (27) by using some backstepping transformation techniques [18]. Such a structure is encountered in several works in the literature, namely in [19], [20] and the references therein.

Similarly to the previous section, by using the transformation (15) and the observer (17), we obtain

$$\dot{e}_\zeta = \tau \Big[\mathcal{A}_{\text{feed}}\big(\tau, \Psi\big) - LC_z\Big] e_\zeta \tag{28}$$

with

$$\mathcal{A}_{\text{feed}}\Big(\tau, \Psi\Big) = A_z + \sum_{i=1}^{n}\sum_{j=i+2}^{n}\Big[\tau^{j-(i+1)}\psi_{ij}^{\text{feed}}\mathbf{e}_n(i)\mathbf{e}_n^\top(j)\Big] \tag{29}$$

where ψ_{ij}^{feed}, independent from τ, comes from [15, Lemma 7] with

$$-\gamma_{f_i^{\text{feed}}} \leq \underline{\gamma}_{\psi_{ij}^{\text{feed}}} \leq \psi_{ij}^{\text{feed}} \leq \bar{\gamma}_{\psi_{ij}^{\text{feed}}} \leq \gamma_{f_i^{\text{feed}}} \tag{30}$$

where $\underline{\gamma}_{\psi_{ij}^{\text{feed}}} \leq 0$ and $\bar{\gamma}_{\psi_{ij}^{\text{feed}}} \geq 0$. From the structure (27), in this case f^{feed}, we have $\psi_{ij}^{\text{feed}} \equiv 0$ for $i = n-1$, $i = n$, and $\forall j = 1, \ldots, n$. It is obvious because from (27), the last two components of f^{feed} are zero. This means that the parameter Ψ belongs to an hyper-rectangle $\mathcal{S}_{f^{\text{feed}}}$ for which the set of vertices $\mathcal{V}_{f^{\text{feed}}}$ is defined as follows:

$$\mathcal{V}_{f^{\text{feed}}} = \Big\{\varphi \in \mathbb{R}^{n\times n} : \varphi_{ij} \in \{\underline{\gamma}_{\psi_{ij}^{\text{feed}}}, \bar{\gamma}_{\psi_{ij}^{\text{feed}}}\} \\ \varphi_{ij} = 0 \text{ for } i = n-1 \text{ and } i = n\Big\}. \tag{31}$$

It follows that

$$\lim_{\tau\to 0}\Big(\mathcal{A}_{\text{feed}}\Big(\tau, \Psi\Big)\Big) = A_z, \ \forall \Psi \in \mathcal{V}_{f^{\text{feed}}} \tag{32}$$

since ψ_{ij}^{feed} is bounded and independent from τ.

Now we are ready to state the following proposition.

Proposition 2: There exists $\tau^{\text{feed}} > 0$, such that the following LMI conditions hold $\forall \tau : 0 < \tau \leq \tau^{\text{feed}}$:

$$\mathcal{A}_{\text{feed}}\big(\tau, w\big)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_{\text{feed}}\big(\tau, w\big) - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0, \\ \forall w \in \mathcal{V}_{f^{\text{feed}}}, \tag{33}$$

where $\mathcal{P} = \mathcal{P}^\top > 0$ and $\mathcal{X}$ are matrices of appropriate dimensions, which are the decision variables of the LMIs (33). Then the observer (17) corresponding to (16), with $L = \mathcal{P}^{-1}\mathcal{X}^\top$, converges asymptotically. Moreover, the estimated state $\hat{x} = \Phi^{-1}\big(\mathbb{T}_\tau^{-1}\hat{\zeta}\big)$ converges asymptotically to the state x of the original system (1) for all τ satisfying $0 < \tau \leq \tau^{\text{feed}}$.

Proof: First, as for the Proposition 1, from observability of (A_z, C_z), we deduce there always exist a matrix $\mathcal{P} = \mathcal{P}^\top > 0$ and a matrix $\mathcal{X}$ such that

$$A_z^\top \mathcal{P} + \mathcal{P}A_z - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0.$$

On the other hand, we have

$$\mathcal{A}_{\text{feed}}\big(\tau, w\big)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_{\text{feed}}\big(\tau, w\big) - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z \\ = \overbrace{A_z^\top \mathcal{P} + \mathcal{P}A_z - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z}^{<0} \\ + \mathbb{S}(\tau, w)\mathcal{P} + \mathcal{P}\mathbb{S}(\tau, w)^\top \tag{34}$$

where

$$\mathbb{S}(\tau, w) \triangleq \sum_{i=1}^{n}\sum_{j=i+2}^{n}\Big[\tau^{j-(i+1)}\psi_{ij}^{\text{feed}}\mathbf{e}_n(i)\mathbf{e}_n^\top(j)\Big].$$

