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# Relaxed controls and measure controls

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***Abstract*— In this paper, we introduce a new idea to generalize the concept of relaxed control to the framework of measure differential equations, recently introduced in [15]. A relaxed control is defined as a probability measure on the space of controls, and, similarly, a measure control is a feedback relaxed control which depends on the measure distribution on the state space representing the state of the system. Relaxed controls are useful to solve optimal control and stabilization problems. On the other side, measure differential equations allow deterministic modeling of uncertainty, finite-speed diffusion, concentration, and other phenomena. Moreover, it represents a natural generalization of Ordinary Differential Equations to measures.**

**We establish regularity properties of measure controls to ensure existence and uniqueness of trajectories and show applications to stabilization problems.**

## I. INTRODUCTION

Most controlled systems can be written as: $\dot{x} = f(x, u)$, where the system state $x$ belongs to a manifold, and the control $u$ to a metric space. The approach of geometric control theory is based on considering the family of vector fields $f_u = f(x, u)$, $u \in U$ and the properties of the Lie Algebra generated by them, [1], [13]. Control problems can be solved by open loops, $t \to u(t)$, and closed loops or feedback, $x \to u(x)$. However, in optimal control and stabilization problems, one has often to resort to relaxed control, see [23].

A relaxed control is a measure $\tilde{u}$ on the control set $U$. Given a relaxed control, we obtain an averaged dynamics by integrating over the control variable $x \to \int f(x, u)\, d\tilde{u}(u)$. The relaxed control may depend on the state $\tilde{u} = \tilde{u}(x)$ generating a feedback by averaging. Such dynamics can be realized by oscillatory signals as the dither, see [22], [24]. The class of control systems admitting a relaxed stabilizing feedback is strictly larger than those admitting a classical stabilizing feedback [3]. One of the most challenging aspects of relaxed feedback is their regularity, which may not be inherited by the regularity of $\tilde{u}$, for instance in weak sense. Our first contribution is the use of optimal transport theory [20] to achieve regularity results for relaxed feedback. More precisely, we show that regularity w.r.t. the Wasserstein distance guarantees Lipschitz continuity of the generated average feedback.

A continuous stabilizing feedback may fail to exist due to topological obstructions as in the famous Brockett's example [6]. Moreover, relaxed feedback may also fail to stabilize systems for which an open loop stabilizing control exists for every initial condition [3]. Thus we resort to more general type of feedback mechanism related to evolution equations for measures. Our starting point is the concept of measure differential equation (briefly MDE), which can be seen as a natural generalization of ODEs to measures. The evolution of a measure is given by a measure vector field, assigning a probability measure on every fiber of the tangent bundle prescribing the velocities along which the measure mass is moving. A fairly complete theory, in terms of existence and uniqueness, was developed in the paper [15]. In particular, solutions are constructed by approximations formed by traveling Dirac deltas on a lattice, called lattice approximate solutions (briefly LAS). Since solutions are interpreted in weak sense, uniqueness for Cauchy problems is not achievable, but it is possible to construct a Lipschitz semigroup. Moreover, extensions are available to measure differential inclusions [4], [10], [14], equations with sources [19], [18], superposition principle [7], biology and other applications [11], [16], [17].

This paper introduces generalization of relaxed controls using this framework. First, we define a random vector field, which is a map assigning to every measure on the state space a measure on the tangent bundle, which is compatible with the given control system. This can be seen as a measure vector field whose values are included in the controlled velocities and generalize the concept of relaxed control. This concept still does not capture a possible nonlocal dependence on feedback mechanisms. Indeed, if the state of the system depends on a measure distributed on the state space, as in multi-agent systems, then the dynamics should depend on the whole measure [8], [9], [18]. We then introduce the definition of measure control, which is a map associating a relaxed feedback to every measure on the state space. The former may depend on the global structure of the measure, not only on local information.

Existence and uniqueness of trajectories to measure controls can be achieved by applying the theory of measure differential equations to measure vector fields compatible with the given control systems. The key regularity assumption is a Lipschitz continuity w.r.t. to the Wasserstein distance of the measure control. More precisely, the relaxed feedback associated to a measure will depend on a Lipschitz fashion on the state variable, and, the relaxed feedback computed at a given point will depend on a Lipschitz fashion on the measure. Such regularity ensures existence of trajectories to measure controls, as well as the existence of a Lipschitz

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semigroup (obtained as limit of LAS-type approximations). Finally, we pass to show the power of the proposed approach. The classical circle stabilization problem is solved by means of a regular measure control.

## II. Measures and Wasserstein distance

We use the standard notation for the Euclidean n-dimensional space $\mathbb{R}^n$, its tangent bundle $T\mathbb{R}^n = \mathbb{R}^n \times \mathbb{R}^n$, and the base projection $\pi_1 : T\mathbb{R}^n \to \mathbb{R}^n$, $\pi_1(x, v) = x$. For every $A \subset \mathbb{R}^n$, we denote by $\chi_A$ the characteristic function of $A$ and by $\mathcal{C}_c^\infty(A)$ the set of compactly supported smooth functions.

