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On an extension of the Friedkin-Johnsen model: The effects of a homophily-based influence matrix

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***Abstract*— In this paper we propose an extended version of the Friedkin-Johnsen (FJ) model that accounts for the effects of homophily mechanisms on the agents' mutual appraisals. The proposed model consists of two difference equations. The first one describes the opinions' evolution, namely how agents modify their opinions taking into account both their personal beliefs and the influences of other agents, as in the standard FJ model. Meanwhile, the second equation models how the influence matrix involved in the opinion formation process updates according to a homophily mechanism, by allowing both positive and negative appraisals. We derive necessary and sufficient conditions for the proposed time-varying version of the classical FJ model to asymptotically converge to a constant solution. In the case of a single discussion topic, asymptotic convergence is always ensured and the limit behavior of the system is derived in closed form.**

I. Introduction

During the last decades, understanding and describing the way we communicate and exchange ideas has been the focus of extensive investigation. Opinion dynamics has become a very lively research field that attracts and combines concepts and techniques from different disciplines, ranging from sociology, psychology and economy, to mathematics and control engineering. Such strong interest resulted in a large number of models trying to capture and mathematically formalize the process of opinion formation in a social network. Despite the clear simplifications that the proposed models have introduced, they have been able to provide many insights into the dynamical processes of diffusion and evolution of opinions in human population [16]. While the initial interest focused mainly on models aimed at explaining consensus [6], more recently a lot of models have been proposed to justify observed behaviors of social groups such as disagreement, polarization and conflict [1], [8], which are even more frequent than consensus in real scenarios. Among them, one of the most famous is surely the Friedkin-Johnsen (FJ) model [8], that captures the fact that the opinion of an individual on a topic evolves under the effects of two main driving forces. On the one hand, the individual (in the following also referred to as "agent") is influenced by the opinions on the same topic of his/her neighbours in the social network, each one weighted by the appraisal that the agent has of them. On the other hand, agents tend to "stick" to their own initial opinions (prejudices), that therefore keep affecting their opinions at each subsequent time. This asymptotically leads to opinions which are closer to each other than the initial opinions, but not identical, namely consensus is no longer reached. In the original FJ model [8] opinions are expressed on a single topic and the influence matrix, that quantifies how much each agent values the opinions of the others, is constant and row stochastic. Later on, the model has been extended to the case of multiple topics [7], [14], [15], with time-varying row stochastic influence matrices [17], and recently a version of the FJ model whose influence matrix has both positive and negative entries has been proposed [9], thus accounting for the fact that relationships among individuals in a network may also be competitive/antagonistic (see [2], [18]).

In all such models the influence matrix is either constant or time-varying, nonnegative or real valued, but it is always assumed to be independent of the dynamics of the agents' opinions. This assumption does not seem to be realistic since in real life very often the interpersonal relationships among the agents depend on the comparison of their opinions, following a homophily mechanism, namely the tendency of individuals to associate and interact more intensively with like-minded people [3], [4], [12], [13]. In other words, agents tend to be influenced by individuals who hold similar opinions and, conversely, give little or even negative weights to the opinions of agents with whom they mostly disagree.

In recent times, an interesting model of the interplay between homophily-based appraisal dynamics and influence-based opinion dynamics has been proposed by F. Liu et al. [11]. The model explores for the first time how the evolution of the opinions of a group of agents on a certain number of issues/topics is influenced by the agents' mutual appraisals and, conversely, the agents' mutual appraisals are updated based on the agents' opinions on the various issues, according to a homophily principle. More recently, a simplified version of the model, that does not quantify the level of mutual appraisal but only its sign, has been proposed in [5]. It has been shown that this model is simpler and yet equally accurate in predicting the asymptotic evolution of the individuals' opinions in small networks, as the ones we will consider in this paper.

In this contribution we propose an extended version of the FJ model whose influence matrix is generated according to a homophily mechanism, by keeping into account only the signs of the agents' appraisals. In the general case, we provide a necessary and sufficient condition for the opinion matrix of a group of n agents on m topics to asymptotically converge to a constant solution, that depends on the agents' initial opinions as well as on the agents' stubbornness coefficients. Finally, we consider the special

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case where there is only one discussion topic and provide an explicit expression of the agents' asymptotic opinions.

The paper is organized as follows: Section II introduces the model explaining the meaning of all the quantities involved. Section III provides the main results about the dynamics of the model. Section IV addresses the single-topic case. Finally, in Section V, a numerical example is proposed.

