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Online Data-Driven Adaptive Control for Unknown Linear Time-Varying Systems

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Abstract— **This paper proposes a novel online data-driven adaptive control for discrete-time unknown linear time-varying systems. Initialized with an empirical feedback gain, the algorithm periodically updates this gain based on the data collected over a short time window before each update. Meanwhile, the stability of the closed-loop system is analyzed in detail, which shows that under some mild assumptions, the proposed online data-driven adaptive control scheme can guarantee practical global exponential stability. Finally, the proposed algorithm is demonstrated by numerical simulations and its performance is compared with other control algorithms for unknown linear time-varying systems.**

I. Introduction

In both classical control and modern control theory, the analysis and design of most controllers rely on the explicit knowledge of the plants. This requirement becomes less practical when the system is complex and highly dimensional. One of the recent research interests in the control community focuses on directly controlling the system by only using the data – that is, the information of inputs/outputs/states – while skipping the modelling step. For linear time-invariant (LTI) systems, Willems *et al.*'s *fundamental lemma* [1] states that if a finite-length input-output trajectory of an LTI system satisfies the so-called *persistence of excitation* condition, then any possible input-output trajectories of this system can be obtained from the aforementioned input-output trajectory. This result was leveraged to avoid system identification in control design, and develop purely data-based methods. For instance, [2] developed data-driven model predictive control methods, [3] deals with classical problems, such as stabilization, [4] with data-driven model reference control, and [5] considers the case where the persistence of excitation condition is not met. Some other related developments are for switched systems (*e.g.*, [6], [7]), delay systems (*e.g.*, [8]), and general nonlinear systems (*e.g.*, [9], [10]).

A particularly interesting class of system is that of linear time-varying (LTV) systems. These appear in many real life applications, for instance, due to changes in operating conditions (such as temperature, pressure, *etc.*) and mechanical wear. Moreover, LTV systems can also be obtained by linearizing nonlinear systems around trajectories of time-varying operating points. As a natural extension of the well-established data-driven control theory for LTI systems, data-driven control methods for LTV systems have also attracted much attention in recent years. In [11], an optimal control scheme for unknown discrete-time LTV systems is proposed, in which the approximate optimal control is obtained via data-driven off-policy *policy iteration*. Based on the ensemble of input-state trajectories collected offline, [12] shows a different method of data-driven control of LTV systems. [13] extends the *fundamental lemma* to linear parameter-varying systems and develops a data-driven predictive control scheme for such systems. Nevertheless, because a sufficient amount of data is required *a priori* to start the control process, the aforementioned methods cannot be run completely online. In [14], an online data-enabled predictive control is modified from the data-enabled predictive control proposed in [2], and it is claimed to be computationally efficient due to the use of fast Fourier transform and the primal-dual formulation in the algorithm. Nevertheless, stability is not guaranteed theoretically for the proposed controller therein. To this end, [15] proposes a different data-driven control method by combining matrix inequalities and the matrix S-lemma. This method is technically for linear parameter-varying (LPV) systems, but is applicable to systems with time-varying system matrices. However, that work requires the assumption of a known range of variations.

This paper proposes a novel online data-driven adaptive control (ODDAC) algorithm to stabilize LTV systems. In contrast to the aforementioned methods in the literature, our algorithm can run completely online. Meanwhile, we do not impose the usual assumptions on the knowledge of the system matrices: they do not need to be affine in a time-varying parameter, and they can be unbounded in time. The control gain is periodically updated based on the data collected over a short time window, aiming to stabilize the system up to the time of the next update. The ODDAC algorithm is also investigated in this paper via a detailed stability analysis, which shows that under some mild assumptions, the closed-loop system is guaranteed to be practically globally exponentially stable.

Notation. Let $\mathbb{R}$ be the real line, $\mathbb{N}$ be the set of all non-negative integers, and $\mathbb{N}_+$ be the set of all positive integers. 0_n and I_n denote the zero matrix and the identity matrix in $\mathbb{R}^{n\times n}$, respectively, and the subscript n is omitted if the dimension can be determined according to the context. For any symmetric matrices $M, N \in \mathbb{R}^{n\times n}$, $M \succ N$ (resp. $M \succeq N, M \prec N$, and $M \preceq N$) means that $M-N$ is positive

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definite (resp. positive semi-definite, negative definite and negative semi-definite). For any $m, n \in \mathbb{N}_+$ and any vector $x \in \mathbb{R}^n$, $|x|$ denotes the 2-norm of x; for any matrix $N \in \mathbb{R}^{n\times m}$, $\|N\|$ denotes the induced 2-norm of N. For two sets of matrices $\Sigma_1, \Sigma_2 \subset \mathbb{R}^{n\times n}$, denote the Minkowski sum as $\Sigma_1 \oplus \Sigma_2 := \{A_1 + A_2 : A_1 \in \Sigma_1, A_2 \in \Sigma_2\}$.

