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On Logical Dynamic Potential Games

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Abstract— **Logical dynamic games are dynamic games with logical dynamics describing the external state evolutionary process, which exist widely in real systems like the Boolean network of lactose operon in *Escherichia coli*. To the best of our knowledge, there is little attention on LDGs. In this paper, we aim at developing a framework for the analysis of dynamic games with logical dynamics under finite-horizon criteria. First, mathematical model of logical dynamic games is provided. A necessary and sufficient condition for the existence of pure feedback Nash equilibrium in a logical dynamic game is derived. Second, rigorous mathematical model of logical dynamic potential game is proposed, which establishes the relationship between logical dynamic games and corresponding optimal control problems. Third, we proved that a logical dynamic game is a logical dynamic potential game, if and only if, all the static sub-games are potential games. Finally, an example is provided to illustrate the theoretical results.**

I. Introduction

Recently, the logical dynamic system (LDS) is becoming a hot research topic, and it is embedded in various real systems, such as the biological, economical, and social systems etc [20], [21], [23]. The essential feature of the LDS is that the state variable belongs to a logic domain [22], which usually consists of a finite number of logic values. A typical LDS is Boolean network [23]. Due to the logical feature, analysis and synthesis of LDS are not easy using traditional mathematical tool until the emergence of a new kind of matrix product, called semi-tensor product (STP) of matrices [24]. STP is an effective tool for dealing with LDSs, which has been successfully applied to many finite-value systems such as Boolean networks [24], [30], [20], finite games [26], [27] and finite automaton [28], to name but a few.

One of the successes of STP in finite games is the verification of potential games, which is converted equivalently into the existence of the solution of linear equations [15], [26]. Moreover, Liu and Zhu proved that the linear equation method presents the lower bound in computational complexity for checking finite potential games [29]. Although the optimal control of LDSs has been studied extensively [20], [30], [31], to the best of our knowledge, works on dynamic games with logical dynamics are still lacking.

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As a successful interdisciplinary research area, dynamic non-cooperative game theory combines game theory and optimal control theory [3]. This paper falls into this category. The key concept in dynamic non-cooperative game theory is the dynamic Nash equilibrium (DNE) strategy profile, describing a situation where no agent has anything to gain by switching only its own strategy when other agents adopt the DNE strategy in the dynamic environment. In stochastic game, the DNE is also called Markov Nash equilibria [1], [11]. Indeed, results on dynamic non-cooperative game theory mainly emerge from investigating the existence of the DNE and developing of algorithms to obtain the DNE.

As the core problem, the existence of DNE has always been the focus of attention [1], [11], [7]. Shapley proved the existence of Markov Nash equilibria for two-person zero-sum stochastic games [1]. Subsequently, similar results were obtained for various stochastic games, such as continuous-time stochastic games [5], open-loop Nash equilibrium of stochastic games [9], infinite state stochastic games [6], just to name a few. However, most previous works are confined to randomized strategies. Compared with randomized strategies, pure strategies are relatively easy to be implemented in physical world. Although some preliminary explorations are reported in [4], [12], [13], determining the existence of the pure strategy Nash equilibrium in a stochastic game is *PSPACE*-hard [14]. Efficient methods are still missing to deal with this challenging problem.

For more than a decade, several attempts in acquiring pure DNE have been made via potential game theory [4], [7], [10]. The importance of potential games lies in three fundamental aspects: (i) existence of pure Nash equilibria, where the potential minimizers provide one of the pure Nash equilibria [16], [19]; (ii) availability of the Nash equilibria by some decentralized iterative strategies, such as logit response [17] or asynchronous myopic best response [18]; (iii) computation of pure Nash equilibria, which is equivalent to the optimization of a potential function [8]. By introducing potential function in the dynamic games, pure DNE were obtained for zero-sum two-person stochastic games and non-zero-sum stochastic games with additive reward functions in [13]. Deterministic Markov Nash equilibria for potential discrete-time stochastic games defined on Borel spaces were proposed in [4]. Just as Potters *et al*. pointed out, “for stochastic games, the existence of a potential function is mostly hard to prove” [13]. However, as far as we known, there are no works on logical dynamic games or logical dynamic potential games.

Motivated by the above analysis, here we aim at developing a framework for the analysis of dynamic potential games with logical dynamics under finite-horizon criteria. Particularly, we are interested in two problems: (i) how to combine the concepts of the potential games and LDSs; (ii) how to verify whether a logical dynamic game is a logical dynamic potential game.

Fortunately, we have solved the above problems successfully. Here we briefly present our main contributions as follows.

- A necessary and sufficient condition for the existence of pure feedback Nash equilibrium in a logical dynamic game is derived (Proposition 4.1). Using dynamic programming theory, it is proved that a logical dynamic game consists of a series of static sub-games. Then the existence of pure feedback Nash equilibrium is found equivalent to the existence of pure Nash equilibria of all the sub-games.
- Rigorous mathematical model of logical dynamic potential game is proposed in Definition 4.2, which establishes the relationship between logical dynamic games and corresponding optimal control problems. This logical dynamic potential game model is formally in parallel with static potential game theory in [16].
- We proved that a logical dynamic game is a logical dynamic potential game, if and only if, all the static sub-games are potential games (Theorem 4.4). Moreover, we find that a logical dynamic game is a logical dynamic potential game if its auxiliary game is a state-based potential game (Theorem 5.3 and 5.5).