A Polish space $(X, d)$ is a complete and separable metric space. We denote by $\mathcal{M}(X)$ the set of positive Radon measures with finite mass on $X$, by $\mathcal{M}_c(X)$ the subset of measures with compact support, by $|\cdot|$ the total variation norm (or total mass), and by $Supp(\cdot)$ the support operator. The subset $\mathcal{P}(X) \subset \mathcal{M}(X)$ indicates the probability measures, i.e. $\mu \in \mathcal{M}(X)$ such that $|\mu| = \mu(X) = 1$. Given a Borel map $\phi : X_1 \to X_2$, with $(X_i, d_i)$ Polish spaces, and $\mu \in \mathcal{M}(X_1)$, we set $\phi\#\mu(A) = \mu(\phi^{-1}(A)) = \mu(\{x \in X_1 : \phi(x) \in A\})$, for every Borel set $A$. The $\otimes$ product allows to combine a measure $\mu \in \mathcal{M}(X_1)$ with a family of measures $\nu_x \in \mathcal{P}(X_2)$, $x \in X_1$, by setting

$$\int_{X_1\times X_2} \phi(x, v)\, d(\mu \otimes_x \nu_x) = \int_{X_1} \int_{X_2} \phi(x, v) d\nu_x(v)\, d\mu(x).$$

Notice that every measure $\Sigma$ on the product $X_1 \times X_2$ can be written by disintegration (see [2]) as $\Sigma = \pi_1\#\Sigma \otimes_x \nu_x$ with $\nu_x$ probability measures on $X_2$.

We are now ready to introduce the Wasserstein distance, which is defined by solving the optimal transport problem among measures. We refer the reader to [20] for the general theory of optimal transport. Consider a Polish space $(X, d)$, $\mu$, $\nu \in \mathcal{M}(X)$, with $|\mu| = |\nu|$, then a transference plan $\tau$ between $\mu$ and $\nu$ is a measure $\tau \in \mathcal{M}(X \times X)$ such that:

$$\tau(A_1 \times \mathbb{R}^n) = \mu(A_1), \quad \tau(\mathbb{R}^n \times A_2) = \nu(A_2),$$

for every Borel sets $A_1$, $A_2 \subset X$. In simple words, $\tau$ is a generalization of a map pushing the measure $\mu$ onto the measure $\nu$. Given a transference plan $\tau$ we can define an associated cost by: $J(\tau) = \int_{X^2} d(x, y)\, d\tau(x, y)$. The Monge-Kantorovich optimal transport problem is formulated as the minimization of $J(\tau)$ over all transference plan and the Wasserstein distance is defined by:

$$W^X(\mu, \nu) = \inf_{\tau\in P(\mu,\nu)} J(\tau).$$

where $P(\mu, \nu)$ is the set of transference plans between $\mu$ and $\nu$. We denote by $P^{opt}(\mu, \nu)$ the set of optimal transference plans (with cost equal to $W(\mu, \nu)$) and endow the space $\mathcal{M}(X)$ with the topology given by the Wasserstein distance, which metrizes the weak convergence over compact sets. The existence of an optimal transference plan and properties of the Wasserstein distance can be found in [20], [21].

For future use, we state the following:

*Lemma 2.1:* Let $(X, d)$ be a polish space, $\mu$, $\nu \in \mathcal{P}(X)$ with compact support and $\varphi : X \to \mathbb{R}^n$ Lipschitz continuous. Then $|\int \varphi(d\mu - d\nu)| \leq Lip(\varphi)\, W(\mu, \nu)$, where $Lip(\varphi)$ is the Lipschitz constant of $\varphi$.

*Proof:* Consider $\tau \in P^{opt}(\mu, \nu)$, then we can write

$$\left| \int \varphi(x)(d\mu - d\nu) \right| \leq \int_{X\times X} |\varphi(x) - \varphi(y)| d\tau(x, y) \leq$$

$$\leq Lip(\varphi) \int_{X\times X} d\tau(x, y) = Lip(\varphi)\, W(\mu, \nu).$$

■

Following the approach of [15], we introduce the following definition.

*Definition 2.1:* A Measure Vector Field (MVF) is a map $V : \mathcal{M}(\mathbb{R}^n) \to \mathcal{M}(T\mathbb{R}^n)$ such that $\pi_1\#V = \mu$.

Notice that, by disintegration ([2]), for every MVF $V$ and $\mu \in \mathcal{M}(\mathbb{R}^n)$ there exists a family of probability measures $\nu_x = \nu_x[V, \mu]$ such that $V[\mu] = \mu \otimes_x \nu_x$. In simple terms, an MVF is a map associating, to every measure on the state space, a measure on the space of controlled velocities, thus naturally linked to the concept of relaxed controls, as illustrated in next section.