The rest of the paper is organized as follows. Section II introduces the preliminaries. In Section III, the mechanism of the ODDAC algorithm and the update law of the feedback gain are explained. In Section IV, the stability properties of the closed-loop system with ODDAC are analyzed. In Section V, a numerical example to demonstrate the ODDAC algorithm is presented and the performance is compared to that of other control algorithms for unknown LTV systems. Section VI concludes the paper with some discussions on future research directions.

II. PRELIMINARIES

In this section, we will introduce the preliminaries, including the system definition, regularity assumptions, and the problem formulation for control design.

Consider a discrete-time LTV system

$$x(t+1) = A(t)x(t) + B(t)u(t), \tag{1}$$

where $x : \mathbb{N} \to \mathbb{R}^n$ is the state; $u : \mathbb{N} \to \mathbb{R}^m$ is the control input; and $A(t), B(t)$ are time-varying matrices of compatible dimensions. Instead of the usual assumption of bounded time-varying system matrices, we assume that these matrices have a bounded rate of variation; that is, they admit a Lipschitz constant as follows.

Assumption 1 (Lipschitz matrix trajectories) *There exists $L \geqslant 0$ such that for all $t, s \in \mathbb{N}$,*

$$\left\| \begin{bmatrix} A(t) - A(s) & B(t) - B(s) \end{bmatrix} \right\| \leqslant L|t - s|. \tag{2}$$

As a result, the system matrices can be potentially unbounded with respect to time. We are interested in stabilizing the system (1) without precisely knowing the matrix trajectories $A(t), B(t)$. Nevertheless, the knowledge of an initial feedback gain K_0 is required to start such a process. K_0 can either be computed via initially known values of $A(0), B(0)$, or empirically assigned based on *a priori* knowledge of the system. We summarize this condition, together with the controllability assumption as follows.

Assumption 2 (Controllability and initial gain) *The matrix pair $(A(t), B(t))$ is controllable for all $t \in \mathbb{N}$. In addition, for some given $\lambda \in (0, 1)$, there exist a known positive definite matrix $P_0 \in \mathbb{R}^{n\times n}$ and a known matrix $K_0 \in \mathbb{R}^{m\times n}$ such that*

$$(A(0) + B(0)K_0)^\top P_0(A(0) + B(0)K_0) \preceq \lambda P_0. \tag{3}$$

Under Assumption 2, $u = K_0x$ is a stabilizing feedback control law for the LTI system obtained by "freezing" the LTV system (1) at $t = 0$.

The main objective of this paper is to solve the following problem:

Problem 1 (Data-driven control of an LTV system) *Assume that the time-varying matrices $A(t), B(t)$ in* (1) *are unknown. Under Assumptions 1 and 2, find a control law which makes the closed-loop system* practically globally exponentially stable *(pGES), namely, there exist positive constants c_1, c_2, and c_3 such that the trajectories of the closed-loop system satisfy*

$$|x(t)| \leqslant c_1 e^{-c_2 t}|x(0)| + c_3, \tag{4}$$

for all $t \geqslant 0$ and $x(0) \in \mathbb{R}^n$.

III. DATA-DRIVEN AND ADAPTATION MECHANISM

We propose a novel ODDAC algorithm in order to solve Problem 1. In this section, we first introduce the timing of data collection and control gain update. To explain in detail how the control gains are selected based on the data, we convert the LTV system (1) into a piece-wise LTI system perturbed by a time-varying term. This allows re-formulating the problem of finding a stabilizing control gain into a linear matrix inequality (LMI) problem, which can be solved efficiently.

A. Periodically updated control law

Due to the time-varying nature of (1), the initial feedback law $u = K_0x$ may not stabilize the system for all $t \in \mathbb{N}$. To this end, the proposed adaptation mechanism periodically updates the feedback gain at time instants $t_i^S = iT, i \in \mathbb{N}_+$, where $T \in \mathbb{N}_+$ is the period of gain update. Our goal is to find a feedback gain $K_i \in \mathbb{R}^{m\times n}$ for each $i \in \mathbb{N}_+$, such that the control law is "essentially" $u(t) = K_ix(t)$ for all $t \in \mathcal{T}_i$. Since the system matrices are unknown, the value of K_i needs to be computed in a way that it only depends on the data over a finite time window prior to time t_i^S. To be precise, let $T^W \in \mathbb{N}_+$ be the length of time window and denote $t_i^W := t_i^S - T^W$. In the sequel, we denote $\mathcal{T}_0 := \{0, 1, \cdots, t_1^S - 1\}$, $\mathcal{T}_i := \{t_i^S, t_i^S + 1, \cdots, t_{i+1}^S - 1\}$, and $\mathcal{T}_i^W := \{t_i^W, t_i^W + 1, \cdots, t_i^S - 1\}$, for all $i \in \mathbb{N}_+$. Following the earlier discussion, we aim to construct a piece-wise control law:

$$u(t) = \begin{cases} K_ix(t), & t \in \mathcal{T}_i \backslash \mathcal{T}_{i+1}^W, \\ K_ix(t) + v(t), & t \in \mathcal{T}_{i+1}^W, \end{cases} \tag{5}$$

where $v(t) \in \{u \in \mathbb{R}^m : |u| \leqslant \bar{v}\}$ for all $t \in \cup_{i\in\mathbb{N}_+}\mathcal{T}_i^W$ and $\bar{v} > 0$ is a design parameter. The reason for introducing the additional signal v to (5) over $\mathcal{T}_{i+1}^W$ is that we need the collected data to be "sufficiently exciting" so that the underlying data-driven problem (Problem 2 to be discussed later) is feasible. Interested readers are referred to [6, Lemma 3] for further details.

The timing of data collection and control gain update is illustrated in Fig. 1. As the system is time-varying, a convergent identification for the matrix trajectories $A(t), B(t)$ is difficult. Instead, based on the data collected over the time window $\mathcal{T}_i^W$, we aim to first conclude a set to which $A(t_i^S), B(t_i^S)$ belong. The controller further predicts what values $A(t), B(t)$ can reach for $t \in \mathcal{T}_i$, under Assumption 1.

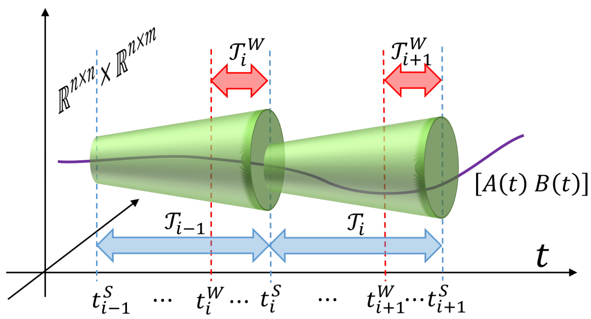

Fig. 1. Illustration of the timing of data collection and control gain update. The purple curve represents a "trajectory" of $[A(t), B(t)] \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$. At each time instant t_i^S, the controller estimates a set to which $[A(t), B(t)]$, $t \in \mathcal{T}_i$, belongs, represented by the green frustum-shaped region. Such estimation is made based on the data collected over the time window $\mathcal{T}_i^W$.

It then finds a *common* stabilizing feedback gain K_i for the system for all possible values of $A(t), B(t), t \in \mathcal{T}_i$.

B. A data-driven model of the LTV system

We exploit the idea of the *congelation of variables* method [16], which treats the original unknown time-varying system as an unknown time-invariant system (namely, the *congealed* part) perturbed by some time-varying terms. More specifically, the time-varying perturbation terms are defined as $\Delta A_i(t) := A(t) - A(t_i^S)$, $\Delta B_i(t) := B(t) - B(t_i^S)$, for $i \in \mathbb{N}_+$. This allows estimating the constant matrices $A(t_i^S), B(t_i^S)$ and attenuating the destabilizing effect of $\Delta A_i(t)$, $\Delta B_i(t)$, separately. To proceed, re-write system (1) as

$$x(t+1) = A_i x(t) + B_i u(t) + w_i(t), \tag{6}$$

for $t \in \mathcal{T}_i^W \cup \mathcal{T}_i$, where $A_i := A(t_i^S)$, $B_i := B(t_i^S)$ and the "virtual disturbance" $w_i(t) := \Delta A_i(t)x(t)+\Delta B_i(t)u(t)$. We now define the following:

$$X_i := \begin{bmatrix} x(t_i^W) & x(t_i^W+1) & \cdots & x(t_i^S-1) \end{bmatrix}, \tag{7a}$$
$$X_i^+ := \begin{bmatrix} x(t_i^W+1) & x(t_i^W+2) & \cdots & x(t_i^S) \end{bmatrix}, \tag{7b}$$
$$U_i := \begin{bmatrix} u(t_i^W) & u(t_i^W+1) & \cdots & u(t_i^S-1) \end{bmatrix}, \tag{7c}$$
$$W_i := \begin{bmatrix} w(t_i^W) & w(t_i^W+1) & \cdots & w(t_i^S-1) \end{bmatrix}. \tag{7d}$$