II. Preliminaries

A. Vector Representation of Finite Set

Vector representation of finite state systems is based on STP of matrices, which is defined as follows.

Definition 2.1: [24] Consider two matrices $A \in \mathbb{R}_{m\times n}$ and $B \in \mathbb{R}_{p\times q}$. Let l be the least common multiple of n and p. The STP of A and B is defined as

$$A \ltimes B := \left(A \otimes I_{l/n}\right)\left(B \otimes I_{l/p}\right) \in \mathbb{R}_{ml/n\times ql/p}.$$

As STP is a generalization of the conventional matrix product, we omit "$\ltimes$" without confusion in the following. Consider a finite set $\mathscr{D}_k = \{1, 2, \cdots, k\}$, the vector form or vector expression of variable $x \in \mathscr{D}_k$ is defined as $\vec{x} := \delta_k^{k-x}$. Then there is a corresponding relationship between finite sets $\mathscr{D}_k$ and Δ_k.

Definition 2.2: (i) A function $f : \prod_{i=1}^n \mathscr{D}_{k_i} \to \mathscr{D}_{k_0}$ is called a logical function.
(ii) A function $g : \prod_{i=1}^n \mathscr{D}_{k_i} \to \mathbb{R}$ is called a pseudo logical function.

Proposition 2.3: [24] Consider the pseudo logical function $g : \prod_{i=1}^n \mathscr{D}_{k_i} \to \mathbb{R}$ and logical function $f : \prod_{i=1}^n \mathscr{D}_{k_i} \to \mathscr{D}_{k_0}$. Let $k = \prod_{i=1}^n k_i$, then using vector expression of finite set, there exists a unique vector $V_g \in \mathbb{R}^k$ and a unique matrix $M_f \in \mathbb{R}_{k_0\times k}$ such that

$$g(x) = V_g \ltimes_{i=1}^n \vec{x}_i,\ \ f(x) = M_f \ltimes_{i=1}^n \vec{x}_i,$$

where V_g and M_f are called the structure vector of g and the structure matrix of f, respectively.

B. Static Potential Games

This subsection provides preliminaries on finite games and potential games, which can be found in many standard textbooks on game theory, for instance, [32], [33].

Definition 2.4: A finite non-cooperative game in strategic-form can be described by a triple $G = \{N, A, C\}$, where

(i) $N = \{1, 2, \cdots, n\}$ is the set of agents;
(ii) $A = \prod_{i=1}^n A_i$ is the set of action profiles and $A_i = \{1, 2, \cdots, k_i\}$ is the set of actions of agent $i \in N$.
(iii) $C = \{c_1, c_2, \cdots, c_n\}$ is the set of all agents' cost functions, $c_i : A \to \mathbb{R}$ is the cost function of agent $i \in N$.

Denote the set of finite non-cooperative games G with $|N| = n$ and $|A_i| = k_i$ by $\mathcal{G}_{[n;k_1,\cdots,k_n]}$. It is obvious that the cost function c_i of a game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$ is a pseudo-logical function. Using the vector representation, we have

$$c_i(a_1, \cdots, a_n) = V_{c_i} \ltimes_{j=1}^n \vec{a}_j, \quad i \in N. \tag{1}$$

One of the core concepts in non-cooperative games is Nash equilibrium, which is defined as follows.

Definition 2.5: [32] Consider a non-cooperative game G, an action profile $a^* = (a_1^*, \cdots, a_n^*) \in A$ is a (pure) Nash equilibrium if for any $i \in N$

$$c_i(a_i^*, a_{-i}^*) \le c_i(a_i, a_{-i}^*), \quad \forall a_i \in A_i,\ a_{-i}^* \in A_{-i}, \tag{2}$$

where $A_{-i} = \prod_{j\ne i} A_j$.

However, pure Nash equilibria may not exist for general games. A special class of games possessing pure Nash equilibria is proposed by Rosenthal [34] and Slade [35], which are called potential games [37].

Definition 2.6: [37] A finite non-cooperative game $G = \{N, A, C\}$ is called an exact potential game if there exists a function $\rho : A \to \mathbb{R}$ such that, for $\forall a_i, b_i \in A_i, \forall a_{-i} \in A_{-i}$

$$c_i(a_i, a_{-i}) - c_i(b_i, a_{-i}) = \rho(a_i, a_{-i}) - \rho(b_i, a_{-i}), \forall i \in N,$$

where ρ is called a potential function of the game G.

However, it is not easy to verify whether a given finite game is a potential game or not [36]. [26] proved that the existence of the potential function equals to the existence of a solution of the following so called potential equation.