## III. Control systems and measure vector fields

Our starting point is a classical control system:

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^n,\ u \in U, \tag{1}$$

where $U \subset (X, d)$, with $(X, d)$ a Polish space, and $f : \mathbb{R}^n \times U \to T\mathbb{R}^n$ is a continuous map. To illustrate the main concepts without dealing with regularity issues, we will make the following assumptions:

(A) $U$ is compact and $f$ is Lipschitz continuous in both variables, i.e. there exists $L > 0$ such that $|f(x, u) - f(y, v)| \leq L(|x - y| + d(u, v))$ for all $x, y, u, v$.

To (1) we associate the differential inclusion:

$$\dot{x} \in F(x), \quad F(x) = \{v : v = f(x, u),\ u \in U\}. \tag{2}$$

For the equivalence of the set of solutions to (1) and (2) we refer the reader to [5].

### A. Relaxed controls

We recall the classical definition of relaxed control [23].

*Definition 3.1:* A relaxed control is a probability measure on the control space, i.e. $\tilde{u} \in \mathcal{P}(U)$.

Every relaxed control gives rise to a vector field $f_{\tilde{u}}(x) = \int f(x, u) d\tilde{u}(u)$.

*Definition 3.2:* A relaxed feedback control is a map $\tilde{u} : \mathbb{R}^n \to \mathcal{P}(U)$. We denote by $RF(U)$ the set of relaxed feedback controls.

Given a relaxed feedback $\tilde{u}$, we can associate a vector field $x \to f_{\tilde{u}(x)}(x) = \int f(x, u) d\big(\tilde{u}(x)\big)(u)$, but, as underlined in [3], the map $x \to f_{\tilde{u}(x)}(x)$ may fail to inherit regularity from $\tilde{u}$ w.r.t. to the weak convergence. Here we show how Lipschitz continuity w.r.t. the Wasserstein distance guarantees the Lipschitz continuity of the associated vector field. This is due to the fact that the Wasserstein distance metrizes the weak convergence (on compact sets.) We have the following:

*Proposition 3.1:* Consider the control system (1) satisfying (A) and a relaxed feedback control $\tilde{u}$. If $\tilde{u}$ is Lipschitz continuous for the Wasserstein distance, then the vector field $x \to f_{\tilde{u}(x)}(x)$ is Lipschitz continuous.

*Proof:* Define the vector field $g(x) = f_{\tilde{u}(x)}(x)$. We can estimate:

$$|g(x) - g(y)| = \left|\int f(x,u)\, d(\tilde{u}(x))(u) - \int f(y,u)\, d(\tilde{u}(y))(u)\right| \leq \left|\int f(x,u)\, d(\tilde{u}(x))(u) - \int f(y,u)\, d(\tilde{u}(x))(u)\right| + \left|\int f(y,u)\, d(\tilde{u}(x))(u) - \int f(y,u)\, d(\tilde{u}(y))(u)\right| = I_1 + I_2.$$

For $I_1$, we have:

$$I_1 \leq \int Lip(f_u)|x-y|\, d(\tilde{u}(x))(u) \leq (\sup_u Lip(f_u))\ |x-y|. \tag{3}$$

On the other side, by Lemma 2.1, we get:

$$I_2 \leq Lip(f_y)W(\tilde{u}(x), \tilde{u}(y)) \leq Lip(f_y)Lip(\tilde{u})\ |x-y|, \tag{4}$$

where $Lip(\tilde{u})$ indicates the Lipschitz constant of $\tilde{u}$ for the Wasserstein distance. By assumption (A), we have $(\sup_u Lip(f_u)) \leq L$, and $(\sup_y Lip(f_y)) \leq L$, thus combining (3) and (4) we conclude. ■

We immediately get the following:

*Proposition 3.2:* Consider (1) satisfying (A) and a relaxed feedback control $\tilde{u}$. If $\tilde{u}$ is Lipschitz continuous for the Wasserstein distance, for every initial datum $x_0$ there exists a unique solution to the Cauchy problem $\dot{x} = f_{\tilde{u}(x)}(x)$, $x(0) = x_0$.

### B. Random and measure vector fields

Here we introduce generalizations of relaxed controls, starting with random vector fields.

*Definition 3.3:* A random vector field (briefly RVF) associated to (1) is a map $v : \mathcal{P}(\mathbb{R}^n) \to \mathcal{P}(T\mathbb{R}^n)$ such that $Supp(v(x)) \subset \{(x,v) : v \in F(x)\} \subset T_x\mathbb{R}^n$.

To every RVF $v$ we can associate an MVF $V^v$ by setting: $V^u[\mu] = \mu \otimes_x v(x)$.