Equation (6) can be re-written into a compact form:

$$X_i^+ = A_i X_i + B_i U_i + W_i. \tag{8}$$

Note that if W_i is known and $\begin{bmatrix} X_i \\ U_i \end{bmatrix}$ is full row rank, we can directly compute the values of A_i, B_i via $\begin{bmatrix} A_i & B_i \end{bmatrix} = (X_i^+ - W_i)\begin{bmatrix} X_i \\ U_i \end{bmatrix}^\dagger$, where $(\cdot)^\dagger$ denotes the right inverse. However, such computation is infeasible, as W_i depends on unknown $\Delta A_i(t)$ and $\Delta B_i(t)$. Nevertheless, we can employ Assumption 1 and estimate that $w_i(t)w_i^\top(t) \preceq |w_i(t)|^2 I \preceq \left\| \begin{bmatrix} \Delta A_i(t) & \Delta B_i(t) \end{bmatrix} \right\|^2 \left| \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right|^2 I \preceq L^2 |t - t_i^S|^2 \left| \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \right|^2 I$, which further implies that $W_i W_i^\top = \sum_{t=t_i^W}^{t_i^S-1} w_i(t)w_i^\top(t) \preceq L^2 \sum_{k=1}^{T^W} k^2 \left| \begin{bmatrix} x(t_i^S-k) \\ u(t_i^S-k) \end{bmatrix} \right|^2 I$, or equivalently

$$\begin{bmatrix} I & W_i \end{bmatrix} \begin{bmatrix} \Pi & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & W_i \end{bmatrix}^\top \succeq 0, \tag{9}$$

where

$$\Pi := L^2 \sum_{k=1}^{T^W} k^2 \left| \begin{bmatrix} x(t_i^S-k) \\ u(t_i^S-k) \end{bmatrix} \right|^2 I. \tag{10}$$

Although we do not know the exact values of A_i, B_i, we can define a set of pairs (A_i, B_i) that are consistent with the collected data X_i, X_i^+, U_i, and "virtual disturbance" W_i (which is not measured), *i.e.*,

$$\begin{aligned}\Sigma_i := \big\{ &(A_i, B_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} : \\ &\exists W_i \text{ s.t. (8) and (9) hold.} \big\}\end{aligned} \tag{11}$$

Note that Σ_i is associated with the state and input data collected over $\mathcal{T}_i^W$ and such a relationship is indicated by the time window index i, for conciseness. This definition is made in the same spirit as the one defined in [5] to characterize *data informativity*. Interested readers may refer to the explanations therein for further detail.

We now proceed to estimate the time-varying matrices ΔA_i and ΔB_i (the time arguments are omitted for conciseness). Recall Assumption 1 and note that (2) yields

$$\begin{bmatrix} \Delta A_i & \Delta B_i \end{bmatrix} \begin{bmatrix} \Delta A_i & \Delta B_i \end{bmatrix}^\top \preceq L^2 |t - t_i^S|^2 I \preceq L^2 T^2 I$$

for any $t \in \mathcal{T}_i$. This inequality can be equivalently written as

$$\begin{bmatrix} I & \Delta A_i & \Delta B_i \end{bmatrix} \begin{bmatrix} L^2T^2 I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} I \\ \Delta A_i^\top \\ \Delta B_i^\top \end{bmatrix} \succeq 0. \tag{12}$$

Similar to the spirit of (11), one can define a set of consistent pairs $(\Delta A_i, \Delta B_i)$, that is

$$\Sigma^D := \{(\Delta A_i, \Delta B_i) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} : \text{(12) holds.}\} \tag{13}$$

The set Σ^D, unlike Σ_i, is a sheer result of Assumption 1. Thus, Σ^D depends on neither the collected data nor the index i.

Since $A(t) = A_i + \Delta A_i(t)$, $B(t) = B_i + \Delta B_i(t)$ for all $t \in \mathcal{T}_i$, we conclude that $(A(t), B(t)) \in \Sigma_i \oplus \Sigma^D$ for all $t \in \mathcal{T}_i$. Thus in order to solve Problem 1, we aim to find a common control gain K_i for the system *uniformly* with respect to the set $\Sigma_i \oplus \Sigma^D$. As such, we arrive at the following formal problem:

Problem 2 (Finding the feedback gain) *For the given λ as in Assumption 2, use the collected data X_i, X_i^+, U_i to find a positive definite matrix $P_i \in \mathbb{R}^{n\times n}$, a matrix $K_i \in \mathbb{R}^{m\times n}$ such that*

$$(A + BK_i)^\top P_i (A + BK_i) \preceq \lambda P_i, \tag{14}$$

for all $(A, B) \in \Sigma_i \oplus \Sigma^D$.