Lemma 2.7: Consider a game $G \in \mathcal{G}_{[n;k_1,\cdots,k_n]}$, set $\Gamma_i = \otimes_{l=1}^{i-1} I_{k_l} \otimes \mathbf{1}_{k_i}^\top \otimes_{l=i+1}^n I_{k_l}, \xi_i \in \mathbb{R}^{k_{-i}}, k_{-i} = \prod_{j\ne i} k_j, i \in N$. Then G is potential, if and only if, the following equation

$$\Gamma_p \xi = V_G^\top, \tag{3}$$

has a solution, where $V_G := [V_{c_2} - V_{c_1}, \cdots, V_{c_n} - V_{c_1}]$ and

$$\Gamma_p = \begin{bmatrix} -\Gamma_1^\top & \Gamma_2^\top & 0 & 0 & \cdots & 0 \\ -\Gamma_1^\top & 0 & \Gamma_3^\top & 0 & \cdots & 0 \\ \vdots & & & & \ddots & \\ -\Gamma_1^\top & 0 & 0 & 0 & \cdots & \Gamma_n^\top \end{bmatrix},\ \xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{bmatrix}. \tag{4}$$

Moreover, if the solution exists, then the potential function $P : A \to \mathbb{R}$ of the game G can be calculated by

$$P(\boldsymbol{a}) = (V_1 - \xi_1^\top \Gamma_1) \ltimes_{i=1}^n \vec{a}_i. \tag{5}$$

Equation (3) is called the potential equation of the game G.

III. Mathematic Model of Logical Dynamic Games

An LDS is described as follows [24]

$$x(t+1) = f(x(t), a_1(t), \cdots, a_n(t)), \quad (6)$$

where $x(t) \in X(=\mathscr{D}_k)$ and $a_j(t) \in A_j(=\mathscr{D}_{r_j})$ are logical variables, which represent the state of system and action of agent j at time t, respectively. Let $A := \prod_{j=1}^n A_j(=\mathscr{D}_r)$ be the set of action profiles, $\boldsymbol{a}(t) = [a_1(t), \cdots, a_n(t)]^\top \in A, r = \prod_{j=1}^n r_j$. And $f: X \times A \to X$ is the logical function.

Using vector representation of finite set, LDS (6) can be expressed as follows

$$\vec{\boldsymbol{x}}(t+1) = M_f \vec{\boldsymbol{x}}(t)\vec{\boldsymbol{a}}(t), \quad (7)$$

where $M_f \in \mathscr{L}_{k \times kr}$ is the structure matrix of logical function f. (7) is called the *algebraic form* of (6). For more details on how to obtain the structure matrix M_f, please refer to [24]. For convenience, denote by $\boldsymbol{x}_t := \boldsymbol{x}(t), \boldsymbol{a}_t := \boldsymbol{a}(t)$ in the sequel.

At each time t, agent $i \in N$ updates his action $a_i(t)$ according to its available information. The information gained by the agents is called the information structure of the LDS (6), which can be classified into different types according to the amount of available information in decision-making process. Here we only list some common information structures in the following, please refer to [3] for more details.

Definition 3.1: Consider the LDS (6), agent i's information structure is a (an)

(i) open-loop pattern if the available information at time t is $\{\boldsymbol{x}_0\}$;

(ii) closed-loop perfect state information pattern if the available information at time t is $\{\boldsymbol{x}_0, \cdots, \boldsymbol{x}_t\}$;

(iii) feedback information pattern if the available information at time t is $\{\boldsymbol{x}_t\}$.

In this paper, we suppose that the information structure of agent i is feedback information pattern. At each time step, agent i updates his action $a_i(t)$ according to the state $\boldsymbol{x}_t$. In other words, the strategy of agent i at time t can be described as $a_i(t) = \mu_t^i(\boldsymbol{x}_t)$, where μ_t^i is a logical function of agent i at time t. Finite-horizon optimization problem over time interval $\{0, 1, \cdots, \tau\}$ is considered in this paper, where τ is the terminal time. We focus on the pure strategy in this paper, i.e., $\mu_t^i : X \to A_i$ is a deterministic logical function. The set of all admissible strategies μ_t^i is denoted by U_t^i. The strategy of agent i is $\mu^i := [\mu_0^i, \mu_1^i, \cdots, \mu_\tau^i]$ and the strategy profile at time t is $\mu_t := [\mu_t^1, \mu_t^2, \cdots, \mu_t^n]$. The strategy of LDS (6) can be defined as

$$\boldsymbol{\mu} := (\mu_0, \mu_1, \cdots, \mu_\tau).$$

The objective function of agent i in this paper is of finite-horizon type. Given an initial state $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and an admissible strategy $\boldsymbol{\mu}$, the objective function of agent $i \in N$ is described as follows

$$J_i(\boldsymbol{x}_0, \boldsymbol{\mu}) = h_i(\boldsymbol{x}_{\tau+1}) + \sum_{t=0}^{\tau} g_i\Big(\boldsymbol{x}_t, \mu_t(\boldsymbol{x}_t)\Big), \quad (8)$$

where $h_i : X \to \mathbb{R}$ is the terminal cost function of agent $i \in N$, $g_i : X \times A \to \mathbb{R}$ is the stage cost function over each time step and τ is the optimization time. By virtue of vector representation of finite set, objective function (8) can be expressed as follows

$$J_i(\boldsymbol{x}_0, \boldsymbol{\mu}) = V_{h_i} \ltimes \vec{\boldsymbol{x}}_{\tau+1} + V_{g_i} \sum_{t=0}^{\tau} \vec{\boldsymbol{x}}_t \vec{\mu}_t, \ i = 1, \cdots, n, \quad (9)$$

where V_{h_i} and V_{g_i} are the structure vectors of h_i and g_i, respectively.

Obviously, the LDS (6) and the objective function (8) constitutes a logical dynamic non-cooperative game under corresponding information structure, which is called *logical dynamic game* (LDG) for convenience. The Nash equilibrium of a dynamic game with feedback information structure is called feedback Nash equilibrium (FNE) [3].

