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Sub-predictors for finite-dimensional observer-based control of stochastic semilinear parabolic PDEs

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Abstract— **We study output-feedback control of 1D stochastic semilinear heat equation with constant input delay and nonlinear multiplicative noise where the nonlinearities satisfy globally Lipschitz condition. We consider the Neumann actuation and nonlocal measurement. To compensate delay r, we construct a chain of $M+1$ sub-predictors in the form of ODEs that correspond to the delay fraction r/M. Differently from the deterministic case, we add an additional sub-predictor to the chain that leads to the closed-loop system with the stochastic infinite-dimensional tail and the finite-dimensional part that consists of non-delayed stochastic equations and delayed deterministic ones. The latter essentially simplifies the Lyapunov-based mean-square L^2 exponential stability analysis of the full-order closed-loop system. We employ corresponding Itô's formulas for stochastic ODEs and PDEs, respectively. Our stability analysis leads to LMIs which are shown to be feasible for any input delay provided M and the observer dimension are large enough and Lipschitz constants are small enough. A numerical example demonstrates the efficiency of the proposed approach.**

I. Introduction

In recent years, estimation and control problems for stochastic PDEs become popular due to their wide applications in many areas of science, engineering, and finance. However, control theory for stochastic PDEs is still at its very beginning stage and many tools and methods, which are effective in the deterministic case, do not work anymore in the stochastic setting [1]. Finite-dimensional controllers for parabolic systems via the modal decomposition approach are very attractive in applications [2], [3]. This approach was extended to the stochastic setting in [4] for additive noise under output-feedback controllers and in [5] for multiplicative noise under state-feedback control. However, in [2], [3], [4], [5], efficient bounds on the observer or controller dimensions were not provided. In recent paper [6], the first constructive LMI-based method for finite-dimensional observer-based controller of deterministic parabolic PDEs was suggested, where the observer dimension was found from simple LMI conditions. In our recent paper [7], the constructive method in [6] was extended to stochastic parabolic PDEs with nonlinear multiplicative noise under boundary control and observer.

Robustness with respect to small delays and/or sampling intervals for deterministic heat equations was studied in [8] for distributed static output-feedback control, in [9] for boundary state-feedback and in [10] for boundary controller based on PDE observer. Delayed implementation of finite-dimensional observer-based controllers for 1D heat equations was introduced in [11] for deterministic case and in [12] for stochastic case. For the estimation of deterministic heat equations with a large input/output delay, a PDE sub-predictor was presented in [13] and a chain of observers was designed in [14]. Finite-dimensional observer-based classical predictors and sub-predictors were introduced in [15], [16], [17] for linear parabolic PDEs. In [18], finite-dimensional observer-based sub-predictors for semilinear parabolic PDEs were explored. However, for stochastic systems, there are few results on predictor-based control, and all existing results are confined to stochastic linear ODEs (see, e.g., [19], [20]). To the best of our knowledge, predictor-based control for stochastic PDEs has not been studied yet.

In the present paper, for the first time, we provide efficient finite-dimensional observer-based sub-predictors design for stochastic heat equations with constant input delay. We consider the 1D stochastic semilinear heat equation with nonlinear multiplicative noise under Neumann actuation and nonlocal measurement, where the nonlinearities satisfy globally Lipschitz condition. To compensate delay r, we construct a chain of $M+1$ ($M \geq 1$) sub-predictors in the form of ODEs that correspond to the delay fraction r/M. Differently from [15], [18] for deterministic heat equations where M sub-predictors were constructed, we add an additional sub-predictor to the chain that leads to the closed-loop system with the stochastic infinite-dimensional tail and the finite-dimensional part that consists of non-delayed stochastic equations and delayed deterministic ones. We construct an appropriate Lyapunov functional for mean-square L^2 exponential stability of full-order closed-loop system and employ corresponding Itô's formulas for stochastic ODEs and PDEs. Note that the Lyapunov functional depends only on the deterministic finite-dimensional part of the closed-loop system. We present LMI conditions for finding M, the observer dimension and Lipschitz constants that preserve the exponential stability. We show that for any input delay, the LMIs are feasible for large enough M and observer dimension, and small enough Lipschitz constants. In the case of one sub-predictor (i.e., $M=0$), our method degenerates into the observer-based control with the delay robustness (as studied in [11] in the deterministic case). We also consider the sub-predictors construction similar to the deterministic case [18] and construct Lyapunov functional that depends both on the deterministic and stochastic parts. A numerical example demonstrates that the two methods lead to complementary

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results, whereas additional sub-predictor for the stochastic case leads to a larger delay for comparatively large M.

Notations: Let $(\Omega,\mathscr{F},\mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ of increasing sub σ-fields of $\mathscr{F}$ and let $\mathbb{E}\{\cdot\}$ be the expectation operator. Denote by $\mathscr{W}(t)$ the 1D standard Brownian motion defined on $(\Omega,\mathscr{F},\mathbb{P})$. For $f\in C([0,1])$, let $\|f\|_{[0,1]}=\max_{x\in[0,1]}|f(x)|$. Denote by $L^2(0,1)$ the space of square integrable functions with inner product $\langle f,g\rangle=\int_0^1 f(x)g(x)\mathrm{d}x$ and induced norm $\|f\|_{L^2}^2=\langle f,f\rangle$. Let $L^2(\Omega;L^2(0,1))$ be the set of all $\mathscr{F}_0$-measurable random variables $z\in L^2(0,1)$ with $\mathbb{E}\|z\|_{L^2}^2<\infty$. $H^1(0,1)$ is the Sobolev space of functions $f:[0,1]\longrightarrow\mathbb{R}$ with a square integrable weak derivative. The norm defined in $H^1(0,1)$ is $\|f\|_{H^1}^2=\|f\|_{L^2}^2+\|f'\|_{L^2}^2$. The Euclidean norm is denoted by $|\cdot|$. For $P\in\mathbb{R}^{n\times n}$, $P>0$ means that P is symmetric and positive definite. The symmetric elements of a symmetric matrix will be denoted by $*$. For $0<P\in\mathbb{R}^{n\times n}$ and $x\in\mathbb{R}^n$, we write $|x|_P^2=x^{\mathrm{T}}Px$. Let $\mathbb{N}$ denote the set of positive integers.

Consider the Sturm-Liouville eigenvalue problem

$$\phi''+\lambda\phi=0,\ x\in(0,1),\quad \phi'(0)=\phi'(1)=0.$$

This problem induces a sequence of eigenvalues with corresponding eigenfunctions given by:

$$\begin{aligned}&\phi_1(x)=1,\quad \lambda_1=0,\\ &\phi_n(x)=\sqrt{2}\cos(\sqrt{\lambda_n}x),\quad \lambda_n=(n-1)^2\pi^2,\ n\geq 2.\end{aligned}\tag{1}$$

The eigenfunctions $\{\phi_n\}_{n=1}^{\infty}$ form a complete orthonormal system in $L^2(0,1)$. Given a positive integer N and $h\in L^2(0,1)$ satisfying $h\overset{L^2}{=}\sum_{n=1}^{\infty}h_n\phi_n$, we denote $\|h\|_N^2=\sum_{n=N+1}^{\infty}h_n^2$.

II. Main results

A. System under consideration

Consider the following stochastic semilinear heat equation under delayed Neumann actuation with known delay $r>0$:

$$\begin{aligned}&\mathrm{d}z(x,t)=[\tfrac{\partial^2}{\partial x^2}z(x,t)+g(z(x,t))]\mathrm{d}t+\sigma(z(x,t))\mathrm{d}\mathscr{W}(t),t\geq 0,\\ &z_x(0,t)=0,\ z_x(1,t)=u(t-r),\\ &z(x,0)=z_0(x),\end{aligned}\tag{2}$$

where $z_0\in L^2(\Omega;L^2(0,1))$, u is the control input to be designed, $\sigma(z(x,t))\mathrm{d}\mathscr{W}(t)$ is the nonlinear multiplicative noise which appears due to the random parameter variation of $g(z(x,t))\mathrm{d}t$. Nonlinear functions $\sigma,g:\mathbb{R}\to\mathbb{R}$ satisfy

$$\begin{aligned}&\sigma(0)=0,\quad |\sigma(z_1)-\sigma(z_2)|\leq\bar{\sigma}|z_1-z_2|,\\ &g(0)=0,\quad |g(z_1)-g(z_2)|\leq\bar{g}|z_1-z_2|,\quad \forall z_1,z_2\in\mathbb{R},\end{aligned}\tag{3}$$

for some $\bar{g},\bar{\sigma}>0$. Here $\bar{\sigma}$ describes the upper bound of noise intensity. We consider the non-local measurement output:

$$y(t)=\langle c,z(\cdot,t)\rangle,\quad c\in L^2(0,1).\tag{4}$$

Following [7], we present the solution to (2) as

$$z(x,t)=\textstyle\sum_{n=1}^{\infty}z_n(t)\phi_n(x),\ z_n(t)=\langle z(\cdot,t),\phi_n\rangle,\tag{5}$$

where $\{\phi_n\}_{n=1}^{\infty}$ are given in (1). By differentiating z_n in (5) and using integration by parts, we arrive at the following infinite stochastic equations

$$\mathrm{d}z_n(t)=[-\lambda_n z_n(t)+g_n(t)+b_nu(t-r)]\mathrm{d}t+\sigma_n(t)\mathrm{d}\mathscr{W}(t),\tag{6}$$

for $n\geq 1$, where

$$\begin{aligned}&g_n(t)=\langle g(\textstyle\sum_{j=1}^{\infty}z_j(t)\phi_j),\phi_n\rangle,\sigma_n(t)=\langle\sigma(\textstyle\sum_{j=1}^{\infty}z_j(t)\phi_j),\phi_n\rangle,\\ &b_1=1,\quad b_n=(-1)^{n-1}\sqrt{2},\ n\geq 2.\end{aligned}\tag{7}$$

By (1) and the integral convergence test, we have

$$\textstyle\sum_{n=N+1}^{\infty}\frac{b_n^2}{\lambda_n}\leq\frac{2}{\pi^2}\left(\frac{1}{N^2}+\int_N^{\infty}\frac{1}{x^2}\mathrm{d}x\right)=\frac{2(N+1)}{\pi^2N^2},\ N\geq 1.\tag{8}$$

Let $\delta>0$ be a desired decay rate and let $N_0\in\mathbb{N}$ satisfy

$$-\lambda_n+\bar{g}+\tfrac{1}{2}\bar{\sigma}^2<-\delta,\ n>N_0,\tag{9}$$

where N_0 is used for the controller design. Let $N\in\mathbb{N}$, $N\geq N_0$, where N will be the dimension of the observer.