*Remark 1:* The idea behind the definition of RVF is as follows. For the classical control system (1), a control value is given by $\bar{u} \in U$. This, in turn, determines a map $\mathbb{R}^n \to T\mathbb{R}^n$, $x \to f(x, \bar{u}) =: f_{\bar{u}}(x)$, i.e. a vector field. This can be generalized by the concept of selection to the differential inclusion (2), that is a map $\mathbb{R}^n \to T\mathbb{R}^n$, $x \to \varphi(x) \in F(x)$. Notice that the map $f_{\bar{u}}$ is a specific selection based on the parametrization of $F$ given by the control system, see [5]. A critical aspect is the regularity of such maps. The control value $\bar{u}$ can be seen as a constant map $u : \mathbb{R}^n \to U$, $u(x) = \bar{u}$, or having the regularity of $f$ w.r.t. to the argument $u$. The same aspect of regularity is critical for selections as well, for instance requiring the continuity of $\varphi$. The concept of RVF is a natural generalization, passing from a map $\mathbb{R}^n \to T\mathbb{R}^n$ to a map $\mathbb{R}^n \to \mathcal{P}(T\mathbb{R}^n)$.

*Remark 2:* Notice that the measure vector field $V^u$ does not take into account the mass distribution of the argument $\mu$. Therefore the RVF generates an MVF depending on the independent variable $x \in \mathbb{R}^n$, but not on the mass distribution of $\mu$. The more general concept of measure control will be introduced in Section V.

Relaxed controls can be seen as a special case of RVF.

*Proposition 3.3:* To every relaxed control $\tilde{u}$ we can associate a RVF by setting $v(\tilde{u}) : \mathbb{R}^n \to T\mathbb{R}^n$, $v(\tilde{u})(x) = f(x, \cdot)\#\tilde{u}$. The vector field associated to a relaxed control is the average vector field obtained integrating the corresponding RVF over the fibers.

*Proof:* We have $f_{\tilde{u}}(x) = \int f(x,u) d\tilde{u}(u) = \int f(x,u) d(v(\tilde{u})(x))(u)$ as stated. ■

The generalization to measures of RVF is captured by the concept of MVF associated to a control system defined as follows.

*Definition 3.4:* An MVF associated to (1) is a map $V : \mathcal{M}(\mathbb{R}^n) \to \mathcal{M}(T\mathbb{R}^n)$ such that $\pi_1\#V[\mu] = \mu$ and $Supp(V[\mu]) \subset \{(x,v) : v \in F(x)\}$.

In simple words an MVF associated to (1) is a measure vector field that is compatible with the velocity constraints given by the control system. Notice that, by disintegration, for an MVF $V$ we can always write $V[\mu] = \mu \otimes_x \nu_x[V, \mu]$. This expression has to be compared with the measure vector field defined by an RVF: $V^v[\mu] = \mu \otimes_x v(x)$. The crucial difference is that the probability measure $v(x)$ does not depend on the argument (measure) $\mu$.

## IV. Trajectories of measure vector fields

In this section, we discuss the construction of trajectories for measure vector fields. Our approach follows [15]. Given a measure vector field $V$ we can associate the measure differential equation:

$$\dot{\mu} = V[\mu]. \tag{5}$$

Since the evolving object is a measure, we need to resort to weak solutions:

*Definition 4.1:* A (weak) solution to (5) is a map $\mu : [0,T] \to \mathcal{M}(\mathbb{R}^n)$ such that for every $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and almost every $t$:

$$\frac{d}{dt}\int_{\mathbb{R}^n} f(x)\, d\mu(t)(x) = \int_{T\mathbb{R}^n} (\nabla f(x) \cdot v)\ dV[\mu(t)](x,v), \tag{6}$$

where we assume that $\int_{T\mathbb{R}^n} (\nabla f(x) \cdot v)\ dV[\mu(t)](x,v)$ is defined for a.e. $t$ and belongs to $L^1([0,T])$, and $\int f\, d\mu(t)$ is absolutely continuous in $t$.

A Cauchy problem for an MDE is obtained assigning an initial condition:

$$\dot{\mu} = V[\mu], \qquad \mu(0) = \mu_0. \tag{7}$$

We now report some results from [15] on existence and uniqueness of solutions to Cauchy problems (7). We start formulating a sublinear growth condition on the MVF:

(H1) There exists $C > 0$ s.t. for every $\mu \in \mathcal{P}(X)$ with compact support it holds:

$$\sup_{(x,v) \in Supp(V[\mu])} |v| \leq C \left(1 + \sup_{x \in Supp(\mu)} |x|\right).$$

Hypothesis (H1) ensures boundedness of weak solutions. Continuity w.r.t. the Wasserstein metrics on $\mathbb{R}^n$ and $T\mathbb{R}^n$ guarantees existence of solutions. We pass directly to formulating assumptions for the existence of a semigroup of solutions. We start introducing a functional to formulate Lipschitz-type continuity for an MVF.