Definition 3.2: Consider the LDG described by (6)-(8), the strategy profile $\boldsymbol{\mu}^* = (\mu^{1*}, \mu^{2*}, \cdots, \mu^{n*})$ is called a (pure strategy) feedback Nash equilibrium if it satisfies the following inequalities recursively for all $i \in N$, $\forall \mu_t^i \in U_t^i$, $t = 0, 1, 2, \cdots, \tau$, $\forall \mu_0, \cdots, \mu_{t-1}$

$$\begin{aligned} &J_i\Big(\boldsymbol{x}_0, \big(\mu_0, \cdots, \mu_{t-1}, (\mu_t^{i*}, \mu_t^{-i*}), \mu_{t+1}^*, \cdots, \mu_\tau^*\big)\Big) \\ &\leq J_i\Big(\boldsymbol{x}_0, \big(\mu_0, \cdots, \mu_{t-1}, (\mu_t^{i}, \mu_t^{-i*}), \mu_{t+1}^*, \cdots, \mu_\tau^*\big)\Big). \end{aligned} \quad (10)$$

Remark 3.3: Obviously, the principle of optimality and Nash equilibrium condition should be satisfied simultaneously for a strategy profile being an FNE.

Problem formulation: In this paper, we aim at combining the concepts of potential games and LDGs. Such LDGs is called logical dynamic potential games (LDPG). We will explore the verification problem for an LDG being an LDPG.

IV. Logical Dynamic Potential Games

A. Decomposition of LDGs

Consider the LDG (6)-(8). According to dynamic programming theory, agent i's value function $V_t^i(\boldsymbol{x}_t)$ at time $t = 0, 1, \cdots, \tau+1$ for the Nash equilibrium strategy $\mu_t^* : X \to A$ $(t = 0, 1, \cdots, \tau)$ is

$$V_t^i(\boldsymbol{x}_t) = \min_{\mu_t^i, \cdots, \mu_\tau^i} \{h_i(\boldsymbol{x}_{\tau+1}) + \sum_{l=t}^{\tau} g_i(\boldsymbol{x}_l, \mu_l^i(\boldsymbol{x}_t), \mu_l^{-i*}(\boldsymbol{x}_t))\}, \quad (11)$$

where $\vec{\boldsymbol{x}}_{l+1} = M_f \vec{\boldsymbol{x}}_l \vec{\boldsymbol{a}}_l, \ l = t, \cdots, \tau$.

The value function $V_t^i(\boldsymbol{x}_t)$ satisfies the following iteration

$$\begin{aligned} V_t^i(\boldsymbol{x}_t) = \min_{\mu_t^i(\boldsymbol{x}_t)} \Big\{ & g_i\big(\boldsymbol{x}_t, \mu_t^i(\boldsymbol{x}_t), \mu_t^{-i*}(\boldsymbol{x}_t)\big) \\ & + V_{t+1}^i\Big(f\big(\boldsymbol{x}_t, \mu_t^i(\boldsymbol{x}_t), \mu_t^{-i*}(\boldsymbol{x}_t)\big)\Big)\Big\}, \end{aligned} \quad (12)$$

with $V_{\tau+1}^i(\boldsymbol{x}_{\tau+1}) = h_i(\boldsymbol{x}_{\tau+1})$. Then the LDG (6)-(8) is equivalent to a series of $\tau+1$ static game problems at time $t = 0, 1, \cdots, \tau$. By using

$$\omega_t^i(\boldsymbol{x}_t, \boldsymbol{a}) := g_i\big(\boldsymbol{x}_t, \boldsymbol{a}\big) + V_{t+1}^i\big(f(\boldsymbol{x}_t, \boldsymbol{a})\big),$$

the corresponding static game $G_t(\boldsymbol{x}_t)$ at time t is

$$G_t(\boldsymbol{x}_t) = \big\{N, \{A_i\}_{i \in N}, \{\omega_t^i(\boldsymbol{x}_t, \cdot)\}_{i \in N}\big\}. \quad (13)$$

We call $G_t\big(\boldsymbol{x}_t\big)$ the sub-game at time t.

Proposition 4.1: Consider the LDG described by (6)-(8), the FNE $\boldsymbol{\mu}^*$ exists, if and only if, each sub-game $G_t\big(\boldsymbol{x}_t\big)$ has at least one pure Nash equilibrium $\boldsymbol{a}_t^*,\ t=0,1,\cdots,\tau$. Furthermore, $\boldsymbol{\mu}^*=(\mu_0^*,\cdots,\mu_\tau^*)$ and $(\boldsymbol{a}_0^*,\boldsymbol{a}_1^*,\cdots,\boldsymbol{a}_\tau^*)$ has the following relationship

$$\boldsymbol{a}_t^*=\mu_t^*(\boldsymbol{x}_t),\ t=0,1,\cdots,\tau.$$

Proof: The conclusion is obvious according to above analysis. So we omit the proof details. ■

B. Structure of LDPGs

This section considers that how to combine the concepts of potential games and LDGs. Recall the definition of static potential games, where the potential minimizers provide one of the pure Nash equilibria for such games. A natural idea is to introduce the concept of potential games into LDGs by associating an optimal control problem with the LDG, such that the solution of the optimal control problem is a DNE of the LDG. An optimal control problem of an LDS can be described as follows

$$\begin{aligned}\min_{\boldsymbol{\mu}} J(\boldsymbol{x}_0,\boldsymbol{\mu})&=\psi(\boldsymbol{x}_{\tau+1})+\sum_{t=0}^{\tau}\phi\big(\boldsymbol{x}_t,\mu_t(\boldsymbol{x}_t)\big)\\ \text{subject to }\ \boldsymbol{x}_{t+1}&=f(\boldsymbol{x}_t,\boldsymbol{a}_t),\end{aligned}\tag{14}$$

where $\phi: X\times A\to\mathbb{R}$ is the stage cost function, $\psi: X\to\mathbb{R}$ is the terminal cost function, and $f: X\times A\to X$ is the logical dynamic function defined in (6).