Introduce

$$\begin{aligned}&z^{N_0}(t)=[z_1(t),\ldots,z_{N_0}(t)]^{\mathrm{T}},\quad B_0=[b_1,\ldots,b_{N_0}]^{\mathrm{T}},\\ &z^{N-N_0}(t)=[z_{N_0+1}(t),\ldots,z_N(t)]^{\mathrm{T}},\ B_1=[b_{N_0+1},\ldots,b_N]^{\mathrm{T}},\\ &A_0=\mathrm{diag}\{-\lambda_n\}_{n=1}^{N_0},\quad A_1=\mathrm{diag}\{-\lambda_n\}_{N_0+1}^{N},\\ &\sigma^{N_0}(t)=\mathrm{col}\{\sigma_n(t)\}_{n=1}^{N_0},\ \sigma^{N-N_0}(t)=\mathrm{col}\{\sigma_n(t)\}_{n=N_0+1}^{N},\\ &G^{N_0}(t)=\mathrm{col}\{g_n(t)\}_{n=1}^{N_0},\ G^{N-N_0}(t)=\mathrm{col}\{g_n(t)\}_{n=N_0+1}^{N}.\end{aligned}$$

From (6) we find that $z^{N_0}(t)$ and $z^{N-N_0}(t)$ satisfy

$$\begin{aligned}&\mathrm{d}z^{N_0}(t)=[A_0z^{N_0}(t)+G^{N_0}(t)+B_0u(t-r)]\mathrm{d}t+\sigma^{N_0}(t)\mathrm{d}\mathscr{W}(t),\\ &\mathrm{d}z^{N-N_0}(t)=[A_1z^{N-N_0}(t)+G^{N-N_0}(t)\\ &\qquad +B_1u(t-r)]\mathrm{d}t+\sigma^{N-N_0}(t)\mathrm{d}\mathscr{W}(t).\end{aligned}\tag{10}$$

Let $c_n=\langle c,\phi_n\rangle$, $C_0=[c_1,\ldots,c_{N_0}]$. Assume that

$$c_n\neq 0,\ 1\leq n\leq N_0.\tag{11}$$

Then, the pair (A_0,C_0) is observable by the Hautus lemma. Choose $L_0=[l_1,\ldots,l_{N_0}]^{\mathrm{T}}$ such that

$$P_o(A_0-L_0C_0)+(A_0-L_0C_0)^{\mathrm{T}}P_o<-2\delta P_o,\tag{12}$$

where $0<P_o\in\mathbb{R}^{N_0\times N_0}$. Furthermore, following [6] we let $l_n=0$, $N_0<n\leq N$. Since $b_n\neq 0$, $n\geq 1$ (see (7)), the pair (A_0,B_0) is controllable by the Hautus lemma. Let $K_0\in\mathbb{R}^{1\times N_0}$ satisfy

$$P_c(A_0-B_0K_0)+(A_0-B_0K_0)^{\mathrm{T}}P_c\leq-2\delta P_c,\tag{13}$$

where $0<P_c\in\mathbb{R}^{N_0\times N_0}$.

B. Sub-predictors

Consider stochastic systems (10). In order to deal with the input delay $r>0$, we fix $M\in\mathbb{N}$ and divide r into M parts of equal size $\frac{r}{M}$. We design a chain of sub-predictors

$$\begin{aligned}&\hat{z}_1^j(t-r)\mapsto\ldots\mapsto z_i^j(t-\tfrac{M-i+1}{M}r)\mapsto\ldots\\ &\mapsto\hat{z}_M^j(t-\tfrac{1}{M}r)\mapsto\hat{z}_{M+1}^j(t)\mapsto z^j(t),\ j\in\{N_0,N-N_0\},\end{aligned}\tag{14}$$

where $\hat{z}_i^j(t-\frac{M-i+1}{M}r)\mapsto\hat{z}_i^j(t-\frac{M-i}{M}r)$ means that $\hat{z}_i^j(t)$ predicts the value of $\hat{z}_i^j(t+\frac{r}{M})$. Similarly, $\hat{z}_{M+1}^j(t)\mapsto z^j(t)$ means that $\hat{z}_{M+1}^j(t)$ predicts the value of $z^j(t)$. The sub-predictors satisfy

$$\begin{aligned}&\mathrm{d}\hat{z}_{M+1}^{N_0}(t)=[A_0\hat{z}_{M+1}^{N_0}(t)+\hat{G}_{M+1}^{N_0}(t)+B_0u(t-r)]\mathrm{d}t\\ &\qquad -L_0[C_0\hat{z}_{M+1}^{N_0}(t)+C_1\hat{z}_{M+1}^{N-N_0}(t)-y(t)]\mathrm{d}t,\\ &\mathrm{d}\hat{z}_{M+1}^{N-N_0}(t)=[A_1\hat{z}_{M+1}^{N-N_0}(t)+\hat{G}_{M+1}^{N-N_0}(t)+B_1u(t-r)]\mathrm{d}t,\\ &\mathrm{d}\hat{z}_i^{N_0}(t)=[A_0\hat{z}_i^{N_0}(t)+\hat{G}_i^{N_0}(t)+B_0u(t-\tfrac{i-1}{M}r)]\mathrm{d}t\\ &\qquad -L_0[C_0\hat{z}_i^{N_0}(t-\tfrac{r}{M})+C_1\hat{z}_i^{N-N_0}(t-\tfrac{r}{M})\\ &\qquad -C_0\hat{z}_{i+1}^{N_0}(t)-C_1\hat{z}_{i+1}^{N-N_0}(t)]\mathrm{d}t,\\ &\mathrm{d}\hat{z}_i^{N-N_0}(t)=[A_1\hat{z}_i^{N-N_0}(t)+\hat{G}_i^{N-N_0}(t)\\ &\qquad +B_1u(t-\tfrac{i-1}{M}r)]\mathrm{d}t,\quad 1\leq i\leq M,\quad t\geq 0,\end{aligned}\tag{15}$$

subject to $\hat{z}_i^{N_0}(t)=0$, $\hat{z}_i^{N-N_0}(t)=0$, $t\leq 0$, $1\leq i\leq M+1$, where $y(t)$ is given by (4), $C_1=[c_{N_0+1},\ldots,c_N]$,

$$\begin{aligned}&\hat{G}_i^{N_0}(t)=\mathrm{col}\{\hat{g}_n^{(i)}(t)\}_{n=1}^{N_0},\ \hat{G}_i^{N-N_0}(t)=\mathrm{col}\{\hat{g}_n^{(i)}(t)\}_{n=N_0+1}^{N},\\ &\hat{g}_n^{(i)}(t)=\langle g(\Phi_1(\cdot)\hat{z}_i^{N_0}(t)+\Phi_2(\cdot)\hat{z}_i^{N-N_0}(t)),\phi_n\rangle,\\ &\Phi_1(x)=[\phi_1(x),\ldots,\phi_{N_0}(x)],\ \Phi_2(x)=[\phi_{N_0+1}(x),\ldots,\phi_N(x)].\end{aligned}$$

Remark 1: Differently form [18], we introduce additional sub-predictor $\hat{z}_{M+1}^j$ ($j\in\{N_0,N-N_0\}$). This splits the stochastic term and delay term into separate systems. See closed-loop system (27) below, where delay term appears only in system

e_i^j, $i=1,\dots,M$ and stochastic term appears only in system z^j, e_{M+1}^j (cf. (25)). The latter allows us to avoid stochastic terms in the corresponding Lyapunov functional (see (28) below). Besides, here (due to $\hat{z}_{M+1}^j$) we have at least two sub-predictors. In Sec. II-F we will present conventional sub-predictors (without $\hat{z}_{M+1}^j$ in (14)), where $M=1$ corresponds to observer-based control with delay robustness and where Lyapunov functional depending on the deterministic and stochastic parts is used (see (57) that follows [21]). From the numerical example in Sec. III, we find that the constructions of sub-predictors with $\hat{z}_{M+1}^j$ and without $\hat{z}_{M+1}^j$ lead to complementary results.

Note that as i increases, the input delay on the right hand-side of (15) decreases by $\frac{r}{M}$. The finite-dimensional observer $\hat{z}(x,t)$ of the state $z(x,t)$, based on (15), is given by

$$\hat{z}(x,t)=\Phi_1(x)\hat{z}_1^{N_0}(t-r)+\Phi_2(x)\hat{z}_1^{N-N_0}(t-r). \quad (16)$$

The controller is further chosen as

$$u|_{[-r,0]}=0,\quad u(t)=-K_0\hat{z}_1^{N_0}(t),\quad t>0, \quad (17)$$

where $K_0\in\mathbb{R}^{1\times N_0}$ is determined by (13).

C. Well-posedness of the closed-loop system

For well-posedness we introduce the change of variables

$$w(x,t)=z(x,t)-\psi(x)u(t-r), \quad (18)$$

where $\psi(x)=-\frac{2}{\pi}\cos(\frac{\pi}{2}x)$ which satisfies

$$\psi''(x)=-\mu\psi(x),\ \mu=\tfrac{\pi^2}{4},\ \psi'(0)=0,\psi'(1)=1. \quad (19)$$

We have the equivalent stochastic heat equation

$$\begin{aligned}&\mathrm{d}w(x,t)=[\tfrac{\partial^2}{\partial x^2}w(x,t)+g(w(x,t)+\psi(x)u(t-r))]\mathrm{d}t\\&\quad-\psi(x)[\mu u(t-r)\mathrm{d}t+\mathrm{d}u(t-r)]\\&\quad+\sigma(w(x,t)+\psi(x)u(t-r))\mathrm{d}\mathscr{W}(t),\ t\ge 0,\\&w_x(0,t)=0,\ w_x(1,t)=0.\end{aligned} \quad (20)$$

Let $\mathscr{A}=\mathrm{diag}\{\mathscr{A}_1,\mathscr{A}_2\}$ where

$$\begin{aligned}&\mathscr{A}_1=I_{M+1}\otimes\mathrm{diag}\{A_0,A_1\}+J_{0,M+1}\otimes\mathscr{C}_0,\\&\mathscr{A}_2:\mathscr{D}(\mathscr{A}_2)\subseteq L^2(0,1)\to L^2(0,1),\quad \mathscr{A}_2h=h'',\\&\mathscr{D}(\mathscr{A}_2)=\{h\in H^2(0,1)|h'(0)=h'(1)=0\}.\end{aligned} \quad (21)$$