Then, from (32) and the continuity of $\mathcal{A}_{\text{feed}}\big(., w\big)$ with respect to τ (we can also use the *Archimedean* property), there exists $\tau^{\text{feed}} > 0$ such that

$$\mathcal{A}_{\text{feed}}\big(\tau^{\text{feed}}, w\big)^\top \mathcal{P} + \mathcal{P}\mathcal{A}_{\text{feed}}\big(\tau^{\text{feed}}, w\big) \\ - C_z^\top \mathcal{X} - \mathcal{X}^\top C_z < 0 \\ \forall w \in \mathcal{V}_{f^{\text{feed}}}. \tag{35}$$

On the other hand, we have

$$\mathcal{A}_{\text{feed}}\big(\tau, w\big) = \mathcal{A}_{\text{feed}}\left(\tau^{\text{feed}}, w^\tau\right)$$

with $w_{ij}^\tau = \left[\frac{\tau}{\tau^{\text{feed}}}\right]^{j-(i+1)} w_{ij}$. Since $\underline{\gamma}_{\psi_{ij}^{\text{feed}}} \leq 0$ and $\bar{\gamma}_{\psi_{ij}^{\text{feed}}} \geq 0$, and we have $w \in \mathcal{V}_{f^{\text{feed}}}$, then $w^\tau \in \mathcal{S}_{f^{\text{feed}}}$ for any $\tau \leq \tau^{\text{feed}}$, i.e.: $\frac{\tau}{\tau^{\text{feed}}} \leq 1$. Hence, from the convexity principle, the inequality (35) is preserved $\forall \tau \leq \tau^{\text{feed}}$, which ends the proof of Proposition 2. ■

IV. Extension to a More General Class of Systems

This section is devoted to a more general class of systems, which does not contain necessarily linear parts like in (13).

Consider the class of systems (1) which can be transformed to the following form through the diffeomorphism $z = \Phi(x)$:

$$\begin{cases} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \vdots \\ \dot{z}_{n-1} \\ \dot{z}_n \end{bmatrix} = \begin{bmatrix} \phi_1(z_1, z_2) \\ \phi_2(z_1, z_2, z_3) \\ \vdots \\ \phi_{n-1}(z_1, z_2, \ldots, z_{n-1}, z_n) \\ \phi_n(z) \end{bmatrix} \\ \quad = f^{\text{nl}}(z) \\ y = \varphi(z_1). \end{cases} \tag{36}$$

This extension can be useful in the sense that some real-world models are not in the form (13), and then they do not need to be transformed into (13) with complex structure of nonlinearities. For a motivating example, we can mention the tumor growth model investigated in [21] and the references therein. Such a tumor growth model is under the form (36). To avoid repetition and cumbersome notations, we consider, in this section, only systems under the nonlinear canonical form. Extension to systems having nonlinear feedforward structures can be straightforwardly obtained.

We use again the same change of variable (15), $\zeta = \mathbb{T}_\tau z$, to design an observer for (36). We then, first, introduce the state observer corresponding to ζ:

$$\dot{\hat{\zeta}} = \mathbb{T}_\tau f^{\mathrm{nl}}\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right) + L\left[y - \varphi\left(\tau\hat{\zeta}_1\right)\right] \tag{37}$$

where we used $\mathbb{T}_\tau^{-1} = \mathbb{T}_{\frac{1}{\tau}}$; $\hat{\zeta}$ is the estimate of ζ. Then the estimation of z is expressed as $\hat{z} = \mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}$. Hence, the estimation of the original state x is given by $\hat{x} = \Phi^{-1}\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right)$. Notice that in the case of the tumor growth model in [21], the system is under the form (36), then there is no need for nonlinear transformation. Therefore, we get directly from (37) an estimation of x as $\hat{x} = \mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}$.