*Definition 4.2:* Consider $V_1, V_2 \in \mathcal{P}(T\mathbb{R}^n)$, set $\mu_i = \pi_1 \# V_i$, $i = 1, 2$, define $\mathcal{T}(V_1, V_2) = \{T \in P(V_1, V_2) : \pi_{13} \# T \in P^{opt}(\mu_1, \mu_2)\}$, where $\pi_{13} : (T\mathbb{R}^n)^2 \to (\mathbb{R}^n)^2$ is the bases projection $\pi_{13}(x, v, y, w) = (x, y)$, and

$$\mathcal{W}(V_1, V_2) = \inf_{T \in \mathcal{T}(V_1, V_2)} \left\{ \int |z - w| \; dT(x, z, y, w) \right\}. \quad (8)$$

Notice that $\mathcal{T}$ is the set of transference plans between $V_1$ and $V_2$ such that the projection on the base is optimal. Unfortunately, $\mathcal{W}$ fails to be a metric since the triangular inequality does not hold, see [15]. We are now ready to formulate the second assumption.

(H2) For every $R > 0$ there exists $K = K(R) > 0$ such that if $Supp(\mu), Supp(\nu) \subset B(0, R)$ then

$$\mathcal{W}(V[\mu], V[\nu]) \le K \; W(\mu, \nu). \quad (9)$$

Hypothesis (H2) (together with (H1)) guarantees the existence of a Lipschitz semigroup of trajectories, defined as:

*Definition 4.3:* For an MVF $V$ satisfying (H1) and $T > 0$, $S : [0, T] \times \mathcal{M}_c(\mathbb{R}^n) \to \mathcal{M}_c(\mathbb{R}^n)$ is a Lipschitz semigroup of trajectories if the following holds for $\mu, \nu \in \mathcal{M}_c(\mathbb{R}^n)$, $|\mu| = |\nu|$, and $t, s \in [0, T]$:

i) $S_0 \mu = \mu$ and $S_t \, S_s \, \mu = S_{t+s} \, \mu$;

ii) the map $t \mapsto S_t \mu$ is a solution to (5);

iii) for every $R > 0$ there exists $C(R) > 0$ such that if $Supp(\mu), Supp(\nu) \subset B(0, R)$ then:

$$Supp(S_t \mu) \subset B(0, e^{Ct}(R + 1)), \quad (10)$$

$$W(S_t \mu, S_t \nu) \le e^{C(R)t} W(\mu, \nu), \quad (11)$$

$$W(S_t \mu, S_s \mu) \le C(R) \; |t - s|. \quad (12)$$

The following holds.

*Theorem 4.1:* If $V$ satisfies (H1)-(H2) then for every $T > 0$ there exists a Lipschitz semigroup of trajectories.

The proof is entirely similar to the proof of the existence of weak solutions to MDEs, see Theorem 2 of [15]. The only difference is the extension from $\mathcal{P}(\mathbb{R}^n)$ to $\mathcal{M}(\mathbb{R}^n)$.

## V. Measure Controls

A measure vector field can be generated by control strategies depending on the datum given by a measure (distribution) on the state space. To capture this possibility we give the following definition.

*Definition 5.1:* A measure control is a map $\tilde{u} : \mathcal{M}(\mathbb{R}^n) \to RF(U)$. We define the associated MVF by $V^{\tilde{u}}[\mu] = \mu \otimes_x \big(f(x, \cdot) \# (\tilde{u}[\mu](x))\big)$.

We are interested in understanding the regularity of measure controls, in particular to ensure the existence of trajectories. This can be achieved via the regularity of the associated MVF and its trajectories. For this, we define the following regularity for measure controls.

*Definition 5.2:* A measure control $\tilde{u}$ is called Wasserstein-Lipschitz continuous if the following holds. Given $\mu$, $\nu \in \mathcal{M}(\mathbb{R}^n)$, $x \in Supp(\mu)$, $y \in Supp(\nu)$, we have:

$$W(\tilde{u}[\mu](x), \tilde{u}[\mu](y)) \le K(\mu) \cdot |x - y|,$$

$$W(\tilde{u}[\mu](x), \tilde{u}[\nu](x)) \le K(x) \cdot W(\mu, \nu).$$

Moreover, the constant $K(\mu)$ is uniformly bounded for uniformly bounded supports, and $K(x)$ is uniformly bounded on compact sets.

### A. Trajectories of measure controls

Here we show that the introduced Wasserstein-Lipschitz regularity is sufficient to ensure the existence of a semigroup of trajectories. More precisely, we have the following:

*Theorem 5.1:* Consider a measure control $\tilde{u}$ and the associated MVF $V^{\tilde{u}}$. If (A) holds and $\tilde{u}$ is Wasserstein-Lipschitz continuous, then $V^{\tilde{u}}$ satisfies (H1) and (H2). Therefore there exists a Lipschitz semigroup of trajectories of $V^{\tilde{u}}$.