If we take $u = K_i x$, then the function $V_i(x) = x^\top P_i x$ has the property that $V_i(x(t+1)) \leqslant \lambda V_i(x(t))$ for all $t \in \mathcal{T}_i$, regardless of the values of $A(t), B(t)$. Clearly, V_i will play a large role in the stability analysis, making the solution of Problem 2 an important step towards solving Problem 1.

The following proposition provides LMI conditions, under which we can solve Problem 2.

Proposition III.1 (An LMI for the feedback gain) *Given a scalar $\lambda \in (0,1)$ and the data X_i, X_i^+, U_i over $\mathcal{T}_i^W$. If there exist scalars $\alpha_1 \geqslant 0$, $\alpha_2 \geqslant 0$, a positive definite matrix $Q_i \in \mathbb{R}^{n\times n}$, and a matrix $L_i \in \mathbb{R}^{m\times n}$ such that*

$$\bar{M} - \alpha_1 \bar{N}_1 - \alpha_2 \bar{N}_2 \succeq 0, \tag{15}$$

where $\bar{M}$, $\bar{N}_1$, and $\bar{N}_2$ are as in (16) *and Π is as defined in* (10)*. Then, the matrices $K_i := L_i Q_i^{-1}$ and $P_i := Q_i^{-1}$ solve Problem 2.*

The proof of Proposition III.1 is based on methods developed in [17] and omitted here.

Remark 1 (Proposition III.1 is only sufficient) *The LMI formulation* (15) *is inspired by the S-procedure in [18, Chapter 2.6]. However, not all solutions of Problem 2 can be found by solving this LMI. This is because the Minkowski sum of two sets of matrices defined via quadratic matrix inequalities (i.e., Σ_i and Σ^D) cannot be expressed by another quadratic matrix inequality in general. Conservativeness introduced by formulating Problem 2 into the LMI* (15) *can be further investigated in future research.*

IV. Stability analysis and algorithmic realization

We summarize here that the proposed ODDAC algorithm applies the control law (5) to the system (1), where the gain K_i can be computed via solving the LMI in Proposition III.1. In this section, we will discuss the stability property of the closed-loop system equipped with ODDAC and present an algorithmic realization of the proposed control scheme.

A. Stability analysis

Note that since the control gain is updated at each time instant t_i^S, the closed-loop system can be viewed as a switched system and hence it is stable if there exists a common Lyapunov function (see [19, Section 2.1]). In other words, stability is guaranteed when P_i's found by solving Problem 2 are the same for all $i \in \mathbb{N}$ (including the one given initially). Such a condition, however, is restrictive as it imposes an equality constraint to the subsequent computation of K_j for all $j \geqslant i$. We therefore adopt an alternative approach to establish stability to allow a different P_i for each control gain update, stated as follows.

Theorem IV.1 (Stability of the closed-loop system) *Consider the discrete-time LTV system* (1) *equipped with the control law* (5)*, under Assumption 1 and Assumption 2. If the LMI* (15) *is feasible for all $i \in \mathbb{N}_+$ and the matrices $P_i := Q_i^{-1}$ satisfy*

$$\sigma_1 I \preceq P_i \preceq \sigma_2 I, \tag{17}$$

$$P_{i+1} \preceq \left(\frac{\hat{\lambda}}{\lambda}\right)^T P_i, \tag{18}$$

for all $i \in \mathbb{N}$ and some $\hat{\lambda} \in [\lambda, 1)$, $\sigma_1 > 0$, $\sigma_2 > 0$, then, the system is pGES. In other words, the solutions of the closed-loop system satisfy that

$$|x(t)| \leqslant \frac{\sigma_2}{\sqrt{\sigma_1}} \hat{\lambda}^{\frac{t}{2}} |x(0)| + \frac{\sqrt{\frac{\sigma_2}{\sigma_1}}}{1-\sqrt{\hat{\lambda}}} \left(\frac{\hat{\lambda}}{\lambda}\right)^{\frac{T}{2}} \bar{B}\bar{v}. \tag{19}$$

The proof of Theorem IV.1 exploits standard arguments of multiple Lyapunov function approach that has been extensively used in the literature (see *e.g.*, [20]). We omit the proof here due to space limitation.