Now let us come back to the convergence analysis of the estimation error $e_\zeta = \zeta - \hat{\zeta}$. We have

$$\begin{aligned}\dot{e}_\zeta = \mathbb{T}_\tau &\overbrace{\left[f^{\mathrm{nl}}\left(\mathbb{T}_{\frac{1}{\tau}}\zeta\right) - f^{\mathrm{nl}}\left(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\right)\right]}^{\Delta f^{\mathrm{nl}}} \\ &+ L\left[\varphi\left(\tau\zeta_1\right) - \varphi\left(\tau\hat{\zeta}_1\right)\right].\end{aligned} \tag{38}$$

By applying [15, Lemma 7] and after isolating the terms corresponding to the $(i+1)^{\mathrm{th}}$ component of the state and the i^{th} component of the nonlinearity, f_i^{nl}, we deduce that there exists functions

$$\psi_{ij}^{\mathrm{nl}} \ : \ \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \ \mathbb{R}$$

$$\varphi_1 \ : \ \mathbb{R} \times \mathbb{R} \longrightarrow \ \mathbb{R}$$

and constants $\underline{\gamma}_{\psi_{ij}^{\mathrm{nl}}}$, $\bar{\gamma}_{\psi_{ij}^{\mathrm{nl}}}$, $\underline{\gamma}_{\varphi_1}$, and $\bar{\gamma}_{\varphi_1}$, such that

$$\begin{aligned}\Delta f^{\mathrm{nl}} = \tau &\left[\sum_{i=1}^{n-1}\psi_{i,i+1}^{\mathrm{nl}}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(i+1)\right]e_\zeta \\ &+ \left[\sum_{i=1}^{n}\sum_{j=1}^{i}\left[\frac{1}{\tau^{i-j}}\psi_{ij}^{\mathrm{nl}}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(j)\right]e_\zeta\right]\end{aligned} \tag{39}$$

$$\varphi\left(\tau\zeta_1\right) - \varphi\left(\tau\hat{\zeta}_1\right) = \tau\varphi_1(t)C_z e_\zeta \tag{40}$$

and

$$\underline{\gamma}_{\psi_{ij}^{\mathrm{nl}}} \leq \psi_{ij}^{\mathrm{nl}}\left(t\right) \leq \bar{\gamma}_{\psi_{ij}^{\mathrm{nl}}} \tag{41}$$

$$\underline{\gamma}_{\varphi_1} \leq \varphi_1\left(t\right) \leq \bar{\gamma}_{\varphi_1} \tag{42}$$

where $\psi_{ij}^{\mathrm{nl}}(t)$ and $\varphi_1(t)$ are independent from τ; $\psi_{ij}^{\mathrm{nl}}(t) \triangleq \psi_{ij}^{\mathrm{nl}}\left(\zeta^{\hat{\zeta}_{j-1}}, \zeta^{\hat{\zeta}_j}\right)$ and $\varphi_1(t) \triangleq \varphi_1\left(\zeta_1, \hat{\zeta}_1\right)$ are introduced for simplification. Furthermore, we assume, as in the previous section, that

$$\underline{\gamma}_{\psi_{ij}^{\mathrm{nl}}} \leq 0 \text{ and } \bar{\gamma}_{\psi_{ij}^{\mathrm{nl}}} \geq 0, \quad \text{for all } j \leq i, i = 1, \ldots, n. \tag{43}$$

More importantly, we need the following assumption for the existence of the observer we propose:

$$\underline{\gamma}_{\psi_{i,i+1}^{\mathrm{nl}}} > 0 \text{ and } \underline{\gamma}_{\varphi_1} > 0. \tag{44}$$

Notice that conditions (44) are introduced first in [2] to guarantee the existence of a high-gain observer for the system (36). Authors in [2, Eq.(75), page 96] used a slightly different, but equivalent, condition, namely

$$0 < \alpha \leq \psi_{i,i+1}^{\mathrm{nl}}(t) \leq \beta \text{ and } \alpha \leq \varphi_1(t) \leq \beta \tag{45}$$

which can be obtained from (41)-(42) and (44) with

$$\alpha = \min\left(\min_{i=1,\ldots,n-1}\underline{\gamma}_{\psi_{i,i+1}^{\mathrm{nl}}}, \underline{\gamma}_{\varphi_1}\right),$$

$$\beta = \max\left(\max_{i=1,\ldots,n-1}\bar{\gamma}_{\psi_{i,i+1}^{\mathrm{nl}}}, \bar{\gamma}_{\varphi_1}\right).$$