Notation. Given two integers k and n, with $k \leq n$, the symbol $[k, n]$ denotes the set $\{k, k+1, \ldots, n\}$. We let $\mathbb{1}_n$ ($\mathbb{0}_n$) denote the n-dimensional vector with all unitary (zero) entries. We denote by $\mathbb{e}_i$ the i-th canonical vector of dimension n, where n is always clear from the context. In the sequel, the (i, j)-th entry of a matrix A is denoted by $[A]_{ij}$, while the i-th entry of a vector v by v_i. The function $\text{sgn} : \mathbb{R}^{n\times m} \to \{-1, 0, 1\}^{n\times m}$ is the function that maps a real matrix A into a matrix taking values in $\{-1, 0, 1\}$, in accordance with the sign of its entries, namely $[\text{sgn}(A)]_{ij} = \text{sgn}([A]_{ij})$ for every i, j. The *max norm of a matrix* $A \in \mathbb{R}^{n\times n}$ is defined as $\|A\|_{max} := \max_{i,j\in[1,n]} |A_{ij}|$. The *Euclidean norm of a vector* $v \in \mathbb{R}^n$ is defined as $\|v\|_2 := \left(\sum_{i=1}^n v_i^2\right)^{1/2}$.

In this paper by an *undirected and signed graph* we mean a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V} = [1, n]$ is the set of nodes (or vertices), $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges (or arcs) and $\mathcal{A} \in \{-1, 0, 1\}^{n\times n}$ is the *adjacency matrix* of the graph $\mathcal{G}$. An arc $(j, i) \in \mathcal{E}$ if and only if $[\mathcal{A}]_{ij} \neq 0$. When so, $[\mathcal{A}]_{ij}$ represents the (positive or negative) weight of the arc. Moreover, due to the fact that the graph is undirected, the matrix $\mathcal{A}$ is symmetric and so $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. Since the adjacency matrix $\mathcal{A}$ uniquely identifies the graph, in the following we will use the notation $\mathcal{G}(\mathcal{A})$. A graph $\mathcal{G}$ is said to be *structurally balanced* [2], [18] if the set of its nodes can be partitioned into two disjoint subsets such that (s.t.) the weights of the edges between nodes belonging to the same subset are nonnegative, and the weights of the edges between nodes belonging to different subsets are nonpositive.

II. The model

Given a group of n agents expressing their opinions on m distinct topics, we denote by $Y(t) \in \mathbb{R}^{n\times m}$ the *opinion matrix at time t*, whose (i, j)-th entry represents the opinion that agent i has about topic j at time $t \in \mathbb{Z}_+$.
We denote by $W(t) \in \mathbb{R}^{n\times n}$ the *influence matrix at time t*, whose (i, j)-th entry represents the influence that agent j has on agent i at time t. Specifically, we assume that:

- $[W(t)]_{ij} > 0 \Leftrightarrow i$ positively regards the opinion of j;
- $[W(t)]_{ij} < 0 \Leftrightarrow i$ negatively regards the opinion of j;
- $[W(t)]_{ij} = 0 \Leftrightarrow i$ neglects the opinion of j.

We assume that at every time t the influence that agent j has on agent i is given by agent i's appraisal of agent j. On the other hand, the appraisal that i has of j at time t is based on a homophily mechanism [3], [12], since it depends on the comparison of the opinions that agents i and j have about all the topics at time $t-1$. As in [5], we consider only the signs of the mutual appraisals, rather than their values. This is motivated by the fact that from a practical viewpoint it is complicated to quantify the appraisals each individual has of the others, but, on the contrary, it is easy to recognize if the relationship between two agents is friendly or hostile. Moreover, this choice is more robust to modelling errors and more realistic, because agent j can influence positively or negatively agent i's opinion about a certain topic, but this influence does not necessarily scale with the absolute value of their mutual appraisal. Furthermore, we have chosen to account also for the fact that two agents decide not to rely on each other's opinions, i.e., $[W(t)]_{ij} = 0$. Indeed, in small-size networks, as the ones we are considering, this corresponds to the case where agent i knows agent j, but decides to neglect his/her opinions, for lack of correlation between their evaluations. Therefore, the fact that the mutual appraisal is zero is an information that should be considered, justifying the choice of dividing each row of the influence matrix by n, instead of by the number of its non-zero entries. However, it is worth noticing that condition $[W(t)]_{ij} = 0$ is a very rare occurrence, as it will be clear in the following, since it corresponds to the case where the (real-valued) opinion vectors of agent i and j at time $t-1$ (i.e., the i-th and j-th rows of $Y(t-1)$) are orthogonal.