We now make some comments to facilitate the understanding of Theorem IV.1. Equation (17) is essentially the "sandwich" condition typically required for the multiple Lyapunov function approach. Meanwhile, if there exists $\mu \geqslant 1$ such that $P_{i+1} \preceq \mu P_i$ for all $i \in \mathbb{N}$, it is sufficient to require

$$T > -\frac{\ln \mu}{\ln \lambda}, \tag{20}$$

in order for (18) to hold, in which case $\hat{\lambda} \in [\lambda \mu^{\frac{1}{T}}, 1)$. Equation (20) imposes a dwell time condition on the switching [21]. This can be intuitively understood as follows: a switched system is stable if it is stable in each mode, and the switching is sufficiently slow. With this controller, we observe in (19) that the solutions can converge to a ball of arbitrarily small radius, by making $\bar{v}$ sufficiently small.

B. The control algorithm

Recall that Proposition III.1 gave an LMI approach to solve Problem 2, which is not directly formulated in terms of P_i, K_i. Yet, our stability results rely on some additional conditions imposed on P_i; namely, the conditions (17) and (18). We remark here that with given values of $\hat{\lambda}$, σ_1,and σ_2, these two conditions can be easily encoded as additional LMIs with respect to variables $Q_i = P_i^{-1}$. To do this, note that (17) is equivalent to

$$\sigma_2^{-1} I \preceq Q_i \preceq \sigma_1^{-1} I. \tag{21}$$

Meanwhile, multiplying (18) by Q_i on both sides, we get $Q_i P_{i+1} Q_i \preceq \left(\frac{\hat{\lambda}}{\lambda}\right)^T Q_i$, which, by exploiting Schur complement, can be equivalently written as

$$\begin{bmatrix} \hat{\lambda}^T Q_i & Q_i \\ Q_i & \lambda^{-T} Q_{i+1} \end{bmatrix} \succeq 0. \tag{22}$$

Based on the timing mechanism sketched in Section III-A, we can now summarize the control algorithm as in Algorithm 1. In the pseudo-code of the algorithm, Lines 1–5 are the initialization procedures. For $t \in \mathcal{T}_i \backslash \mathcal{T}_{i+1}^W$, which leads to the branch starting from Line 11, the control $u(t) = K_i x(t)$ is applied to the system, where the gain K_i is either initialized at the beginning of the control process, or computed in the last while-loop. This is consistent with the first line in (5). When the time reaches t_{i+1}^W, which

$$
\bar{M} := \begin{bmatrix} \lambda Q_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & Q_i \\ 0 & 0 & 0 & 0 & 0 & L_i \\ 0 & 0 & 0 & 0 & 0 & Q_i \\ 0 & 0 & 0 & 0 & 0 & L_i \\ 0 & Q_i & L_i^\top & Q_i & L_i^\top & Q_i \end{bmatrix}, \bar{N}_1 := \begin{bmatrix} I & X_i^+ \\ 0 & -X_i \\ 0 & -U_i \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \Pi & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & X_i^+ \\ 0 & -X_i \\ 0 & -U_i \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}^\top, \bar{N}_2 := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L^2 T^2 I & 0 & 0 \\ 0 & -I & 0 \\ 0 & 0 & -I \end{bmatrix} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}^\top. \tag{16}
$$

Algorithm 1 Online Data-Driven Adaptive Control

Input: K_0, P_0, λ, T, T^W

1: $i \leftarrow 0$, $t \leftarrow 0$
2: Set $\hat{\lambda} \in (\lambda, 1)$, $\sigma_1 > 0$, $\sigma_2 > 0$, and $\bar{v} > 0$.
3: $u(0) \leftarrow K_0 x(0)$
4: $Q_0 \leftarrow P_0^{-1}$
5: Set X_i, X_i^+, U_i as empty matrices
6: **while** system is running **do**
7: $\quad i_{\text{new}} = \lfloor \frac{t}{T} \rfloor$
8: $\quad$ **if** $t - i_{\text{new}} T \geqslant T - T^W$ **then** $\triangleright$ When $t \in \mathcal{T}_{i+1}^W$
9: $\quad\quad$ Append $x(t), x(t+1), u(t)$ to X_i, X_i^+, U_i $\triangleright$ Collect the new data
10: $\quad\quad$ $u(t) \leftarrow K_i x(t) + v(t)$ $\triangleright$ Update the control input with the exciting signal v
11: $\quad$ **else** $\triangleright$ When $t \in \mathcal{T}_i \backslash \mathcal{T}_{i+1}^W$
12: $\quad\quad$ **if** $i_{\text{new}} \neq i$ **then** $\triangleright$ When $t = t_{i_{new}}^S$
13: $\quad\quad\quad$ $i \leftarrow i_{\text{new}}$
14: $\quad\quad\quad$ Solve the LMIs (15), (21), and (22) for $Q_i, L_i, \alpha_1, \alpha_2$
15: $\quad\quad\quad$ $K_i \leftarrow L_i Q_i^{-1}$ $\triangleright$ Update the control gain
16: $\quad\quad\quad$ Reset X_i, X_i^+, U_i as empty matrices
17: $\quad\quad$ $u(t) \leftarrow K_i x(t)$ $\triangleright$ Update the control input without the exciting signal v
18: $\quad$ $t \leftarrow t + 1$