Definition 4.2: The LDG (6)-(8) is called an LDPG, if there exists an optimal control problem (14) satisfying the following conditions for all $i\in N,\ \forall\tilde{\mu}_t^i,\mu_t^i\in U_t^i,\ t=0,1,2,\cdots,\tau,\ \forall\mu_0,\cdots,\mu_{t-1}$

$$J_i\big(\boldsymbol{x}_0,M_t\big)-J_i\big(\boldsymbol{x}_0,\tilde{M}_t\big)=J\big(\boldsymbol{x}_0,M_t\big)-J\big(\boldsymbol{x}_0,\tilde{M}_t\big),\tag{15}$$

where $\boldsymbol{\mu}^*=(\mu_0^*,\mu_1^*,\cdots,\mu_\tau^*)$ is the solution of the optimal control problem (14), and

$$\begin{array}{rcl}M_t&=&\big(\mu_0,\cdots,\mu_{t-1},(\mu_t^i,\mu_t^{-i}),\mu_{t+1}^*,\cdots,\mu_\tau^*\big),\\ \tilde{M}_t&=&\big(\mu_0,\cdots,\mu_{t-1},(\tilde{\mu}_t^i,\mu_t^{-i}),\mu_{t+1}^*,\cdots,\mu_\tau^*\big).\end{array}\tag{16}$$

Remark 4.3: Rigorous mathematical model of LDPG is proposed in Definition 4.2, which establishes the relations between LDGs and corresponding optimal control problems. This LDPG model is formally in parallel with static weighted potential game model proposed in [16].

According to the dynamic programming theory, the value function $V_t(\boldsymbol{x}_t),\ t=0,1,\cdots,\tau$ of problem (14) is

$$V_t(\boldsymbol{x}_t)=\min_{\mu_t(\boldsymbol{x}_t),\cdots,\mu_\tau(\boldsymbol{x}_\tau)}\left\{\psi(\boldsymbol{x}_{\tau+1})+\sum_{p=t}^{\tau}\phi\big(\boldsymbol{x}_p,\mu_p(\boldsymbol{x}_p)\big)\right\},$$

where $\boldsymbol{x}_{p+1}=f(\boldsymbol{x}_p,\mu_p),\ p=t,\cdots,\tau$. The value function $V_t(\boldsymbol{x}_t)$ satisfies the dynamic programming formulation

$$V_t(\boldsymbol{x}_t)=\min_{\mu_t}\big\{\phi\big(\boldsymbol{x}_t,\mu_t\big)+V_{t+1}\big(f(\boldsymbol{x}_t,\mu_t(\boldsymbol{x}_p))\big)\big\},\tag{17}$$

and $V_{\tau+1}(\boldsymbol{x}_{\tau+1})=\psi(\boldsymbol{x}_{\tau+1})$.

Denote by $\Omega_t(\boldsymbol{x}_t,\boldsymbol{a}) := \phi\big(\boldsymbol{x}_t,\boldsymbol{a}\big)+V_{t+1}\big(f(\boldsymbol{x}_t,\boldsymbol{a})\big)$. Then the finite-horizon optimal control problem (14) can be regarded as a series of τ static minimization problems with $\Omega_t(\boldsymbol{x}_t,\boldsymbol{a})$ as the cost function at time $t=0,1,\cdots,\tau$.

Theorem 4.4: The LDG (6)-(8) is an LDPG with (14) as its associated optimal control problem, if and only if, each sub-game $G_t\big(\boldsymbol{x}_t\big)$ is a static potential game with $\Omega_t(\boldsymbol{x}_t,\boldsymbol{a})$ as its potential function, i.e., for every $\boldsymbol{x}_t\in X,\ \forall i\in N,\ \forall a_{-i}\in A_{-i},\ \forall a_i,b_i\in A_i$,

$$\omega_t^i(\boldsymbol{x}_t,a_i,a_{-i})-\omega_t^i(\boldsymbol{x}_t,b_i,a_{-i})=\Omega_t(\boldsymbol{x}_t,a_i,a_{-i})-\Omega_t(\boldsymbol{x}_t,b_i,a_{-i}).$$

Furthermore, the FNE $\boldsymbol{\mu}^*$ is

$$\mu_t^*(\boldsymbol{x}_t)=\arg\min_{\boldsymbol{a}\in A}\Omega_t(\boldsymbol{x}_t,\boldsymbol{a}).\tag{18}$$

Proof: The proof is obvious according to above analysis and we omit the details to meet the page limitation. ■

V. Verification of LDPGs

Consider the LDG (6)-(8). Construct an auxiliary state-based game as $G:=\{N,X,\{A_i\}_{i\in N},\{g_i(\boldsymbol{a},\boldsymbol{x})\}_{i\in N}\}$.