Here $\mathscr{C}_0=\begin{bmatrix}L_0C_0 & L_0C_1\\ 0 & 0\end{bmatrix}$, $J_{0,M}$ is an upper triangular Jordan block of order M with zero diagonal and $\otimes$ is the Kronecker product. Let $\xi(t)=\mathrm{col}\{\hat{Z}(t),w(\cdot,t)\}$ where $\hat{Z}=\mathrm{col}\{\hat{z}_1^{N_0},\hat{z}_1^{N-N_0},\dots,\hat{z}_{M+1}^{N_0},\hat{z}_{M+1}^{N-N_0}\}$. Without loss of generality we assume $z(\cdot,t)=z_0(\cdot)$ for $t<0$. Then (15) and (20) subject to the control input (17) can be presented as

$$\mathrm{d}\xi(t)=[\mathscr{A}\xi(t)+G(t)+f_1(t)]\mathrm{d}t+\Sigma(t)\mathrm{d}\mathscr{W}(t),$$

where

$$\begin{aligned}&f_1(t)=\begin{bmatrix}\Sigma_{i=1}^{M+1}\mathbb{B}_iK_0\hat{z}_1^{N_0}(t-\frac{i-1}{M}r)-\mathbb{C}_0\hat{Z}(t-\frac{r}{M})+\mathbb{C}_1\hat{Z}(t)+\mathbb{L}_0\langle c,w(\cdot,t)\rangle-\mathbb{L}_0\langle c,\psi\rangle\mathbb{K}_0\hat{Z}(t-r)\\ \psi(\cdot)K_0f_2(t-r)\end{bmatrix},\\&G(t)=\begin{bmatrix}\mathrm{col}\{\hat{G}_1(t),\dots,\hat{G}_M(t)\}\\ \hat{G}_{M+1}^{N_0}(t)-L_0(C_0\hat{z}_{M+1}^{N_0}(t)+C_1\hat{z}_{M+1}^{N-N_0}(t)-\langle c,z(t)\rangle)\\ \hat{G}_{M+1}^{N-N_0}(t)\\ g(w(t)-\psi(\cdot)K_0\hat{z}_1^{N_0}(t-r))\end{bmatrix},\ \mathbb{L}_0=\begin{bmatrix}0_{MN\times 1}\\ L_0\\ 0_{(N-N_0)\times 1}\end{bmatrix},\\&\mathbb{C}_0=\left[\begin{array}{c|c}I_M\otimes\mathscr{C}_0 & 0\\ \hline 0 & 0_{N\times N}\end{array}\right],\ \mathbb{C}_1=\left[\begin{array}{c|c}I_M\otimes\mathscr{C}_0 & 0\\ \hline 0 & -\mathscr{C}_0\end{array}\right],\\&f_2(t)=(\mu I+A_0-B_0K_0)\hat{z}_1^{N_0}(t)+\hat{G}_1^{N_0}(t)-L_0[C_0\hat{z}_1^{N_0}(t-\tfrac{r}{M})\\&\quad+C_1\hat{z}_1^{N-N_0}(t-\tfrac{r}{M})-C_0\hat{z}_2^{N_0}(t)-C_1\hat{z}_2^{N-N_0}(t)],\\&\mathbb{B}_i=\mathrm{col}\{0_{(i-1)N\times 1},B_0,B_1,0_{(M-i+1)\times 1}\},\ i=0,1,\dots,M+1,\\&\hat{G}_i(t)=\begin{bmatrix}\hat{G}_i^{N_0}(t)\\ \hat{G}_i^{N-N_0}(t)\end{bmatrix},\quad \Sigma(t)=\begin{bmatrix}0_{(M+1)N\times 1}\\ \sigma(w(t)-\psi(\cdot)K_0\hat{z}_1^{N_0}(t-r))\end{bmatrix}.\end{aligned}$$

Let $\mathscr{H}=\mathbb{R}^{(M+1)N}\times L^2(0,1)$ be a Hilbert space with norm $\|\cdot\|_{\mathscr{H}}^2=|\cdot|^2+\|\cdot\|_{L^2}^2$. Consider $\mathscr{V}=\mathbb{R}^{(M+1)N}\times H^1(0,1)$ with norm $\|\cdot\|_{\mathscr{V}}^2=|\cdot|^2+\|\cdot\|_{H^1}^2$, and $\mathscr{V}'=\mathbb{R}^{(M+1)N}\times H^{-1}(0,1)$. Hence, $\mathscr{V}\subset\mathscr{H}\subset\mathscr{V}'$. The duality scalar product between $\mathscr{V}'$ and $\mathscr{V}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathscr{V}',\mathscr{V}}$. Then $\mathscr{A}:\mathscr{V}\to\mathscr{V}'$ is a closed linear operator with domain $\mathscr{D}(\mathscr{A})$ dense in $\mathscr{H}$. For any $\xi_i\in\mathscr{V}$, $i=1,2$, we can easily check that $\mathscr{A}$ satisfies $|\langle\mathscr{A}\xi_1,\xi_2\rangle_{\mathscr{V}',\mathscr{V}}|\le\alpha\|\xi_1\|_{\mathscr{V}}\|\xi_2\|_{\mathscr{V}}$ and $\langle\mathscr{A}\xi_1,\xi_1\rangle_{\mathscr{V}',\mathscr{V}}\le-\beta\|\xi_1\|_{\mathscr{V}}^2+\gamma\|\xi_1\|_{\mathscr{H}}^2$ for some $\alpha,\beta>0$ and $\gamma\in\mathbb{R}$. Since σ,g satisfy the global Lipschitz condition (3), by the step method on $[\frac{i}{M}r,\frac{i+1}{M}r]$ $(i=0,1,\dots)$ with initial conditions $\xi(\frac{i}{M}r)\in\mathscr{D}(\mathscr{A})$ (see [12]), we obtain, for $z_0\in\mathscr{D}(\mathscr{A}_2)$ almost surely, existence of a unique solution $\xi\in L^2(\Omega;C([0,\infty)\setminus\mathscr{J};\mathscr{H}))\cap L^2(\Omega\times[0,\infty)\setminus\mathscr{J};\mathscr{V})$, where $\mathscr{J}=\{\frac{i}{M}r\}_{i=0}^{\infty}$, such that $\xi(t)\in\mathscr{D}(\mathscr{A})$, $t\ge 0$, almost surely.

D. Mean-square L^2 stability analysis

Define the estimation errors as follows

$$\begin{aligned}&e_i^j(t)=\hat{z}_{i+1}^j(t-\tfrac{M-i}{M}r)-\hat{z}_i^j(t-\tfrac{M-i+1}{M}r),1\le i\le M,\\&e_{M+1}^j(t)=z^j(t)-\hat{z}_{M+1}^j(t),\quad j\in\{N_0,N-N_0\}.\end{aligned} \quad (22)$$

Then the last term on the right-hand-side of system $\hat{z}_{M+1}^{N_0}(t)$ in (15) can be presented as

$$\begin{aligned}&C_0\hat{z}_{M+1}^{N_0}(t)+C_1\hat{z}_{M+1}^{N-N_0}(t)-y(t)\\&\overset{(4)}{=}-[C_0e_{M+1}^{N_0}(t)+C_1e_{M+1}^{N-N_0}(t)+\zeta(t)].\end{aligned} \quad (23)$$

where $\zeta(t)=\sum_{n=N+1}^{\infty}c_nz_n(t)$. Furthermore, by (22), we get

$$\hat{z}_1^{N_0}(t-r)+\textstyle\sum_{i=1}^{M+1}e_i^{N_0}(t)=z^{N_0}(t). \quad (24)$$

In particular, if the errors $e_i^{N_0}(t)$, $1\le i\le M+1$ converge to zero, from (24) we have $\hat{z}_1^{N_0}(t)\to z^{N_0}(t+r)$, meaning that $\hat{z}_1^{N_0}(t)$ predicts the future system state $z^{N_0}(t+r)$. Using (10), (15) and (23), we arrive at

$$\begin{aligned}&\mathrm{d}e_{M+1}^{N_0}(t)=[(A_0-L_0C_0)e_{M+1}^{N_0}(t)+H_{M+1}^{N_0}(t)\\&\quad-L_0C_1e_{M+1}^{N-N_0}(t)-L_0\zeta(t)]\mathrm{d}t+\sigma^{N_0}(t)\mathrm{d}\mathscr{W}(t),\\&\mathrm{d}e_{M+1}^{N-N_0}(t)=[A_1e_{M+1}^{N-N_0}(t)+H_{M+1}^{N-N_0}(t)]\mathrm{d}t+\sigma^{N-N_0}(t)\mathrm{d}\mathscr{W}(t),\\&\mathrm{d}e_M^{N_0}(t)=[(A_0-L_0C_0)e_M^{N_0}(t)+L_0C_0\Upsilon_{M,r}^{N_0}(t)\\&\quad+H_M^{N_0}(t)-L_0C_1e_M^{N-N_0}(t)+L_0C_1\Upsilon_{M,r}^{N-N_0}(t)\\&\quad+L_0(C_0e_{M+1}^{N_0}(t)+C_1e_{M+1}^{N-N_0}(t)+\zeta(t))]\mathrm{d}t,\\&\mathrm{d}e_i^{N_0}(t)=[(A_0-L_0C_0)e_i^{N_0}(t)+L_0C_0\Upsilon_{i,r}^{N_0}(t)+H_i^{N_0}(t)\\&\quad+L_0C_0(e_{i+1}^{N_0}(t)-\Upsilon_{i+1,r}^{N_0}(t))-L_0C_1(e_i^{N-N_0}(t)-\Upsilon_{i,r}^{N-N_0}(t))\\&\quad+L_0C_1(e_{i+1}^{N-N_0}(t)-\Upsilon_{i+1,r}^{N-N_0}(t))]\mathrm{d}t,\ 1\le i\le M-1,\\&\mathrm{d}e_i^{N-N_0}(t)=[A_1e_i^{N-N_0}(t)+H_i^{N-N_0}(t)]\mathrm{d}t,\ 1\le i\le M.\end{aligned} \quad (25)$$