triggers the condition in Line 8, the controller activates the exciting signal and starts to collect the data. As the time reaches t_i^S, which triggers the condition in Line 12, a new feedback gain is computed, by solving the LMIs (15), (21) and (22). Meanwhile, the index i is updated, indicating that a new period has started, and the data buffers X_i, X_i^+, U_i are reset. Thanks to Theorem IV.1, we have the following result regarding the functionality of Algorithm 1.

Corollary IV.2 (Functionality of the ODDAC algorithm) *Consider the discrete-time LTV system* (1) *under Assumption 1 and Assumption 2. Let the ODDAC Algorithm 1 be applied to the system and assume that Line 14 of the algorithm is always feasible. Then the closed-loop system is pGES, in the sense that* (19) *holds on all solutions.*

Remark 2 (Space and computation complexity) *In view of* (10)*, a total of T^W number of state and control variables are needed for the update of K_i. Meanwhile, the formula of $\bar{N}_1$ in* (16) *involves matrix multiplication of dimension* $\max\{n, T^W\}$*. Thus, by picking a small T^W, the ODDAC algorithm is more efficient in both memory use and online computation. We also remark here that once $\bar{N}_1, \bar{N}_2$ are computed, the LMI* (15) *is of dimension $4n + 2m$, which is independent of the window length T^W.*

V. Simulation

In this section, we present a numerical example to which our controller is applied and compare the performance of the proposed controller to that of some control schemes in the literature.

Consider an LTV system in the form of (1) with $n = 5, m = 2$. The time-varying $A(t)$ and $B(t)$ are generated by the (element-wise) cubic interpolation of the three pairs of matrices:

$$
A(0) = \begin{bmatrix} -0.5 & -0.4 & 0.1 & -0.8 & -0.2 \\ -0.5 & -0.1 & 0.2 & 0.7 & 0 \\ -0.4 & -0.9 & 0.6 & -0.3 & 0.4 \\ 0.2 & -0.3 & -1.2 & 0 & -0.1 \\ -0.6 & 0.8 & -0.5 & -0.1 & -0.1 \end{bmatrix}, B(0) = \begin{bmatrix} -1.4 & 2.2 \\ 0.9 & 1.4 \\ 2.7 & 0.5 \\ -0.7 & 1.5 \\ 0.6 & -1.9 \end{bmatrix},
$$

$$
A(500) = \begin{bmatrix} -0.5 & -0.7 & 0.3 & -0.6 & 0 \\ 0 & 0 & 0 & 0.8 & 0.4 \\ -0.7 & -1.0 & 0.7 & 0.1 & 0.2 \\ -0.2 & -0.2 & -1.1 & 0.3 & 0.3 \\ -0.9 & 0.7 & -0.9 & 0.5 & 0.4 \end{bmatrix}, B(500) = \begin{bmatrix} -1.5 & 2.4 \\ 0.9 & 1.3 \\ 2.9 & 0.7 \\ -0.7 & 1.5 \\ 0.4 & -1.9 \end{bmatrix},
$$

$$
A(1000) = \begin{bmatrix} 0 & -0.6 & -0.2 & -0.7 & 0.5 \\ 0 & 0.1 & 0.4 & 1.1 & 0.7 \\ -1.4 & -0.9 & 0.5 & 0.5 & 0.5 \\ -0.2 & -0.2 & -1.5 & -0.3 & 0.5 \\ -0.9 & 0.5 & -0.6 & 0.7 & 0.5 \end{bmatrix}, B(1000) = \begin{bmatrix} -1.4 & 2.4 \\ 0.9 & 1.5 \\ 3.0 & 0.6 \\ -0.8 & 1.5 \\ 0.5 & -1.9 \end{bmatrix}.
$$