Now, we introduce the following notations:

$$\boldsymbol{\psi}_t \triangleq \begin{bmatrix}\psi_{1,2}^{\mathrm{nl}}(t)\\ \psi_{2,3}^{\mathrm{nl}}(t)\\ \vdots\\ \psi_{n-1,n}^{\mathrm{nl}}(t)\end{bmatrix} \in \mathbb{R}^{n-1},\ \boldsymbol{\varphi}_t \triangleq \begin{bmatrix}\psi_{11}^{\mathrm{nl}}(t)\\ \psi_{21}^{\mathrm{nl}}(t)\\ \psi_{22}^{\mathrm{nl}}(t)\\ \psi_{31}^{\mathrm{nl}}(t)\\ \vdots\\ \psi_{n1}^{\mathrm{nl}}(t)\\ \vdots\\ \psi_{nn}^{\mathrm{nl}}(t)\end{bmatrix} \in \mathbb{R}^{\frac{n(n+1)}{2}} \tag{46}$$

$$\begin{aligned}\boldsymbol{A}\left(\boldsymbol{\psi}_t\right) &\triangleq \sum_{i=1}^{n-1}\psi_{i,i+1}^{\mathrm{nl}}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(i+1)\\ &= \begin{bmatrix}0 & \psi_{1,2}^{\mathrm{nl}}(t) & 0 & \ldots & 0 & 0\\ 0 & 0 & \psi_{2,3}^{\mathrm{nl}}(t) & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \ldots & \psi_{n-1,n}^{\mathrm{nl}}(t)\\ 0 & 0 & 0 & 0 & \ldots & 0\end{bmatrix}\end{aligned} \tag{47}$$

$$\boldsymbol{\mathcal{A}}_{\mathrm{nl}}\left(\tau, \boldsymbol{\varphi}_t\right) \triangleq \sum_{i=1}^{n}\sum_{j=1}^{i}\frac{1}{\tau^{1+i-j}}\psi_{ij}^{\mathrm{nl}}(t)\mathbf{e}_n(i)\mathbf{e}_n^\top(j) \tag{48}$$

$$\boldsymbol{C}\left(\varphi_1(t)\right) \triangleq \varphi_1(t)C_z.$$

The dynamic of the estimation error (38) can then be expressed under the following compact form:

$$\dot{e}_\zeta = \tau\Big[\boldsymbol{A}\left(\boldsymbol{\psi}_t\right) + \boldsymbol{\mathcal{A}}_{\mathrm{nl}}\left(\tau, \boldsymbol{\varphi}_t\right) - L\boldsymbol{C}\left(\varphi_1(t)\right)\Big]e_\zeta. \tag{49}$$

By definition (46) and assumption (41), the time-varying parameters $\boldsymbol{\psi}_t$ and $\boldsymbol{\varphi}_t$ belong to bounded convex sets for which the sets of vertices are respectively, given as follows:

$$\begin{aligned}\mathcal{V}_{\boldsymbol{\psi}} = \Big\{&\varphi \in \mathbb{R}^{n-1} : \varphi_i \in \{\underline{\gamma}_{\psi_{i,i+1}^{\mathrm{nl}}}, \bar{\gamma}_{\psi_{i,i+1}^{\mathrm{nl}}}\},\\ &\text{where } \underline{\gamma}_{\psi_{i,i+1}^{\mathrm{nl}}} > 0, i = 1, \ldots, n-1\Big\},\end{aligned} \tag{50}$$

$$\mathcal{V}_{\boldsymbol{\varphi}} = \Big\{ \boldsymbol{\rho} \in \mathbb{R}^{\frac{n(n+1)}{2}} : \boldsymbol{\rho} = \begin{bmatrix} \boldsymbol{\rho}_1 \\ \vdots \\ \boldsymbol{\rho}_n \end{bmatrix}, \boldsymbol{\rho}_i = \begin{bmatrix} \rho_{i1} \\ \vdots \\ \rho_{ii} \end{bmatrix}, \\ \rho_{ij} \in \{ \underline{\gamma}_{\psi_{ij}^{\mathrm{nl}}}, \bar{\gamma}_{\psi_{ij}^{\mathrm{nl}}} \}, \text{ s.t (43)} \Big\}. \quad (51)$$

Now we are ready to state the main proposition of this section.