*Proof:* Consider a measure $\mu$, fix $(x, v) \in Supp(V^{\tilde{u}}[\mu])$ and $u_0 \in U$, then (assuming w.l.o.g. $L \ge 1$):

$$|v| \le \sup_{v \in Supp\left(f(x,\cdot) \# (\tilde{u}[\mu](x))\right)} |v| \le$$

$$\le L \cdot \Big( |f(x, u_0)| + \sup_{u \in Supp\left(\tilde{u}[\mu](x)\right)} d(u, u_0) \Big) \le$$

$$\le L \cdot \big( |f(x, u_0)| + diam(U, u_0) \big),$$

where $diam(U, u_0) = \max_{u \in U} d(u_0, u)$, thus (H1) holds true.

Let us now pass to (H2). Given $\mu$, $\nu \in \mathcal{M}(\mathbb{R}^n)$, set $V_1 = V^{\tilde{u}}[\mu])$ and $V_2 = V^{\tilde{u}}[\nu])$. We need to estimate:

$$\mathcal{W}(V_1, V_2) = \inf_{T \in \mathcal{T}(V_1, V_2)} \left\{ \int |z - w| \; dT \right\}.$$

Fix $T \in \mathcal{T}(V_1, V_2)$ and assume that $T$ is a Dirac mass centered at $(\bar{x}, \bar{z}, \bar{y}, \bar{w})$. Then we can estimate:

$$\int |z - w| \; dT = |\bar{z} - \bar{w}| = |f(\bar{x}, \bar{u}) - f(\bar{y}, \bar{v})|$$

for some $\bar{u}$, $\bar{v} \in U$. Here we used the equality: $V^{\tilde{u}}[\mu] = \mu \otimes_x \big(f(x, \cdot) \# (\tilde{u}[\mu](x))\big)$, and the same for $\nu$. By assumption (A), we have:

$$|f(\bar{x}, \bar{u}) - f(\bar{y}, \bar{v})| \le L \cdot \big( |\bar{x} - \bar{y}| - |\bar{u} - \bar{v}| \big).$$

Since we can approximate every plan in $\mathcal{T}$ with a finite sum of Dirac masses, we obtain:

$$\mathcal{W}(V_1, V_2) \le \inf_{T \in \mathcal{T}(V_1, V_2)} \left\{ \int L(|x - y| + |u - v|) \; dT \right\}$$

$$\le L \cdot \inf_{T \in \mathcal{T}} \Big( W(\mu, \nu) + \int |u - v| \, d\tau(T) \Big), \quad (13)$$

where $\tau(T) \in P(\tilde{u}[\mu](x), \tilde{u}[\nu](y))$ is obtained from $T$ by taking measurable inverses of the maps $f(x, \cdot)$, $f(y, \cdot)$, and we have used the fact that $\pi_{13} \# T \in P^{opt}(\mu, \nu)$. Notice that for every $\pi \in P^{opt}(\mu, \nu)$ we can, by disintegration, construct a $T \in \mathcal{T}$ such that the fiber component is given by an arbitrary $\tau \in P(\tilde{u}[\mu](x), \tilde{u}[\nu](y))$. Since $\tilde{u}$ is Wasserstein-Lipschitz continuous, we can estimate:

$$\inf_{\tau \in P(\tilde{u}[\mu](x),\tilde{u}[\nu](y))} \int |u - v| \, d\tau \leq$$

$$W(\tilde{u}[\mu](x), \tilde{u}[\mu](y)) + W(\tilde{u}[\mu](y), \tilde{u}[\nu](y)) \leq$$

$$K(\mu) \cdot |x - y| + K(x) \cdot W(\mu, \nu).$$

Therefore we get:

$$\inf_{T \in \mathcal{T}} \Big( \int |u - v| \, d\tau(T) \Big) \leq \big( K(\mu) W(\mu, \nu) + K(x) W(\mu, \nu) \big). \tag{14}$$

Now, combing (13) and (14), we get:

$$\mathcal{W}(V_1, V_2) \leq (1 + K(\mu) + K(x)) \cdot W(\mu, \nu).$$

For $\mu$, $\nu$ with compact support, the constants $K(\mu)$ and $K(y)$ are uniformly bounded, thus assumption (H2) holds true. ■

*Remark 3:* The assumption (A) on (uniform) Lipschitz continuity of the map $f$ in Theorem 5.1 can be relaxed to local Lipschitz continuity. In fact, to prove assumption (H2) we need only a local estimate w.r.t. the support of the measures.

### *B. Stabilization via measure controls: the circle problem*

A long standing problem is the existence of a continuous stabilizing feedback, under the assumption that every single state can be asymptotically driven to the origin. There are well-known topological obstructions [6] preventing such a feedback to exist.