Here $\Upsilon_{i,r}^j(t)=e_i^j(t)-e_i^j(t-\frac{r}{M})$, $H_{M+1}^j(t)=G^j(t)-\hat{G}_{M+1}^j(t)$, $H_i^j(t)=\hat{G}_{i+1}^j(t-\frac{M-i}{M}r)-G_i^j(t-\frac{M-i+1}{M}r)$, $1\le i\le M$, $j\in\{N_0,N-N_0\}$. Introduce the notations

$$\begin{aligned}&X_z(t)=\mathrm{col}\{z^{N_0}(t),z^{N-N_0}(t)\},\ \mathscr{B}=\mathrm{col}\{B_0,B_1\},\\&X_e(t)=\mathrm{col}\{e_1^{N_0}(t),\dots,e_{M+1}^{N_0}(t),e_1^{N-N_0}(t),\dots,e_{M+1}^{N-N_0}(t)\},\\&\Upsilon_r(t)=\mathrm{col}\{\Upsilon_{1,r}^{N_0}(t),\dots,\Upsilon_{M,r}^{N_0}(t),\Upsilon_{1,r}^{N-N_0}(t),\dots,\Upsilon_{M,r}^{N-N_0}(t)\},\\&H(t)=\mathrm{col}\{H_1^{N_0}(t),\dots,H_{M+1}^{N_0}(t),H_1^{N-N_0}(t),\dots,H_{M+1}^{N-N_0}(t)\},\\&\sigma(t)=\begin{bmatrix}\sigma^{N_0}(t)\\ \sigma^{N-N_0}(t)\end{bmatrix},\ G(t)=\begin{bmatrix}G^{N_0}(t)\\ G^{N-N_0}(t)\end{bmatrix},\quad F_z=\begin{bmatrix}A_0-B_0K_0 & 0\\ -B_1K_0 & A_1\end{bmatrix},\\&\mathscr{I}_2=\begin{bmatrix}0_{MN_0\times N_0} & 0\\ I_{N_0} & 0\\ 0_{M(N-N_0)\times N_0} & 0\\ 0 & I_{N-N_0}\end{bmatrix},\ \mathscr{L}_\zeta=\begin{bmatrix}0_{(M-1)N_0\times 1}\\ L_0\\ -L_0\\ 0_{(M+1)(N-N_0)\times 1}\end{bmatrix},\\&F_e=\begin{bmatrix}I_{M+1}\otimes(A_0-L_0C_0)+J_{0,M+1}\otimes L_0C_0 & -I_{M+1}\otimes L_0C_1+J_{0,M+1}\otimes L_0C_1\\ 0 & I_{M+1}\otimes A_1\end{bmatrix},\\&\Lambda_e=[I_M\otimes L_0C_0-J_{0,M}\otimes L_0C_0,I_M\otimes L_0C_1-J_{0,M}\otimes L_0C_1],\end{aligned}$$

$$\mathscr{I}_1 = \mathrm{col}\{I_{MN_0}, 0_{((M+1)N-MN_0)\times MN_0}\}, \mathscr{K}_0 = [K_0, 0_{1\times(N-N_0)}],$$
$$\mathscr{I}_0 = [I_{N_0}, \ldots, I_{N_0}, 0_{N_0\times(M+1)(N-N_0)}] \in \mathbb{R}^{N_0\times(M+1)N}. \tag{26}$$

Then from (6), (10), (17), (24), (25) and (26), we arrive at the following system for $t \geq 0$,

$$\mathrm{d}X_z(t) = [F_z X_z(t) + \mathscr{B}K_0\mathscr{I}_0 X_e(t) + G(t)]\mathrm{d}t + \sigma(t)\mathrm{d}\mathscr{W}(t), \tag{27a}$$
$$\mathrm{d}X_e(t) = \mathbf{F}(t)\mathrm{d}t + \mathscr{I}_2\sigma(t)\mathrm{d}\mathscr{W}(t), \tag{27b}$$
$$\begin{aligned} &\mathbf{F}(t) = F_e X_e(t) + \mathscr{L}_\zeta \zeta(t) + \mathscr{I}_1 \Lambda_e \Upsilon_r(t) + H(t),\\ &\mathrm{d}z_n(t) = [-\lambda_n z_n(t) + g_n(t) - b_n\mathscr{K}_0 X_z(t)\\ &\qquad + b_n K_0 \mathscr{I}_0 X_e(t)]\mathrm{d}t + \sigma_n(t)\mathrm{d}\mathscr{W}(t),\ n > N. \end{aligned} \tag{27c}$$

For mean-square L^2-stability analysis of (27), we consider the Lyapunov functional:

$$\begin{aligned} &V(t) = V_{\mathrm{nom}}(t) + V_{P_e}(t) + V_{S_e}(t) + V_{R_e}(t),\\ &V_{\mathrm{nom}}(t) = V_{P_z}(t) + \rho\textstyle\sum_{n=N+1}^{\infty} z_n^2(t),\ V_{P_z}(t) = |X_z(t)|^2_{P_z},\\ &V_{P_e}(t) = |X_e(t)|^2_{P_e}, V_{S_e}(t) = \textstyle\int_{t-\frac{r}{M}}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathscr{I}_3 X_e(s)|^2_{S_e}\mathrm{d}s,\\ &\mathscr{I}_3 = \begin{bmatrix} I_{MN_0} & 0_{MN_0\times N_0} & 0 & 0_{MN_0\times(N-N_0)} \\ 0 & 0 & I_{M(N-N_0)} & 0 \end{bmatrix},\\ &V_{R_e}(t) = \tfrac{r}{M}\textstyle\int_{-\frac{r}{M}}^{0}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathscr{I}_3\mathbf{F}(s)|^2_{R_e}\mathrm{d}s\mathrm{d}\theta, \end{aligned} \tag{28}$$

where P_z, P_e, S_e, R_e are positive matrices of appropriate dimensions and $\rho > 0$ is a scalar. Without loss of generality we assume that $z(\cdot,t) = z(\cdot,0)$ for $t < 0$. In this regard, $X_z(t)$ for $t < 0$ is well-defined. The terms $V_{S_e}(t)$, $V_{R_e}(t)$ compensate the delay term $\Upsilon_r(t)$ in (27b). Note that $V_{R_e}(t)$ has the same form as in [18], since it compensates delay in e_i^j ($i = 1,\ldots,M$), whereas ODEs for these e_i^j do not contain noise. The stochastic term appears only in systems z^j, e_{M+1}^j where no delay term appears. Therefore, we do not need to construct the noise-dependent functional as introduced in [12], [21].

By Parseval's equality and the change of variables (18), we present $V_{\mathrm{nom}}(t)$ in (28) as

$$\begin{aligned} &V_{\mathrm{nom}}(t) = V_{P_z}(t) - V_1(t) + V_2(w(t),t), V_1(t) = \rho|X_z(t)|^2,\\ &V_2(t) = \rho\|w(\cdot,t) + \psi(\cdot)u(t-r)\|^2_{L^2}. \end{aligned} \tag{29}$$

For functions V_{P_z} and V_1, calculating the generator $\mathscr{L}$ along stochastic ODE (27a) (see [22, P. 149]), we have

$$\begin{aligned} &\mathscr{L}V_{P_z}(t) + 2\delta V_{P_z}(t) = X_z^{\mathrm{T}}(t)[P_z F_z + F_z^{\mathrm{T}} P_z + 2\delta P_z]X_z(t)\\ &+\sigma^{\mathrm{T}}(t)P_z\sigma(t) + 2X_z^{\mathrm{T}}(t)P_z\mathscr{B}K_0\mathscr{I}_0 X_e(t) + 2X_z^{\mathrm{T}}(t)P_z G(t),\\ &\mathscr{L}V_1(t) + 2\delta V_1(t) = \rho\textstyle\sum_{n=1}^{N} 2(-\lambda_n + \delta)z_n^2(t) + \rho|\sigma(t)|^2\\ &+\rho\textstyle\sum_{n=1}^{N} 2z_n(t)[g_n(t) - b_n\mathscr{K}_0 X_z(t) + b_n K_0\mathscr{I}_0 X_e(t)]. \end{aligned} \tag{30}$$

From (15) and (17) we have $\mathrm{d}u(t-r) = F_u(t-r)\mathrm{d}t$, where

$$\begin{aligned} &F_u(t) = -K_0(A_0 - B_0K_0)\hat{z}_1^{N_0}(t) - K_0\hat{G}_M^{N_0}(t) - K_0L_0C_0\hat{z}_2^{N_0}(t)\\ &\quad + K_0L_0C_0\hat{z}_1^{N_0}(t-\tfrac{r}{M}) + K_0L_0C_1[\hat{z}_1^{N-N_0}(t-\tfrac{r}{M}) - \hat{z}_2^{N-N_0}(t)]. \end{aligned}$$

Recalling $\mathscr{A}_2$ in (21), we can rewrite (20) subject to (17) as

$$\begin{aligned} &\mathrm{d}w(t) = [\mathscr{A}_2 w(t) + g(w(t) + \psi(x)u(t-r)) - \psi(\cdot)\mu u(t-r)\\ &\quad - \psi(\cdot)F_u(t-r)]\mathrm{d}t + \sigma(w(t) + \psi(\cdot)u(t-r))\mathrm{d}\mathscr{W}(t), \end{aligned} \tag{31}$$

where $w(t) = w(\cdot,t)$. Note that $w(t)$ is a strong solution to (31) (see Sec. II-C). For $V_2(t)$, calculating the generator $\mathscr{L}$ along (31) (see [23, P. 228]) we obtain

$$\begin{aligned} &\mathscr{L}V_2(t) \overset{(3),(18)}{\leq} 2\rho\langle\mathscr{A}_2 w(t) + g(z(t)), z(t)\rangle_{L^2}\\ &-2\rho\langle\mu\psi(\cdot)u(t-r), z(t)\rangle_{L^2} + \rho\bar{\sigma}^2\|z(t)\|^2_{L^2}\\ &= 2\rho\textstyle\sum_{n=1}^{\infty} z_n(t)\langle\mathscr{A}_2 w(t), \phi_n\rangle + 2\rho\sum_{n=1}^{\infty} z_n(t)g_n(t)\\ &+2\rho\textstyle\sum_{n=1}^{\infty} z_n(t)\langle-\mu\psi(\cdot)u(t-r), \phi_n\rangle + \rho\sum_{n=1}^{\infty}\bar{\sigma}^2 z_n^2(t). \end{aligned} \tag{32}$$

Using integration by parts, (1) and (19), we arrive at

$$\begin{aligned} &\langle\mathscr{A}_2 w(t), \phi_n\rangle = -\lambda_n w_n(t) = -\lambda_n z_n(t) + \lambda_n\langle\psi, \phi_n\rangle u(t-r),\\ &\langle-\mu\psi(\cdot)u(t-r), \phi_n\rangle = [b_n - \lambda_n\langle\psi, \phi_n\rangle]u(t-r). \end{aligned} \tag{33}$$

Substituting (33) into (32) and using (17), (24), we arrive at

$$\begin{aligned} &\mathscr{L}V_2(w(t),t) + 2\delta V_2(w(t),t) = \rho\textstyle\sum_{n=1}^{\infty} 2(-\lambda_n + \delta + \frac{\bar{\sigma}^2}{2})z_n^2(t)\\ &\quad + \rho\textstyle\sum_{n=1}^{\infty} 2z_n(t)[g_n(t) - b_n\mathscr{K}_0 X_z(t) + b_n K_0\mathscr{I}_0 X_e(t)]. \end{aligned} \tag{34}$$