Proposition 3: There exists $\tau_{\mathrm{nl}}^{\star} > 0$, such that $\forall \tau \geq \tau_{\mathrm{nl}}^{\star}$, there exist matrices $\mathcal{P} = \mathcal{P}^{\top} > 0$ and $\mathcal{X}$ of appropriate dimensions, and a scalar $\lambda > 0$, such that the following LMI holds:

$$\big[\boldsymbol{A}(v) + \boldsymbol{\mathcal{A}}_{\mathrm{nl}}(\tau, w)\big]^{\top} \mathcal{P} + \mathcal{P}\left[\boldsymbol{A}(v) + \boldsymbol{\mathcal{A}}_{\mathrm{nl}}(\tau, w)\right] \\ - \boldsymbol{C}(\kappa)^{\top}\mathcal{X} - \mathcal{X}^{\top}\boldsymbol{C}(\kappa) < -\lambda \mathbb{I}_n, \\ \forall v \in \mathcal{V}_{\boldsymbol{\psi}}, \forall w \in \mathcal{V}_{\boldsymbol{\varphi}}, \forall \kappa \in \{\underline{\gamma}_{\varphi_1}, \bar{\gamma}_{\varphi_1}\}. \quad (52)$$

Then the observer (37) with $L = \mathcal{P}^{-1}\mathcal{X}^{\top}$, converges exponentially. Moreover, the estimated state $\hat{x} = \Phi^{-1}\big(\mathbb{T}_{\frac{1}{\tau}}\hat{\zeta}\big)$ converges exponentially to the state x of the original system (1), $\forall \tau \geq \tau_{\mathrm{nl}}^{\star} > 0$.

Proof: Since $v \in \mathcal{V}_{\boldsymbol{\psi}}$, $\kappa \in \{\underline{\gamma}_{\varphi_1}, \bar{\gamma}_{\varphi_1}\}$, and taking in mind (44), it follows from [2, Lemma 2.1, page 96] that there exist matrices $\mathcal{P} = \mathcal{P}^{\top} > 0$ and $\mathcal{X}$ of appropriate dimensions, and a scalar $\lambda > 0$, such that

$$\boldsymbol{A}(v)^{\top}\mathcal{P} + \mathcal{P}\boldsymbol{A}(v) - \boldsymbol{C}(\kappa)^{\top}\mathcal{X} - \mathcal{X}^{\top}\boldsymbol{C}(\kappa) < -2\lambda\mathbb{I}_n, \\ \forall v \in \mathcal{V}_{\boldsymbol{\psi}}, \forall w \in \mathcal{V}_{\boldsymbol{\varphi}}, \forall \kappa \in \{\underline{\gamma}_{\varphi_1}, \bar{\gamma}_{\varphi_1}\}. \quad (53)$$

Also, since

$$\lim_{\tau \to 0} \big(\boldsymbol{\mathcal{A}}_{\mathrm{nl}}(\tau, w)\big) = 0_{n\times n}, \ \forall w \in \mathcal{V}_{\boldsymbol{\varphi}}$$

then it is obvious that $\exists \tau_{\mathrm{nl}}^{\star} > 0$ large enough, such that

$$\boldsymbol{\mathcal{A}}_{\mathrm{nl}}(\tau, w)^{\top}\mathcal{P} + \mathcal{P}\boldsymbol{\mathcal{A}}_{\mathrm{nl}}(\tau, w) < \lambda\mathbb{I}_n, \\ \forall w \in \mathcal{V}_{\boldsymbol{\varphi}}, \ \forall \tau \geq \tau_{\mathrm{nl}}^{\star}. \quad (54)$$

Hence, summing (53) and (54), the relation (52) is inferred. ■

Remark 3: Notice that the condition (44) may also be replaced by

$$\bar{\gamma}_{\psi_{i,i+1}^{\mathrm{nl}}} < 0 \text{ and } \bar{\gamma}_{\varphi_1} < 0 \quad (55)$$

and the result still remains valid. Indeed, the condition for the existence of solution to (53) is the strict monotonicity of the nonlinearities ϕ_i and φ with respect to the variables z_{i+1} and z_1, respectively.

V. Conclusion and Future Work

In this paper, we demonstrated that LMI-based approaches can also guarantee the design of nonlinear observers for a large class of nonlinear systems. We proposed LMI conditions for the synthesis of nonlinear observers and we showed that the feasibility of such LMIs is guaranteed for some families of nonlinear systems. While the feasibility of the LMIs is not ensured for arbitrary structure of the nonlinearities, it is shown that such feasibility guarantee is applicable to important families of systems, namely triangular systems and systems having feedforward structures.

In future work, we aim to apply the results to real-world models to show the effectiveness and superiority of the proposed techniques. We also aim to deepen the methods to propose new synthesis approaches based on LMI techniques combined with high-gain observer-based methodology, for systems having arbitrary structure of the nonlinearities, namely non-triangular systems.

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