Here we illustrate how measure controls can overcome the classical circle stabilization problem. This is defined as follows. Consider the control system:

$$\dot{\theta} = u, \quad u \in \{-1, +1\}, \tag{15}$$

where $\theta \in \mathbb{S}^1$, the unit circle in $\mathbb{R}^2$. We consider an arclength parametrization of $\mathbb{S}^1$ so it can be identified with the interval $[0, 2\pi]$ (assuming 0 coincides with $2\pi$.) Thus the system evolves either clockwise or counterclockwise at an angular speed equal to 1. A continuous stabilizing feedback $u(\theta)$ to $\theta = 0$ does not exist. This is a simple case of a cut locus for $\mathbb{S}^1$ geodesics starting form $\theta = 0$. Stabilizing controls can be found for instance using hybrid controls, see [12].

We now define a regular measure control stabilizing the system (15) to $\theta = 0$. First define the sector $A = \{\theta : |\theta - \pi| < \eta\}$, where $0 < \eta < \pi/2$. Let $\psi : A \to [0, 1]$ be a smooth function such $\psi(\pi \pm \eta) = 0$, $\psi(\pi) = 1$, $\psi(\pi - \theta) = \psi(\pi + \theta)$, and $\psi$ increasing on $A \cap [\pi/2, \pi]$. Then given $\mu \in \mathcal{M}(\mathbb{S}^1)$ we define the cumulative distribution of $\mu$ on the sector $A$ as follows. For $\theta \in A$, set:

$$F_\mu(\theta) = \mu([\pi - \eta, \theta]).$$

Let $m_\mu = F_\mu(\pi + \eta) = \mu(A)$ and define the dynamics on the sector $A$ as follows. Set $\theta_\mu = \inf\{\theta \in A : F_\mu(\theta) = \frac{1}{2} m_\mu\}$ and $c_1 = F_\mu(\theta_\mu) - \frac{m_\mu}{2}$, $c_2 = \frac{m_\mu}{2} - F_\mu(\theta_\mu-)$ so that $\mu(\{\theta_\mu\}) = c_1 + c_2$. We set:

$$\tilde{u}[\mu] = \begin{cases} \delta_1 & \theta < \theta_\mu \\ \delta_{-1} & \theta > \theta_\mu \\ \frac{1}{\mu(\{\theta_\mu\})} (c_1 \delta_1 + c_2 \delta_{-1}) & \text{if } x = \theta_\mu, \mu(\{\theta_\mu\}) > 0 \end{cases}. \tag{16}$$

In simple words, $\theta_\mu$ is the mid point of the mass of $\mu$ located in $A$ and we move half mass to the left and half to the right. This results in a regular measure control stabilizing the system to $\theta = 0$. Indeed, we have the following:

*Lemma 5.1:* The measure control $\tilde{u}$ given by (16) satisfies the Wasserstein-Lipschitz continuity property.

*Proof:* Notice that the quantities $m_\mu$, $\theta_\mu$, $c_1$, and $c_2$ are Lipschitz continuous w.r.t. the Wasserstein distance. Therefore, we conclude using standard Lipschitz regularity w.r.t. the state space of the controlled dynamics. ■

We are ready to prove that the measure control $\tilde{u}$ given by (16) stabilizes the system. However, we are now dealing with measures distributed on the state space $\mathbb{S}^1$, thus we need to define the appropriate concept of stabilization.

*Definition 5.3:* Consider a control system (1) satisfying (A). We say that a measure control $\tilde{u}$ stabilized the system to the origin if the following holds. For every trajectory $\mu(\cdot)$ of the corresponding MVF $V^{\tilde{u}}$ we have $W(\mu(t), \delta_0) \to 0$, with convergence estimates uniform for uniformly compact support of the initial data.

This definition implies that the convergence to the origin is guaranteed for every initial datum for the Wasserstein distance. In particular, the whole mass of the measure must converge to zero as the time tends to infinity. We have the following:

*Lemma 5.2:* Let $\tilde{u}$ be the measure control on $\mathbb{S}^1$ given by (16). Then there exists $\delta t > 0$ such that if $\mu(\cdot)$ is a trajectory of the corresponding MVF $V^{\tilde{u}}$ then