Let $\alpha_1, \alpha_2, \alpha_3 > 0$. By the Young inequalities we have

$$\begin{aligned} &\textstyle\sum_{n=N+1}^{\infty} 2z_n(t)g_n(t) \leq \sum_{n=N+1}^{\infty}\frac{z_n^2(t)}{\alpha_1} + \alpha_1\sum_{n=N+1}^{\infty} g_n^2(t)\\ &\quad \overset{(7)}{\leq} \textstyle\sum_{n=N+1}^{\infty}\frac{1}{\alpha_1}z_n^2(t) - \alpha_1|G(t)|^2 + \alpha_1\sum_{n=1}^{\infty} g_n^2(t),\\ &\textstyle\sum_{n=N+1}^{\infty} 2z_n(t)[-b_n\mathscr{K}_0 X_z(t) + b_n K_0\mathscr{I}_0 X_e(t)]\\ &\quad \overset{(8)}{\leq} \textstyle\sum_{n=N+1}^{\infty}\frac{\lambda_n}{\alpha_2}z_n^2(t) + \frac{2\alpha_2(N+1)}{N^2\pi^2}|\mathscr{K}_0 X_z(t)|^2\\ &\qquad + \textstyle\sum_{n=N+1}^{\infty}\frac{\lambda_n}{\alpha_3}z_n^2(t) + \frac{2\alpha_3(N+1)}{N^2\pi^2}|K_0\mathscr{I}_0 X_e(t)|^2. \end{aligned} \tag{35}$$

By Parseval's equality we have

$$\textstyle\sum_{n=1}^{\infty} g_n^2(t) \overset{(3)}{\leq} \bar{g}^2|X_z(t)|^2 + \bar{g}^2\sum_{n=N+1}^{\infty} z_n^2(t). \tag{36}$$

Combination of (30), (34), (35) and (36) yields

$$\begin{aligned} &\mathscr{L}V_{\mathrm{nom}}(t) + 2\delta V_{\mathrm{nom}}(t) \leq X_z^{\mathrm{T}}(t)[P_z F_z + F_z^{\mathrm{T}} P_z + 2\delta P_z\\ &\quad + \tfrac{2\rho\alpha_2(N+1)}{N^2\pi^2}\mathscr{K}_0^{\mathrm{T}}\mathscr{K}_0 + \rho(\bar{\sigma}^2 + \alpha_1\bar{g}^2)I]X_z(t)\\ &\quad + \sigma^{\mathrm{T}}(t)[P_z - \rho I]\sigma(t) + \textstyle\sum_{n=N+1}^{\infty}\chi_n z_n^2(t)\\ &\quad + 2X_z^{\mathrm{T}}(t)P_z[\mathscr{B}K_0\mathscr{I}_0 X_e(t) + G(t)]\\ &\quad + \tfrac{2\rho\alpha_3(N+1)}{N^2\pi^2}|K_0\mathscr{I}_0 X_e(t)|^2 - \rho\alpha_1|G(t)|^2, \end{aligned} \tag{37}$$

where $\chi_n = 2\rho(-\lambda_n + \delta + \frac{\bar{\sigma}^2}{2} + \frac{\alpha_1}{2}\bar{g}^2 + \frac{1}{2\alpha_1} + \frac{\lambda_n}{2\alpha_2} + \frac{\lambda_n}{2\alpha_3})$. For V_{P_e}, V_{S_e}, V_{R_e}, calculating the generator $\mathscr{L}$ along (27b) (see [22, P. 149]), we have

$$\begin{aligned} &\mathscr{L}V_{P_e}(t) + 2\delta V_{P_e}(t) = X_e^{\mathrm{T}}(t)[P_e F_e + F_e^{\mathrm{T}} P_e + 2\delta P_e]X_e(t)\\ &\quad + 2X_e^{\mathrm{T}}(t)P_e[\mathscr{L}_\zeta\zeta(t) + \mathscr{I}_1\Lambda_e\Upsilon_r(t) + H(t)] + |\mathscr{I}_2\sigma(t)|^2_{P_e},\\ &\mathscr{L}V_{S_e}(t) + 2\delta V_{S_e}(t) \leq |\mathscr{I}_3 X_e(t)|^2_{S_e} - \varepsilon_M|\mathscr{I}_3 X_e(t) - \Upsilon_r(t)|^2_{S_e},\\ &\mathscr{L}V_{R_e}(t) + 2\delta V_{R_e}(t) \leq \tfrac{r^2}{M^2}|\mathscr{I}_3\mathbf{F}(t)|^2_{R_e} - \tfrac{r\varepsilon_M}{M}\textstyle\int_{t-\frac{r}{M}}^{t}|\mathscr{I}_3\mathbf{F}(s)|^2_{R_e}\mathrm{d}s, \end{aligned} \tag{38}$$

where $\varepsilon_M = \mathrm{e}^{-2\delta r/M}$. By Jensen's inequality, we obtain

$$\tfrac{r}{M}\textstyle\int_{t-\frac{r}{M}}^{t}|\mathscr{I}_3\mathbf{F}(s)|^2_{R_e}\mathrm{d}s \geq |\int_{t-\frac{r}{M}}^{t}\mathscr{I}_3\mathbf{F}(s)\mathrm{d}s|^2_{R_e} \overset{(27b)}{=} |\Upsilon_r(t)|^2_{R_e}. \tag{39}$$

By Parseval's equality we have

$$\begin{aligned} &|H_{M+1}^{N_0}(t)|^2 + |H_{M+1}^{N-N_0}(t)|^2 = \textstyle\sum_{n=1}^{N}|g_n(t) - \hat{g}_n^{(M+1)}(t)|^2\\ &\leq \textstyle\int_0^1|g(z(x,t)) - g(\Phi_1(x)\hat{z}_{M+1}^{N_0}(t) + \Phi_2(x)\hat{z}_{M+1}^{N-N_0}(t))|^2\mathrm{d}x\\ &\leq \bar{g}^2\textstyle\int_0^1|z(x,t) - \Phi_1(x)\hat{z}_{M+1}^{N_0}(t) - \Phi_2(x)\hat{z}_{M+1}^{N-N_0}(t)|^2\mathrm{d}x\\ &\leq \bar{g}^2|e_{M+1}^{N_0}(t)|^2 + \bar{g}^2|e_{M+1}^{N-N_0}(t)|^2 + \bar{g}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t),\\ &|H_i^{N_0}(t)|^2 + |H_i^{N-N_0}(t)|^2 \leq \bar{g}^2|e_i^{N_0}(t)|^2 + \bar{g}^2|e_i^{N-N_0}(t)|^2, \end{aligned}$$

where $1 \leq i \leq M$, which implies

$$|H(t)|^2 \leq \bar{g}^2|X_e(t)|^2 + \bar{g}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t). \tag{40}$$

Besides, from Parseval's equality and (3) we have

$$|\sigma(t)|^2 \leq \|\sigma(z(\cdot,t))\|^2_{L^2} \leq \bar{\sigma}^2|X_z(t)|^2 + \bar{\sigma}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t). \tag{41}$$

By Cauchy-Schwarz inequality, we have $\zeta^2(t) \leq \|c\|_N^2\sum_{n=N+1}^{\infty} z_n^2(t)$. Let $\hat{\chi}_n = \chi_n + \beta_1\bar{g}^2 + \beta_2\bar{\sigma}^2$, where $\beta_1, \beta_2 > 0$. Then from the monotonicity of λ_n, we find

$$\textstyle\sum_{n=N+1}^{\infty}\hat{\chi}_n z_n^2(t) \leq \hat{\chi}_{N+1}\|c\|_N^{-2}\zeta^2(t) \tag{42}$$

provided $\hat{\chi}_{N+1} < 0$.

Let $\eta(t) = \mathrm{col}\{X_z(t), X_e(t), \Upsilon_r(t), \zeta(t), G(t), H(t)\}$. By (37)-(42) and the S-procedure [24, Sec 3.2.3], we get

$$\begin{aligned} &\mathscr{L}V(t) + 2\delta V(t) + \beta_1[\bar{g}^2|X_e(t)|^2 + \bar{g}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t)\\ &-|H(t)|^2] + \beta_2[\bar{\sigma}^2|X_z(t)|^2 + \bar{\sigma}^2\textstyle\sum_{n=N+1}^{\infty} z_n^2(t) - |\sigma(t)|^2]\\ &\leq \sigma^{\mathrm{T}}(t)\Psi_1\sigma(t) + \eta^{\mathrm{T}}(t)\Psi_2\eta(t) < 0 \end{aligned} \tag{43}$$

provided

$$\Psi_1 = P_z + \mathscr{I}_2^{\mathrm{T}} P_e\mathscr{I}_2 - \rho I - \beta_2 I < 0, \tag{44a}$$
$$\Psi_2 = \Xi + (r/M)^2\Theta^{\mathrm{T}} R_e\Theta < 0, \tag{44b}$$

where

$$\Theta = [0,\ \mathscr{I}_3 F_e,\ \mathscr{I}_3\mathscr{I}_1\Lambda_e,\ \mathscr{I}_3\mathscr{L}_\zeta,\ 0,\ \mathscr{I}_3],$$

$$\Xi = \begin{bmatrix} \Psi_{11} & P_z\mathscr{B}K_0\mathscr{I}_0 & 0 & 0 & P_z & 0 \\ * & \Psi_{22} & P_e\mathscr{I}_1\Lambda_e+\varepsilon_M\mathscr{I}_3^{\mathrm{T}}S_e & P_e\mathscr{L}_\zeta & 0 & P_e \\ * & * & -\varepsilon_M(S_e+R_e) & 0 & 0 & 0 \\ * & * & * & \hat{\chi}_{N+1}\|c\|_N^{-2} & 0 & 0 \\ * & * & * & * & -\alpha_1\rho I & 0 \\ * & * & * & * & * & -\beta_1 I \end{bmatrix}, \quad (45)$$

$$\Psi_{11} = P_zF_z + F_z^{\mathrm{T}}P_z + 2\delta P_z + \frac{2\rho\alpha_2(N+1)}{N^2\pi^2}\mathscr{K}_0^{\mathrm{T}}\mathscr{K}_0 + \rho(\bar{\sigma}^2+\alpha_1\bar{g}^2)I + \beta_2\bar{\sigma}^2 I,$$

$$\Psi_{22} = P_eF_e + F_e^{\mathrm{T}}P_e + 2\delta P_e + (1-\varepsilon_M)\mathscr{I}_3^{\mathrm{T}}S_e\mathscr{I}_3 + \frac{2\rho\alpha_3(N+1)}{N^2\pi^2}\mathscr{I}_0^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathscr{I}_0 + \beta_1\bar{g}^2 I.$$

Applying Schur complement, we find that (44b) holds iff

$$\left[\begin{array}{cc|c} \Xi_1 & \frac{r}{M}\Theta^{\mathrm{T}}R_e & \begin{bmatrix} 0_{N(2M+2)\times 3} \\ 1 \quad 1 \quad 1 \\ 0_{N(M+2)\times 3}\end{bmatrix} \\ * & -R_e & 0 \\ \hline * & * & -\mathrm{diag}\{\frac{\alpha_1\|c\|_N^2}{\rho}, \frac{\alpha_2\|c\|_N^2}{\rho\lambda_{N+1}}, \frac{\alpha_3\|c\|_N^2}{\rho\lambda_{N+1}}\} \end{array}\right] < 0, \quad (46)$$

where $\Xi_1 = \Xi$ defined in (45) with $\hat{\chi}_{N+1}$ therein replaced by $2\rho\|c\|_N^{-2}(-\lambda_{N+1}+\delta+\frac{\bar{\sigma}^2}{2}+\frac{\alpha_1}{2}\bar{g}^2)+\beta_1\bar{g}^2\|c\|_N^{-2}+\beta_2\bar{\sigma}^2\|c\|_N^{-2}$. Summarizing, we obtain:

Theorem 1: Consider system (2) with control law (17), measurement (4) with $c \in L^2(0,1)$ satisfying (11), $z_0 \in \mathscr{D}(\mathscr{A}_2)$ almost surely and $z_0 \in L^2(\Omega; L^2(0,1))$. Let $N_0 \in \mathbb{N}$ satisfy (9) and $N \in \mathbb{N}$ satisfy $N \geq N_0$. Assume that L_0 and K_0 are obtained from (12) and (13), respectively. Given $r > 0$, if there exist positive definite matrices P_z, P_e S_e, R_e, positive scalars α_1, α_2, α_3, β_1, β_2 and tuning parameter $\rho > 0$ such that LMIs (44a) and (46) hold, then the solution $z(x,t)$ to (2) subject to the control law (17) and the corresponding observer $\hat{z}(x,t)$ given by (16) satisfy

$$\mathbb{E}[\|z(\cdot,t)\|_{L^2}^2 + \|\hat{z}(\cdot,t)\|_{L^2}^2] \leq D\mathrm{e}^{-2\delta t}\mathbb{E}\|z(\cdot,0)\|_{L^2}^2, \quad t \geq 0, \quad (47)$$

for some $D \geq 1$. Given $r > 0$, LMIs (44a) and (46) are always feasible for M, N large enough and $\bar{\sigma}, \bar{g} > 0$ small enough.

Proof: First, by arguments similar to the proof of Theorem 2.1 in [7], we can obtain from (43) that $\mathbb{E}V(t) \leq \mathrm{e}^{-2\delta t}\mathbb{E}V(0)$, $t \geq 0$. By the definition of $V(t)$ in (28), we can get (47).

For any given $r > 0$, to prove the feasibility of (44) for large enough N, M and small enough $\bar{\sigma}, \bar{g} > 0$, we take $\bar{\sigma}, \bar{g} \to 0^+$, $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $P_z = \mathrm{diag}\{\hat{P}_z, p_1 I_{N-N_0}\}$, $P_e = \mathrm{diag}\{\hat{P}_e, p_2 I_{(M+1)(N-N_0)}\}$, $S_e = \mathrm{diag}\{\hat{S}_e, I_{M(N-N_0)}\}$, $R_e = \mathrm{diag}\{\hat{R}_e, I_{M(N-N_0)}\}$, with $0 < \hat{P}_z \in \mathbb{R}^{N_0\times N_0}$, $0 < \hat{P}_e \in \mathbb{R}^{(M+1)N_0\times(M+1)N_0}$, $0 < \hat{S}_e, \hat{R}_e \in \mathbb{R}^{MN_0\times MN_0}$ and $p_1, p_2 > 0$. By using suitable congruent transformation to Ψ_2, applying Schur complement repeatedly and letting $p_1 \to 0^+$, $p_2, \beta_1 \to \infty$, we find that $\Psi_2 < 0$ if

$$\left[\begin{array}{c|ccc|cc} \hat{\psi}_{11} & \hat{P}_zB_0K_0 & \hat{P}_zB_0K_0\hat{\mathscr{I}}_0 & 0 & \hat{P}_z\mathbb{I}_1 & 0 \\ \hline * & \multicolumn{3}{c|}{\phi(\hat{P}_e,\hat{S}_e,\hat{R}_e)} & \begin{matrix} 0 \\ 0 \end{matrix} & \begin{matrix} \hat{P}_e\hat{L}_{02} \\ 0 \end{matrix} \\ \hline * & \multicolumn{3}{c|}{*} & \multicolumn{2}{c}{\mathrm{diag}\{-\rho I, 2\rho\hat{\chi}_{N+1}\|c\|_N^{-2}\}} \end{array}\right] + (\tfrac{r}{M})^2\hat{\Theta}^{\mathrm{T}}\hat{R}_e\hat{\Theta} \quad (48)$$

$$+ \frac{2\rho(N+1)}{N^2\pi^2}\mathrm{diag}\{\hat{\mathscr{I}}_4^{\mathrm{T}}K_0^{\mathrm{T}}K_0\hat{\mathscr{I}}_4, \hat{\mathscr{I}}_0^{\mathrm{T}}K_0^{\mathrm{T}}K_0\hat{\mathscr{I}}_0, 0, 0\} < 0,$$

where

$$\phi(\hat{P}_e,\hat{S}_e,\hat{R}_e) = \begin{bmatrix} \hat{\psi}_{22} & \hat{P}_e\hat{\Lambda}_e+\varepsilon_M\hat{S}_e \\ * & -\varepsilon_M(\hat{S}_e+\hat{R}_e)\end{bmatrix}, \hat{\Theta} = [0, \hat{L}_{02}C_0, F_{11}, \hat{\Lambda}_e, 0, \hat{L}_{02}],$$

$$F_{e11} = I_M \otimes (A_0 - L_0C_0) + J_{0,M}\otimes L_0C_0,$$

$$\hat{\psi}_{11} = \hat{P}_z(A_0 - B_0K_0) + (A_0-B_0K_0)^{\mathrm{T}}\hat{P}_z + 2\delta\hat{P}_z,$$

$$\hat{\psi}_{22} = \hat{P}_eF_{e11} + F_{e11}^{\mathrm{T}}\hat{P}_e + 2\delta\hat{P}_e + (1-\varepsilon_M)\hat{S}_e,$$

$$\hat{\mathscr{I}}_0 = [I_{N_0}, \ldots, I_{N_0}] \in \mathbb{R}^{N_0\times(M+1)N_0},\ \hat{\mathscr{I}}_4 = [-I_{N_0}, I_{N_0}],$$

$$\hat{\Lambda}_e = I_M \otimes L_0C_0 - J_{0,M}\otimes L_0C_0, \hat{L}_{02} = \mathrm{col}\{0_{(M-1)N_0\times 1}, L_0 - L_0\}.$$

For any given $r > 0$, we first fix very large $M > 0$. From Proposition 1 in [15] we can obtain for suitable positive matrices $\hat{P}_e$, $\hat{S}_e$, $\hat{R}_e$,

$$\phi(\hat{P}_e,\hat{S}_e,\hat{R}_e) + (\tfrac{r}{M})^2[F_{e11}, \hat{\Lambda}_e]^{\mathrm{T}}\hat{R}_e[F_{e11}, \hat{\Lambda}_e] < 0.$$

We replace $\phi(\hat{P}_e,\hat{S}_e,\hat{R}_e)$ with $\phi(\beta\hat{P}_e,\beta\hat{S}_e,\beta\hat{R}_e) = \beta\phi(\hat{P}_e,\hat{S}_e,\hat{R}_e)$, $\beta > 0$. Let $\hat{P}_z = P_c$, given in (13), resulting in $\hat{\psi}_{11} < 0$. Setting $\beta > 0$ to be large enough, then choosing $\rho = \sqrt{N}$ large enough, $\beta_2 = p_2^2$ and applying Schur complement three times in (48), we find that (44a) and (48) hold. Fixing such M and N and using continuity, we have that (44) are feasible provided $\bar{\sigma}, \bar{g} > 0$ are small enough. ■

E. Observer-based design: delay robustness

For the case of $M = 0$ in (14), $\hat{z}^j(t) = \hat{z}^j_{M+1}(t)$, $j \in \{N_0, N-N_0\}$ satisfy

$$\begin{aligned} \mathrm{d}\hat{z}^{N_0}(t) &= [A_0\hat{z}^{N_0}(t) + \hat{G}^{N_0}(t) + B_0u(t-r)]\mathrm{d}t \\ &\quad - L_0[C_0\hat{z}^{N_0}(t) + C_1\hat{z}^{N-N_0}(t) - y(t)]\mathrm{d}t, \\ \mathrm{d}\hat{z}^{N-N_0}(t) &= [A_1\hat{z}^{N-N_0}(t) + \hat{G}^{N-N_0}(t) + B_1u(t-r)]\mathrm{d}t, \end{aligned} \quad (49)$$

where $\hat{G}^j(t) = \hat{G}^j_{M+1}(t)$ is defined above Remark 1. Then our method degenerates into the observer-based control with the delay robustness as studied in [11] for deterministic PDEs. Following [11], we construct a finite-dimensional observer of the form

$$\hat{z}(x,t) = \Phi_1(x)\hat{z}^{N_0}(t) + \Phi_2(x)\hat{z}^{N-N_0}(t), \quad (50)$$

where Φ_1 and Φ_2 are defined below (15). We propose the controller

$$u(t) = K_0\hat{z}^{N_0}(t), \quad (51)$$

where $K_0 \in \mathbb{R}^{1\times N_0}$ is determined by (13).

Let $e^j(t) = z^j(t) - \hat{z}^j(t)$, $j \in \{N_0, N-N_0\}$. Using (10) and (49), we obtain

$$\begin{aligned} \mathrm{d}e^{N_0}(t) &= [(A_0 - L_0C_0)e^{N_0}(t) + H^{N_0}(t) \\ &\quad - L_0C_1e^{N-N_0}(t) - L_0\zeta(t)]\mathrm{d}t + \sigma^{N_0}(t)\mathrm{d}\mathscr{W}(t), \\ \mathrm{d}e^{N-N_0}(t) &= [A_1e^{N-N_0}(t) + H^{N-N_0}(t)]\mathrm{d}t + \sigma^{N-N_0}(t)\mathrm{d}\mathscr{W}(t), \end{aligned} \quad (52)$$

where $H^j(t) = G^j(t) - \hat{G}^j(t)$, $j \in \{N_0, N-N_0\}$. Introduce the following notations:

$$X(t) = \mathrm{col}\{\hat{z}^{N_0}(t), \hat{z}^{N-N_0}(t), e^{N_0}(t), e^{N-N_0}(t)\},$$

$$F = \begin{bmatrix} A_0 - B_0K_0 & 0 & L_0C_0 & L_0C_1 \\ -B_1K_0 & A_1 & 0 & 0 \\ 0 & 0 & A_0 - L_0C_0 & -L_0C_1 \\ 0 & 0 & 0 & A_1 \end{bmatrix}, \hat{G}(t) = \begin{bmatrix} \hat{G}^{N_0}(t) \\ \hat{G}^{N-N_0}(t) \end{bmatrix},$$

$$H(t) = \begin{bmatrix} H^{N_0}(t) \\ H^{N-N_0}(t) \end{bmatrix},\ \ \mathbf{I}_1 = \begin{bmatrix} I_N \\ 0_{N\times N} \end{bmatrix},\ \ \mathbf{I}_2 = \begin{bmatrix} 0_{N\times N} \\ I_N \end{bmatrix},$$

$$\mathscr{B}_1 = \mathrm{col}\{B_0, B_1, 0_{N\times 1}\},\ \mathscr{L}_0 = \mathrm{col}\{L_0, 0, -L_0, 0\},$$

$$\mathscr{K}_1 = [K_0, 0_{1\times(2N-N_0)}],\ \ \Upsilon_r^{N_0}(t) = \hat{z}^{N_0}(t) - \hat{z}^{N_0}(t-r).$$