Definition 5.1: A state-based game G is called a state-based potential game, if there exists a function $\phi: A\times X\to\mathbb{R}$ the following two conditions for any agent $i\in N,\ \forall a_i,a_i'\in A_i,\ \forall\boldsymbol{x},\boldsymbol{x}'\in X$ are satisfied

$$g_i(a_i',a_{-i},\boldsymbol{x})-g_i(a_i,a_{-i},\boldsymbol{x})=\phi(a_i',a_{-i},\boldsymbol{x})-\phi(a_i,a_{-i},\boldsymbol{x}),$$
$$g_i(\boldsymbol{a},\boldsymbol{x})-g_i(\boldsymbol{a},\boldsymbol{x}')=\phi(\boldsymbol{a},\boldsymbol{x})-\phi(\boldsymbol{a},\boldsymbol{x}').$$

Remark 5.2: State-based potential games have been discussed in many works, such as [2], [7]. The definition of state-based potential game in this paper is a little different from existing definitions in [2]. We do not impose any restrictions on the logical dynamics (6).

Theorem 5.3: Consider the LDG (6)-(8). If the auxiliary game $G=\{N,X,\{A_i\}_{i\in N},\{g_i(\boldsymbol{a},\boldsymbol{x})\}_{i\in N}\}$ is a state-based potential game and the terminal cost function $h_i(\boldsymbol{x})$ satisfies $h_i(\boldsymbol{x})=h_j(\boldsymbol{x}),\ \forall i,j\in N$, then the LDG is an LDPG.

Proof: We omit the proof details to save the space. ■

A natural question is how to determine whether a state-based game is state-based potential game or not. The following result provides a method to deal with this problem.

Lemma 5.4: A state-based game $G=\big\{N,X,\{A_i\}_{i\in N},\{g_i(\boldsymbol{a},\boldsymbol{x})\}_{i\in N}\big\}$ is a state-based potential game, if and only if, there exists a function $\phi: A\times X\to\mathbb{R}$ and $n+1$ functions $d_i: A_{-i}\times X\to\mathbb{R}, i=1,2,\cdots,n,\ d_{n+1}: A\to\mathbb{R}$, such that

$$\begin{cases}g_i(\boldsymbol{a},\boldsymbol{x})=\phi(\boldsymbol{a},\boldsymbol{x})+d_i(a_{-i},\boldsymbol{x}),\ i\in N,\\ g_i(\boldsymbol{a},\boldsymbol{x})=\phi(\boldsymbol{a},\boldsymbol{x})+d_{n+1}(\boldsymbol{a}).\end{cases}\tag{19}$$

Proof: The details are omitted to save the space. ■

Theorem 5.5: A state-based game $G=\big\{N,X,\{A_i\}_{i\in N},\{g_i(\boldsymbol{a},\boldsymbol{x})\}_{i\in N}\big\}$ is a state-based potential game, if and only if, the following linear equation has a solution

$$\Gamma_{sp1}\xi=0,\tag{20}$$

where $\dot{\Gamma}_i=\otimes_{p=1}^{i-1}I_{r_p}\otimes\mathbf{1}_{r_i}^\top\otimes_{q=i+1}^{n}I_{r_q}\otimes I_k$ and

$$\Gamma_{sp1}=\begin{bmatrix}-\dot{\Gamma}_1^\top&0&\cdots&0&(I_k\otimes\mathbf{1}_r^\top)\\0&-\dot{\Gamma}_2^\top&\cdots&0&(I_k\otimes\mathbf{1}_r^\top)\\\vdots&&\ddots&&\vdots\\0&0&\cdots&-\dot{\Gamma}_n^\top&(I_k\otimes\mathbf{1}_r^\top)\end{bmatrix}.\tag{21}$$

Furthermore, if the solution exists, the potential function can be expressed as

$$\phi(\boldsymbol{a},\boldsymbol{x}) = (V_{g_1} - \xi_1^\top \dot{\Gamma}_1^\top) \ltimes \vec{\boldsymbol{a}}\vec{\boldsymbol{x}}, \tag{22}$$

where ξ_1 is the first $k\prod_{i=2}^n r_i$ elements of ξ.

Proof: The details are omitted to save the space. ■

VI. Examples

Consider a reduced Boolean model for the lactose operon in *Escherichia coli* (Chapter 1.3, page 15, [38]), which models the gene regulation dynamics via two glucose control mechanisms: inducer exclusion and catabolite repression. The detailed dynamics is described as follows

$$\begin{cases} x_1(t+1) = \neg a_1(t) \wedge (x_2(t) \vee x_3(t)), \\ x_2(t+1) = \neg a_1(t) \wedge a_2(t) \wedge x_1(t), \\ x_3(t+1) = \neg a_1(t) \wedge [a_2(t) \vee (a_3(t) \wedge x_1(t))], \end{cases} \tag{23}$$

where $x_1, x_2, x_3 \in \{0,1\}$ are state variables which denote the lac mRNA, lactose in high concentrations and lactose in medium concentrations, respectively; $a_1, a_2, a_3 \in \{0,1\}$ are control inputs which represent the extracellular glucose, high extracellular lactose and medium extracellular lactose, respectively. The state set and action profile set are

$$\begin{aligned} X &= \{000, 001, 010, 011, 100, 101, 110, 111\}; \\ A &= \{000, 001, 010, 011, 100, 101, 110, 111\}. \end{aligned}$$