$$\mu(t + \delta t)(A) \leq \frac{3}{4} \mu(t)(A).$$

*Proof:* Assume $\mu$ absolutely continuous and define $\theta_{1/4}$ such that $F_\mu(\theta_{1/4}) = \frac{m_{\mu(t)}}{4}$ and, similarly, $\theta_{3/4}$ such that $F_\mu(\theta_{3/4}) = \frac{3m_{\mu(t)}}{4}$. Assume $|\theta_{1/4} - (\pi - \eta)| \leq |\theta_{3/4} - (\pi + \eta)|$ being the opposite case entirely similar. By definition of $\tilde{u}$ we have that the mass on $A \cap [\pi - \eta, \theta_{1/4}]$ rotates counterclockwise (angular speed $-1$) for small times. We claim that this mass exits the sector $A$ within time $\delta t = \theta_{1/4} - (\pi - \eta)$. Indeed, since $|\theta_{1/4} - (\pi - \eta)| \leq |\theta_{3/4} - (\pi + \eta)|$ on the time interval $[t, t + \delta t]$ at most mass of $\frac{m_{\mu(t)}}{4}$ can exit the sector $A$ from the point $\pi + \eta$. Therefore on the time interval $[t, t + \delta t]$ the mass not yet exited from $A$ from the point $\pi - \eta$ is at most $\frac{m_{\mu(t)}}{3}$, thus $\tilde{u}$ is defined to be equal to $-1$ on the time interval $[t, t + \delta t]$ on $A \cap [\pi - \eta, \theta(t)]$, where $\theta(t) = \theta_{1/4} - t$. Finally, at least $\frac{m_{\mu(t)}}{4}$ exits the sector $A$ within time $\delta t$ as claimed. By approximation we can treat the case of a general measure $\mu$. ■

We are now ready to prove the following:

*Theorem 5.2:* Let $\tilde{u}$ be the measure control on $\mathbb{S}^1$ given by (16). If $\mu(\cdot)$ is a trajectory of the corresponding MVF

$V^{\tilde{u}}$ then $W(\mu(t), \delta_0) \to 0$ as $t \to \infty$. In particular, $\tilde{u}$ asymptotically stabilized the system to 0.

*Proof:* Consider a trajectory $\mu(\cdot)$ of the MVF $V^{\tilde{u}}$. From Lemma 5.2, we have:

$$\mu(t + k\delta t)(A) \leq \left(\frac{3}{4}\right)^k \mu(t)(A),$$

for every $k \in \mathbb{N}$. Therefore, for every $\epsilon$ we can choose $k$ such that $\mu(t + k\delta t)(A) \leq \epsilon$. In turn, the mass outside $A$ will reach 0 within time $2\pi$, thus we immediately obtain $W(\mu(k\delta t + 2\pi), \delta_0) \leq \epsilon$ which proves the claim. ∎

### C. Stabilization via measure controls: two-dimensional case

We consider a two-dimensional example (see [3]), which can be seen as a generalization of the circle problem . Let $x = (x_1, x_2) \in \mathbb{R}^2$, for $a \neq 0$ define $C(a) = \{(x_1, x_2) : x_1^2 + (x_2 - a)^2 = a^2\}$ the circle centered in $(0, a)$ with radius $a$, and let $t(x)$ the unit tangent vector to $C(a)$ at $x \in C(a)$ rotating counterclockwise in the upper half plane and clockwise in the lower half plane. We complete the definition of $t$ setting $t(x, 0) = (1, 0)$. Consider the control system:

$$\dot{x} = u\, e^{-\frac{1}{|x|^2}} t(x), \;\; u \in [-1, 1]. \tag{17}$$

System (17) is smooth and for every initial condition there exists an open-loop control driving the system to zero (just rotate on a circle $C(a)$). But there exist no continuous stabilizing feedback, classical or relaxed, even excluding $x = 0$. Indeed, on every $C(a)$ we have a circle problem. For each circle $C(a)$ define a measure control as in (16), replacing the sector $A$ with a sector $A'(a)$ around the point $(0, 2a)$. The quantities $m_\mu$, $\theta_\mu$, $c_1$, and $c_2$ are Lipschitz continuous w.r.t. the variable $a$. Thus we can define a measure control $\tilde{u}$ on the whole plane, which is Wasserstein-Lipschitz continuous and stabilizes the system on each circle $C(a)$. If $\mu_0 \in \mathcal{M}_c(\mathbb{R}^2)$, then the sector $A'(a)$ intersects the measure support only for $a$ on a bounded set. Therefore, if $\mu(\cdot)$ is a trajectory starting from $\mu_0$ we conclude that $W(\mu(t), 0) \to 0$ reasoning as in the proofs of Lemma 5.2 and Theorem 5.2. Hence $\tilde{u}$ is a regular stabilizing measure control.

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## VII. Conclusions and future directions

Classical control problems, such as optimal control and stabilization ones, are difficult to solve using open-loop or feedback control. The classical concept of relaxed control, which is a probability measure on the control set, allows to solve a larger set of problems. However, it may be insufficient in many instances.

In this paper, we introduced the concept of measure control, which is a generalization of relaxed feedback control. This new concept is inspired by recent work on differential equations for measures. In simple terms, the state of the system is described by a measure distributed on the state space, and a measure control is a relaxed feedback depending on the global structure of the measure. After proving the existence of a Lipschitz semigroup for regular measure controls, we showed how to solve the classical circle problem via a regular measure control. Future developments will include applications to other optimal control and stabilization problems.

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