Then, from (4), (49)-(52), we have the closed-loop systems:

$$\begin{aligned} \mathrm{d}X(t) &= \mathbf{F}(t)\mathrm{d}t + \mathbf{I}_2\sigma(t)\mathrm{d}\mathscr{W}(t), \\ \mathrm{d}z_n(t) &= [-\lambda_nz_n(t) + g_n(t) - b_n\mathscr{K}_1X(t) \\ &\quad + b_nK_0\Upsilon_r^{N_0}(t)]\mathrm{d}t + \sigma_n(t)\mathrm{d}\mathscr{W}(t), \end{aligned} \quad (53)$$

where $\mathbf{F}(t) = FX(t) + \mathbf{I}_1\hat{G}(t) + \mathbf{I}_2H(t) + \mathscr{B}_1K_0\Upsilon_r^{N_0}(t) + \mathscr{L}_0\zeta(t)$, $\zeta(t) = \sum_{n=N+1}^{\infty} c_nz_n(t)$. Note that in (53), the stochastic term appears only in system e^{N_0}, e^{N-N_0}, whereas the delay term appears only in system $\hat{z}^{N_0}, \hat{z}^{N-N_0}$. So there is still a separation between stochastic term and delay term. For system (53), we consider the Lyapunov functional:

$$\begin{aligned} V(t) &= V_{\mathrm{nom}}(t) + V_S(t) + V_R(t), \\ V_{\mathrm{nom}}(t) &= V_P(t) + \rho\textstyle\sum_{n=N+1}^{\infty} z_n^2(t),\ V_P(t) = |X(t)|_P^2, \\ V_S(t) &= \textstyle\int_{t-r}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathscr{I}_1X(s)|_S^2\mathrm{d}s, \\ V_R(t) &= r\textstyle\int_{-r}^{0}\int_{t+\theta}^{t} \mathrm{e}^{-2\delta(t-s)}|\mathscr{I}_1\mathbf{F}(s)|_R^2\mathrm{d}s\mathrm{d}\theta, \end{aligned} \quad (54)$$

where $\mathscr{I}_1 = [I_{N_0}, 0_{N_0\times(2N-N_0)}]$, P, S, R are positive matrices with appropriate dimensions. The terms V_S and V_R are introduced to compensate $\Upsilon_r^{N_0}$ in system $\hat{z}^{N_0}$. Following the arguments similar to (29)-(46), we have

$$\begin{aligned}&\mathscr{L}V(t)+2\delta V(t)+\beta_1[\bar{g}^2|\mathbf{I}_2^{\mathrm{T}}X(t)|^2+\bar{g}^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)-|H(t)|^2]\\&+\beta_2[\bar{\sigma}^2|\mathbf{I}_3X(t)|^2+\bar{\sigma}^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)-|\sigma(t)|^2]\le 0\end{aligned}$$

provided

$$\begin{aligned}&\mathbf{I}_2^{\mathrm{T}}P\mathbf{I}_2-\rho I-\beta_2 I<0,\\&\left[\begin{array}{cc|c}\Xi_1 & \frac{r}{M}\Theta^{\mathrm{T}}R & \begin{bmatrix}0_{(4N+N0)\times 3}\\ 1\quad 1\quad 1\end{bmatrix}\\ * & -R & 0\\ \hline * & * & -\mathrm{diag}\{\frac{\alpha_1\|c\|_N^2}{\rho},\frac{\alpha_2\|c\|_N^2}{\rho\lambda_{N+1}},\frac{\alpha_3\|c\|_N^2}{\rho\lambda_{N+1}}\}\end{array}\right]<0,\end{aligned}\tag{55}$$

where $\Theta = \mathscr{I}_1[F,\ \mathscr{B}_1K_0,\ \mathbf{I}_1,\ \mathbf{I}_2,\ \mathscr{L}_0]$,

$$\begin{aligned}&\Xi=\begin{bmatrix}\Psi_{11} & P\mathscr{B}_1K_0+\varepsilon_r\mathscr{I}_1^{\mathrm{T}}S & P\mathbf{I}_1 & P\mathbf{I}_2 & P\mathscr{L}_0\\ * & \Psi_{22} & 0 & 0 & 0\\ * & * & -\rho\alpha_1 I & -\rho\alpha_1 I & 0\\ * & * & * & -(\rho\alpha_1+\beta_1)I & 0\\ * & * & * & * & \chi_{N+1}\|c\|_N^{-2}\end{bmatrix},\\&\Psi_{11}=PF+F^{\mathrm{T}}P+2\delta P+\tfrac{2\rho\alpha_2(N+1)}{N^2\pi^2}\mathscr{K}_1^{\mathrm{T}}\mathscr{K}_1+\beta_1\bar{g}^2\mathbf{I}_2\mathbf{I}_2^{\mathrm{T}}\\&\qquad+(\rho\bar{\sigma}^2+\rho\alpha_1\bar{g}^2+\beta_2\bar{\sigma}^2)\mathbf{I}_3\mathbf{I}_3^{\mathrm{T}}+(1-\varepsilon_r)\mathscr{I}_1^{\mathrm{T}}S\mathscr{I}_1,\\&\Psi_{22}=\tfrac{2\rho\alpha_3(N+1)}{N^2\pi^2}K_0^{\mathrm{T}}K_0-\varepsilon_r(S+R),\\&\chi_{N+1}=2\rho(-\lambda_{N+1}+\delta+\tfrac{\bar{\sigma}^2}{2}+\tfrac{\alpha_1}{2}\bar{g}^2)+\beta_1\bar{g}^2+\beta_2\bar{\sigma}^2,\\&\mathbf{I}_3=\begin{bmatrix}I_{N_0} & 0 & I_{N_0} & 0\\ 0 & I_{N-N_0} & 0 & I_{N-N_0}\end{bmatrix}.\end{aligned}$$

Summarizing, LMIs (55) guarantee the mean-square L^2 exponential stability of system (53) with a decay rate δ.

F. Sub-predictor construction without $\hat{z}_{M+1}^{N_0}$, $\hat{z}_{M+1}^{N-N_0}$.

If we ignore $\hat{z}_{M+1}^j(t)$ in the construction of sub-predictors, i.e., $\hat{z}_M^j(t-\frac{1}{M}r)\mapsto z^j(t)$ in (14) (see, e.g., [18]), the last term for system $\hat{z}_M^{N_0}$ in (15) becomes $-L_0[C_0\hat{z}_M^{N_0}(t-\frac{r}{M})+C_1\hat{z}_M^{N-N_0}(t-\frac{r}{M})-y(t)]$. Define $e_M^j(t)=z^j(t)-\hat{z}_M^j(t-\frac{r}{M})$. Then from (6), (10), (17), (24)-(26) (ignoring $\hat{z}_{M+1}^j(t)$), we have the closed-loop system ($M\ge 2$):

$$\begin{aligned}&\mathrm{d}X_z(t)=[F_zX_z(t)+\mathscr{B}K_0\mathscr{I}_0X_e(t)+G(t)]\mathrm{d}t+\sigma(t)\mathrm{d}\mathscr{W}(t),\\&\mathrm{d}X_e(t)=\mathbf{F}_e(t)\mathrm{d}t+\mathbf{I}\sigma(t)\mathrm{d}\mathscr{W}(t),\\&\mathrm{d}z_n(t)=[-\lambda_nz_n(t)+g_n(t)-b_n\mathscr{K}_0X_z(t)\\&\qquad+b_nK_0\mathscr{I}_0X_e(t)]\mathrm{d}t+\sigma_n(t)\mathrm{d}\mathscr{W}(t),\quad n>N,\end{aligned}\tag{56}$$

where $\mathbf{F}_e(t)=F_eX_e(t)+H(t)+\mathscr{I}_1\Lambda_e\Upsilon_r(t)+\mathscr{L}_\zeta\zeta(t-\frac{r}{M})$, $X_z(t)$, $\Upsilon_r(t)$, F_z, Λ_e, $G(t)$, $\sigma(t)$, $\mathscr{B}$, $\mathscr{K}_0$ are defined in (26),

$$\begin{aligned}&X_e(t)=\mathrm{col}\{e_1^{N_0}(t),\dots,e_M^{N_0}(t),e_1^{N-N_0}(t),\dots,e_M^{N-N_0}(t)\},\\&H(t)=\mathrm{col}\{H_1^{N_0}(t),\dots,H_M^{N_0}(t),H_1^{N-N_0}(t),\dots,H_M^{N-N_0}(t)\},\\&F_e=\begin{bmatrix}I_M\otimes(A_0-L_0C_0)+J_{0,M}\otimes L_0C_0 & -I_M\otimes L_0C_1+J_{0,M}\otimes L_0C_1\\ 0 & I_M\otimes A_1\end{bmatrix},\\&\mathbf{I}=\begin{bmatrix}0_{(M-1)N_0\times N_0} & 0\\ I_{N_0} & 0\\ 0_{(M-1)(N-N_0)\times N_0} & 0\\ 0 & I_{N-N_0}\end{bmatrix},\quad \mathscr{L}_\zeta=\begin{bmatrix}0_{(M-2)N_0\times 1}\\ L_0\\ -L_0\\ 0_{M(N-N_0)\times 1}\end{bmatrix},\\&\mathscr{I}_0=[I_{N_0},\dots,I_{N_0},0_{N_0\times M(N-N_0)}]\in\mathbb{R}^{N_0\times MN},\\&\mathscr{I}_1=\mathrm{col}\{I_{MN_0},0_{M(N-N_0)\times MN_0}\}.\end{aligned}$$

For $M=1$, we have closed-loop system (56) with F_e, Λ_e, $\mathscr{L}_\zeta$ replaced by

$$F_e=\begin{bmatrix}A_0-L_0C_0 & -L_0C_1\\ 0 & A_1\end{bmatrix},\ \mathscr{L}_\zeta=\begin{bmatrix}-L_0\\ 0_{(N-N_0)\times 1}\end{bmatrix},\ \Lambda_e=[L_0C_0,L_0C_1].$$