Using the vector representation, we can obtain

$$\begin{aligned} M_f = \delta_8[&8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8, \\ &8,8,8,8,8,8,8,8,8,8,1,1,1,1,5,3,3,3,7,1,1,1,5, \\ &3,3,3,7,3,3,3,7,4,4,4,8,4,4,4,8,4,4,4,8]. \end{aligned}$$

The information structure is feedback pattern. For a given strategy $\boldsymbol{\mu}$, the objective function of agent $i\ (=1,2,3)$ is

$$J_i(\boldsymbol{x}_0, \boldsymbol{\mu}) = h_i(\boldsymbol{x}_{\tau+1}) + \sum_{t=0}^{\tau} g_i\Big(\boldsymbol{x}_t, \mu_t(\boldsymbol{x}_t)\Big), \ \tau = 10,$$

where the terminal cost function are set as follows

$$\begin{aligned} h_1(\boldsymbol{x}) &= [3,2,1,4,6,2,2,5] \ltimes \vec{\boldsymbol{x}} := V_{h_1} \ltimes \vec{\boldsymbol{x}}, \\ h_2(\boldsymbol{x}) &= [6,7,5,0,3,3,4,2] \ltimes \vec{\boldsymbol{x}} := V_{h_2} \ltimes \vec{\boldsymbol{x}}, \\ h_3(\boldsymbol{x}) &= [4,0,2,2,1,3,2,5] \ltimes \vec{\boldsymbol{x}} := V_{h_3} \ltimes \vec{\boldsymbol{x}}. \end{aligned}$$

The stage cost function $g_i(\boldsymbol{x},\boldsymbol{a}),\ i=1,2,3$ are omitted here. According to Theorem 4.1 and Algorithm 1, we have

$$\begin{aligned} V_{11}^i(\boldsymbol{x}) &= V_{h_i} \ltimes \vec{\boldsymbol{x}} \\ \omega_{10}^i(\boldsymbol{x},\boldsymbol{a}) &= g_i(\boldsymbol{x},\boldsymbol{a}) + V_{11}^i(f(\boldsymbol{x},\boldsymbol{a})) \\ &= g_i(\boldsymbol{x},\boldsymbol{a}) + h_i(f(\boldsymbol{x},\boldsymbol{a})) \\ &= (V_{g_i} + V_{h_i}M_f) \ltimes \vec{\boldsymbol{x}} \ltimes \vec{\boldsymbol{a}},\ i=1,2,3. \end{aligned} \tag{24}$$

The sub-game at time $t=10$ is

$$G_{10}(\boldsymbol{x}) = \big\{N, \{A_i\}_{i\in N}, \{\omega_{10}^i(\boldsymbol{x},\cdot)\}_{i\in N}\big\}.$$

The feedback Nash equilibrium of $G_{10}(\boldsymbol{x})$ is $\boldsymbol{a}_{10}^* = \mu_{10}^*(\boldsymbol{x})$, which can be calculated as

$$\begin{aligned} &\mu_{10}^*(\boldsymbol{x}) = 000, \text{ if } \boldsymbol{x}=000; && \mu_{10}^*(\boldsymbol{x}) = 000, \text{ if } \boldsymbol{x}=001; \\ &\mu_{10}^*(\boldsymbol{x}) = 000, \text{ if } \boldsymbol{x}=010; && \mu_{10}^*(\boldsymbol{x}) = 111, \text{ if } \boldsymbol{x}=011; \\ &\mu_{10}^*(\boldsymbol{x}) = 011, \text{ if } \boldsymbol{x}=100; && \mu_{10}^*(\boldsymbol{x}) = 000, \text{ if } \boldsymbol{x}=101; \\ &\mu_{10}^*(\boldsymbol{x}) = 111, \text{ if } \boldsymbol{x}=110; && \mu_{10}^*(\boldsymbol{x}) = 111, \text{ if } \boldsymbol{x}=111. \end{aligned}$$

The logical expression of μ_{10}^* is

$$\begin{cases} \mu_{10}^{1*}(\boldsymbol{x}) = (x_1 \wedge x_3) \vee x_2 \\ \mu_{10}^{2*}(\boldsymbol{x}) = (x_1 \wedge x_3) \vee (\neg x_3 \wedge x_2) \\ \mu_{10}^{3*}(\boldsymbol{x}) = (x_1 \wedge x_3) \vee (\neg x_3 \wedge x_2), \end{cases}$$

whose algebraic expression is

$$\vec{\boldsymbol{a}}_{10}^* = K_{10}^* \ltimes \vec{\boldsymbol{x}} = \delta_8[1,1,1,8,4,1,8,8] \ltimes \vec{\boldsymbol{x}}.$$

The value function and the sub-game is calculated by

$$\begin{aligned} &V_t^i = V_{g_i}(I_8 \otimes K_t^*)R_8 + V_{t+1}^i M_f (I_8 \otimes K_t^*)R_8, \\ &V_{\omega_t^i} = V_{g_i} + V_{t+1}^i M_f,\ t = 0,1,2,\cdots,9,10. \end{aligned}$$

Then we can obtain the FNE via Algorithm 1.