One can numerically verify that Assumption 1 holds with the Lipschitz constant $L = 0.0037$. Assumption 2 also holds with initial values of K_i, Q_i given by

$$
K_0 = \begin{bmatrix} 0.13 & 0.26 & -0.25 & 0.04 & -0.13 \\ 0.08 & 0.28 & 0.13 & 0.05 & 0.01 \end{bmatrix},
$$

$$
Q_0 = \begin{bmatrix} 0.75 & -0.13 & 0.03 & -0.26 & -0.08 \\ -0.13 & 0.88 & -0.08 & -0.12 & 0.36 \\ 0.03 & -0.08 & 0.21 & 0.01 & -0.01 \\ -0.26 & -0.12 & 0.01 & 0.43 & 0.14 \\ -0.08 & 0.36 & -0.01 & 0.14 & 1.13 \end{bmatrix}.
$$

Let the initial state be $x(0) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^\top$. We consider four different control methods to stabilize the system: 1) static state feedback control $u = K_0 x$, 2) a discrete-time version of the model reference adaptive control (MRAC) in [22, Section 5.2.6] with normalization [23, Section 4.3], 3) online data-enabled predictive control (ODeePC) as recently proposed in [14], and 4) the proposed ODDAC method. Assume that the matrix trajectories $A(t), B(t)$ are unknown to all four controllers. In particular, the parameters of ODDAC are selected such that $T = 100$, $T^W = 10$, $\lambda = 0.9$, $\hat{\lambda} = 0.91$, $\sigma_1 = 0.001$, $\sigma_2 = 1000$, $\bar{v} = 10^{-10}$. The semi-logarithmic time histories of $|x(t)|$ are shown in Fig. 2. We also apply the four control schemes with the same parameters to the LTI system (*i.e.*, $(A(t), B(t)) = (A(0), B(0))$ for

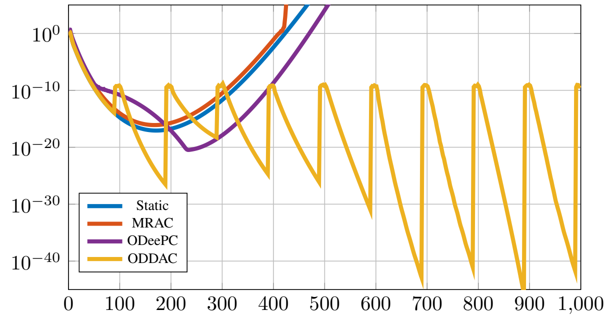

Fig. 2. Semi-logarithmic time histories of $|x(t)|$ of the LTV system in the four control schemes.

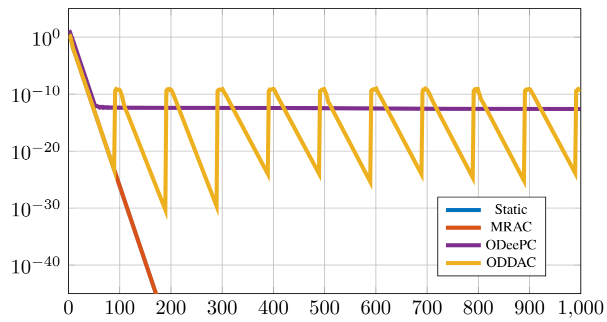

Fig. 3. Semi-logarithmic time histories of $|x(t)|$ of the LTI system in the four control schemes.

all $t \in \mathbb{N}$). The semi-logarithmic time histories of $|x(t)|$ of the LTI system are plotted in Fig. 3, where it is seen that both static state feedback control and MRAC make the closed-loop system exponentially stable, and both ODeePC and ODDAC make the closed-loop system pGES.

From Fig. 2, one can observe that all the controllers can guarantee that the solutions of the LTV system evolve close to the origin over a short period of time. Nevertheless, only ODDAC can maintain this for the entire simulated time interval. For all other methods, the solutions of the closed-loop system diverge after some time. Note that the "sawtooth" pattern of $\log |x(t)|$ in the ODDAC time history has a time period of 100 time units. This is caused by the exciting signal $v(t)$, which drives $|x(t)|$ to an acceptable amplitude (depends on the magnitude of $\bar{v}$) to collect the data for computing the new feedback gain. This is the "price" paid to guarantee practical stability for an arbitrarily long time. The simulation results are hence consistent with the practical stability property established in the paper and reveal the advantage of the proposed control algorithm in time-varying systems.

VI. Conclusion

This paper has proposed and discussed a novel control algorithm for LTV systems: ODDAC. Based on the data collected over the preceding short time window, the algorithm periodically updates a feedback gain, aiming to stabilize the system up to the time of the next update. In order to guarantee pGES of the closed-loop system equipped with ODDAC, we assume that the LMIs for finding the feedback gains always have solutions. Although feasibility of these LMIs is expected to be true when the system varies sufficiently slowly, this assumption needs to be further investigated in detail.

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