Note that in closed-loop system (56), both the stochastic term and the delay term appear in system $e_M^{N_0}$, which requires us to construct a Lyapunov functional that depends on the deterministic and stochastic parts for the mean-square L^2 exponential stability of system (56):

$$\begin{aligned}&V(t)=V_{\mathrm{nom}}(t)+V_q(t)+V_{P_e}(t)+V_{S_e}(t)+V_{R_e}(t)+V_{Q_e}(t),\\&V_q(t)=q\textstyle\int_{t-\frac{r}{M}}^{t}\mathrm{e}^{-2\delta(t-s)}\zeta^2(s)\mathrm{d}s,\\&V_{S_e}(t)=\textstyle\int_{t-\frac{r}{M}}^{t}\mathrm{e}^{-2\delta(t-s)}|X_e(s)|_{S_e}^2\mathrm{d}s,\\&V_{R_e}(t)=\tfrac{r}{M}\textstyle\int_{-\frac{r}{M}}^{0}\int_{t+\theta}^{t}\mathrm{e}^{-2\delta(t-s)}|\mathbf{F}_e(s)|_{R_e}^2\mathrm{d}s\mathrm{d}\theta,\\&V_{Q_e}(t)=\textstyle\int_{-\frac{r}{M}}^{0}\int_{t+\theta}^{t}\mathrm{e}^{-2\delta(t-s)}|\mathbf{I}\sigma(t)|_{Q_e}^2\mathrm{d}s\mathrm{d}\theta,\end{aligned}\tag{57}$$

with $V_{\mathrm{nom}}, V_{P_z}, V_{P_e}$ defined in (28), where P_z, P_e, S_e, R_e, Q_e are positive matrices of appropriate dimensions and $\rho, q>0$ are scalars. Note that V_{S_e} and V_{R_e} are utilized to compensate delay term $\Upsilon_r(t)$ in system X_e, whereas functional $V_{Q_e}(t)$ (depends on $\mathbf{I}\sigma(t)$) is introduced to compensate the stochastic part in systems $e_M^{N_0}$ and $e_M^{N-N_0}$. Let $\eta(t)=\mathrm{col}\{X_z(t),X_e(t),\Upsilon_r(t),\xi(t),\zeta(t-\frac{r}{M}),G(t),H(t)\}$. By arguments similar to (29)-(46) and using Itô integral properties (see, e.g., [12, Eq. (46)]), we have

$$\begin{aligned}&\mathbb{E}[\mathscr{L}V(t)+2\delta V(t)]+\beta_1\mathbb{E}[\bar{g}^2|X_e(t)|^2+\bar{g}^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)\\&-|H(t)|^2]+\beta_2\mathbb{E}[\bar{\sigma}^2|X_z(t)|^2+\bar{\sigma}^2\textstyle\sum_{n=N+1}^{\infty}z_n^2(t)-|\sigma(t)|^2]\\&\le\mathbb{E}[\sigma^{\mathrm{T}}(t)\Psi_1\sigma(t)]+\mathbb{E}[\eta^{\mathrm{T}}(t)\Psi_2\eta(t)]+\tfrac{\Psi_3\mathbb{E}\zeta^2(t)}{\|c\|_N^2}<0\end{aligned}$$

provided

$$\begin{aligned}&\Psi_1=P_z-\rho I-\beta_2I+\tfrac{r}{M}\mathbf{I}^{\mathrm{T}}Q_e\mathbf{I}+\mathbf{I}^{\mathrm{T}}P_e\mathbf{I}<0,\\&\Psi_2=\Xi+(r/M)^2\Theta^{\mathrm{T}}R_e\Theta<0,\quad \Psi_3=q\|c\|_N^2+\hat{\chi}_{N+1}<0,\end{aligned}\tag{58}$$

where $\hat{\chi}_n$ is defined below (41), $\Theta=[0,F_e,\Lambda_e,0,\mathscr{L}_\zeta,0,I]$,

$$\begin{aligned}&\Xi=\begin{bmatrix}\Psi_{11} & P_zBK_0\mathscr{I}_0 & 0 & 0 & 0 & P_z & 0\\ * & \Psi_{22} & P_e\Lambda_e+\varepsilon_MS_e & 0 & P_e\mathscr{L}_\zeta & 0 & P_e\\ * & * & -\varepsilon_M(S_e+R_e) & \varepsilon_MR_e & 0 & 0 & 0\\ * & * & * & -\varepsilon_M(Q_e+R_e) & 0 & 0 & 0\\ * & * & * & * & -\varepsilon_Mq & 0 & 0\\ * & * & * & * & * & -\alpha_1\rho I & 0\\ * & * & * & * & * & * & -\beta_1 I\end{bmatrix},\\&\Psi_{11}=P_zF_z+F_z^{\mathrm{T}}P_z+2\delta P_z+\tfrac{2\rho\alpha_2(N+1)}{N^2\pi^2}\mathscr{K}_0^{\mathrm{T}}\mathscr{K}_0\\&\qquad+\rho(\bar{\sigma}^2+\alpha_1\bar{g}^2)I+\beta_2\bar{\sigma}^2I,\\&\Psi_{22}=P_eF_e+F_e^{\mathrm{T}}P_e+2\delta P_e+(1-\varepsilon_M)S_e\\&\qquad+\tfrac{2\rho\alpha_3(N+1)}{N^2\pi^2}\mathscr{I}_0^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathscr{I}_0+\beta_1\bar{g}^2I.\end{aligned}$$

Applying Schur complement, we find that $\Psi_3<0$ iff

$$\left[\begin{array}{c|ccc}q\|c\|_N^2+2\rho(-\lambda_{N+1}+\delta+\frac{\bar{\sigma}^2}{2}+\frac{\alpha_1}{2}\bar{g}^2)+\beta_1\bar{g}^2+\beta_2\bar{\sigma}^2 & 1 & 1 & 1\\ \hline * & \multicolumn{3}{c}{-\frac{1}{\rho}\mathrm{diag}\{\alpha_1,\frac{\alpha_2}{\lambda_{N+1}},\frac{\alpha_3}{\lambda_{N+1}}\}}\end{array}\right]<0.$$

Feasibility of (58) guarantees the mean-square L^2 exponential stability of the closed-loop system (56) with a decay rate δ.

III. A NUMERICAL EXAMPLE

In this section, we consider system (2) where g satisfies (3) with $\bar{g}=0.5$, which results in an unstable open-loop system for $\sigma(z)\equiv 0$. Let $N_0=1$ and $c(x)=\chi_{[0,0.9]}(x)$ (an indicator function). Take $\delta=4$. The observer and controller gains L_0 and K_0 are found from (12) and (13) and are given by $L_0=5$, $K_0=4.5$.

First, we consider the observer-based control for delay robustness (i.e., Sec. II-E with delay robustness and Sec. II-F with 1 sub-predictor). Take $\delta=0$, $\bar{\sigma}\in\{0.3,0.4,0.5\}$, $\rho=1$. The LMIs were verified, respectively, for $N\in\{4,6,8\}$. The results are given in Table I, which show that Sec. II-E with simpler LMIs (no stochastic-dependent terms in Lyapunov functional) allows slightly larger delays.

Next, we consider the sub-predictors for any delays. Take $\delta=0$, $\bar{\sigma}\in\{0.3,0.4,0.5\}$, $\rho=1$. The LMIs in Theorem 1 and Sec. II-F were verified, respectively, for $N\in\{4,6,8\}$, number of sub-predictors chosen as $2,3,4,5,6$ to obtain the maximal values of r which preserve the feasibility of LMIs. The results are given in Table II for LMIs in Theorem 1 (with $\hat{z}_{M+1}^{N_0}$

and $\hat{z}_{M+1}^{N-N_0}$) and in Table III for LMIs in Sec. II-F (without $\hat{z}_{M+1}^{N_0}$ and $\hat{z}_{M+1}^{N-N_0}$). From Tables II and III we can see that Theorem 1 and Sec. II-F lead to complementary results, whereas Theorem 1 leads to a larger delay for comparatively large M and has less computational complexity for the same number of sub-predicts and observer dimensions. Similar to the deterministic case in [15], [18] for large number of sub-predictors, due to the term $\frac{2\rho\alpha_3(N+1)}{N^2\pi^2}\mathscr{I}_0^{\mathrm{T}}K_0^{\mathrm{T}}K_0\mathscr{I}_0$, we need much larger N to guarantee the feasibility of LMIs in (44) and (58). Note that differently from the deterministic case, for larger M, we require smaller upper bounds $\bar{\sigma}$ on the noise intensity to guarantee the feasibility of LMIs.

TABLE I

MAXIMAL r FOR FEASIBILITY: SEC.II-E VS SEC.II-F WITH $M = 1$.

	$N=4$		$N=6$		$N=8$	
$\bar{\sigma}$	Sec.II-E	Sec.II-F	Sec.II-E	Sec.II-F	Sec.II-E	Sec.II-F
0.3	0.235	0.231	0.240	0.237	0.241	0.239
0.4	0.225	0.220	0.230	0.225	0.231	0.227
0.5	0.215	0.208	0.219	0.213	0.221	0.215

TABLE II

MAXIMAL r FOR FEASIBILITY OF LMIS IN THEOREM 1.

	$N=4$			$N=6$			$N=8$		
$M+1\backslash\bar{\sigma}$	0.3	0.4	0.5	0.3	0.4	0.5	0.3	0.4	0.5
2	0.23	0.22	0.21	0.24	0.23	0.22	0.24	0.23	0.22
3	0.39	0.37	0.35	0.39	0.38	0.35	0.40	0.38	0.36
4	0.50	0.47	0.41	0.51	0.48	0.43	0.51	0.48	0.43
5	0.59	0.52	0.35	0.60	0.54	0.40	0.61	0.55	0.42
6	0.63	0.41	—	0.66	0.42	0.02	0.67	0.43	0.11

TABLE III

MAXIMAL r FOR FEASIBILITY OF LMIS IN SEC.II-F.

	$N=4$			$N=6$			$N=8$		
$M\backslash\bar{\sigma}$	0.3	0.4	0.5	0.3	0.4	0.5	0.3	0.4	0.5
2	0.36	0.34	0.31	0.37	0.35	0.32	0.37	0.35	0.33
3	0.46	0.43	0.39	0.48	0.44	0.40	0.48	0.45	0.41
4	0.55	0.50	0.41	0.56	0.52	0.44	0.58	0.52	0.45
5	0.60	0.48	0.21	0.63	0.53	0.33	0.64	0.55	0.36
6	0.60	0.23	—	0.65	0.40	—	0.67	0.42	0.02

IV. CONCLUSIONS

In this paper, we considered output-feedback control of 1D stochastic semilinear heat equation with constant input delay and nonlinear noise under Neumann actuation and nonlocal measurement. To compensate delay we constructed a nonlinear sequential sub-predictor. Improvements and extension of predictor-based control to various stochastic PDEs may be topics for future research.

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