$$\begin{cases} \vec{\boldsymbol{a}}_t^* = \vec{\boldsymbol{a}}_{10}^* = K_{10}^* \ltimes \vec{\boldsymbol{x}},\ t=1,2,\cdots,9. \\ \vec{\boldsymbol{a}}_0^* = K_0^* \ltimes \vec{\boldsymbol{x}} = \delta_8[8,4,1,8,4,1,8,8] \ltimes \vec{\boldsymbol{x}}. \end{cases}$$

Therefore, the FNE of the LDG is

$$\boldsymbol{\mu}^* = [\mu_0^*(\boldsymbol{x}), \mu_1^*(\boldsymbol{x}), \cdots, \mu_9^*(\boldsymbol{x}), \mu_{10}^*(\boldsymbol{x})],\ \forall \boldsymbol{x} \in X.$$

According to (24) and using Lemma 2.7, it is easy to verify that all sub-games $G_t(\boldsymbol{x}),\ \forall t$ are potential games with the common potential function $P_t(\boldsymbol{x},\boldsymbol{a}) = P(\boldsymbol{x},\boldsymbol{a})$ in Table I.

TABLE I

Potential function vector

$P(\boldsymbol{x},\boldsymbol{a})$	000	001	010	011	100	101	110	111
000	-3	-1	-1	0	-1	0	0	-1
001	-5	0	0	-1	0	-1	-1	1
010	-10	-9	-9	-4	-9	-4	-4	-1
011	3	-1	-1	0	-1	0	0	-3
100	8	1	1	-3	1	-3	-3	2
101	-18	-12	-12	-1	-12	-1	-1	5
110	-4	0	0	-2	0	-2	-2	-6
111	5	8	8	5	8	5	5	-2

Therefore, we conclude that the LDG (23) is an LDPG.

Denote accumulated costs $J_i(t)$ of each agent i as follows

$$\begin{aligned} J_i(t) &:= \sum_{p=0}^{t} g_i(\boldsymbol{x}_p, \boldsymbol{a}_p^*),\ t=0,1,\cdots,10; \\ J_i(11) &:= \sum_{p=0}^{10} g_i(\boldsymbol{x}_p, \boldsymbol{a}_p^*) + h_i(\boldsymbol{x}_11),\ i=1,2,3. \end{aligned}$$

The dynamics of accumulated costs for each agent from the initial state $\boldsymbol{x}(0) \in \{010,100,101,110\}$ is shown in Fig. 1.

Denote accumulated potentials $J_p(t)$ of each agent $i = 1,2,3$ under the DNE strategy as follows

$$J_p(t) := \sum_{p=0}^{t} P(\boldsymbol{x}_p, \boldsymbol{a}_p^*),\ t=0,1,\cdots,10.$$

The dynamics of accumulated potentials from the initial state $\boldsymbol{x}(0) \in \{010,100,101,110\}$ is shown in Fig. 2.

VII. Concluding Remarks

A framework for the analysis of LDGs under finite-horizon criteria is established in this paper. We have focused on the mathematical model of LDG, existence of FNE, and verification of LDPGs. First, mathematical model of logical dynamic

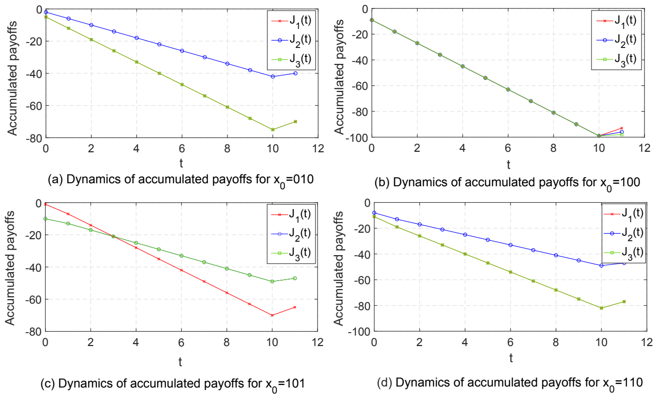

Fig. 1. Dynamics of accumulated costs $J_i(t)$ from initial state $\boldsymbol{x}(0) \in \{010, 100, 101, 110\}$

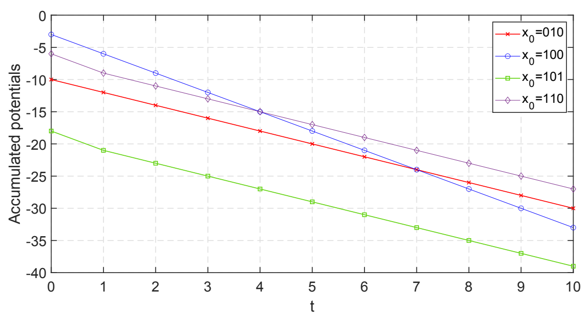

Fig. 2. Dynamics of accumulated potentials $J_p(t)$ from initial state $\boldsymbol{x}(0) \in \{010, 100, 101, 110\}$

games is provided. A necessary and sufficient condition for the existence of pure feedback Nash equilibrium in a logical dynamic game is derived. Second, rigorous mathematical model of logical dynamic potential game is proposed, which establishes the relationship between logical dynamic games and corresponding optimal control problems. Third, a logical dynamic game is proved to be a logical dynamic potential game, if and only if, all static sub-games are potential games.

Further works focus on the analysis and synthesis of LDGs with uncertainty and infinite-horizon criteria.

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