Based on these premises, in this paper we propose the following model, representing the intertwining between an FJ-type opinion dynamics and a homophily-based appraisal mechanism:

$$Y(t+1) = (I_n - \Theta)W(t+1)Y(t) + \Theta Y(0), \quad (1)$$

$$W(t+1) = \frac{1}{n}\text{sgn}\left(Y(t)Y(t)^\top\right), \quad (2)$$

where $\Theta \in \mathbb{R}^{n\times n}$ is a diagonal matrix. For every $i \in [1, n]$, the nonnegative diagonal entry $\theta_i = [\Theta]_{ii}$ of Θ represents the stubbornness of agent i in preserving the original opinion. In the paper we will steadily assume:

Assumption 1. For every $i \in [1, n]$ the stubbornness of agent i satisfies $0 < \theta_i < 1$.

It is easy to see that if the i-th row of $Y(0)$ is zero, then the i-th row of $Y(t)$ is zero for every $t \geq 0$. This corresponds to the case where the i-th agent has no interest in any of the topics, and gives zero weight to the others' opinions. We rule out this pathological case.

Assumption 2. The matrix $Y(0) \in \mathbb{R}^{n\times m}$ is devoid of zero rows.

Finally, it is worth noticing that the influence matrix $W(t+1)$, as defined, is a symmetric matrix for every $t \geq 0$.

III. General results

In order to investigate the asymptotic behavior of the opinion matrix, we first provide an alternative way to express the opinion matrix at time t, by introducing the transition matrix $M(t)$, relating $Y(t)$ to $Y(0)$. In the following we will steadily resort to the following notation:

$$S_0 := Y(0)Y(0)^\top. \quad (3)$$

Proposition 1. *For every $Y(0) \in \mathbb{R}^{n\times m}$, at every time $t \geq 0$ we have*

$$Y(t+1) = M(t+1)Y(0), \quad (4)$$

where

$$\begin{aligned} M(t+1) &= (I_n - \Theta)W(t+1)M(t) + \Theta, && (5)\\ M(0) &= I_n, && (6)\\ W(t+1) &= \frac{1}{n}\mathrm{sgn}(M(t)S_0M(t)^\top). && (7) \end{aligned}$$

Proof. We prove the result by induction on t. We first show that the result is true for $t = 0$. We observe that

$$W(1) = \frac{1}{n}\mathrm{sgn}(Y(0)Y(0)^\top) = \frac{1}{n}\mathrm{sgn}(M(0)S_0M(0)^\top),$$

and hence

$$\begin{aligned} Y(1) &= [(I_n - \Theta)W(1) + \Theta]\, Y(0)\\ &= [(I_n - \Theta)W(1)M(0) + \Theta]\, Y(0) = M(1)Y(0), \end{aligned}$$

where

$$M(1) = (I_n - \Theta)W(1)M(0) + \Theta.$$

Now we assume that equations (4), (5) and (7) are true for $t < \bar{t}$ and prove that they hold true also for $t = \bar{t}$.
From $W(\bar{t}+1) = \frac{1}{n}\mathrm{sgn}(Y(\bar{t})Y(\bar{t})^\top)$, by the inductive assumption (on the expression of Y), we obtain $W(\bar{t}+1) = \frac{1}{n}\mathrm{sgn}(M(\bar{t})S_0M(\bar{t})^\top)$. On the other hand,

$$\begin{aligned} Y(\bar{t}+1) &= (I_n - \Theta)W(\bar{t}+1)Y(\bar{t}) + \Theta Y(0)\\ &= [(I_n - \Theta)W(\bar{t}+1)M(\bar{t}) + \Theta]\, Y(0)\\ &= M(\bar{t}+1)Y(0), \end{aligned}$$

where $M(\bar{t}+1) = (I_n - \Theta)W(\bar{t}+1)M(\bar{t}) + \Theta$. □

Based on Proposition 1, we now derive the main result regarding the asymptotic behavior of the sequence $\{M(t)\}_{t\in\mathbb{Z}_+}$.

Theorem 2. *For every $Y(0) \in \mathbb{R}^{n\times m}$, the solution of the system in (5)-(6)-(7) is bounded, and specifically[1] $\|M(t)\|_{max} \leq 1$ for all $t \in \mathbb{Z}_+$. Moreover, the following conditions are equivalent:*

(i) There exists $M_\infty := \lim_{t\to+\infty} M(t)$;
(ii) There exists $T \in \mathbb{Z}_+, T \geq 1$, such that $W(t) = W(T) =: W_\infty$, for every $t \geq T$.

If either of the above equivalent conditions holds, then

$$M_\infty = (I_n - \Theta)W_\infty M_\infty + \Theta. \quad (8)$$

Proof. We prove the result by induction on $t \in \mathbb{Z}_+$. We first observe that $\|M(0)\|_{max} = \|I_n\|_{max} = 1$. We now assume that $\|M(t)\|_{max} \leq 1$ and prove that $\|M(t+1)\|_{max} \leq 1$. From (5) we get

$$\begin{aligned} \|M(t+1)\|_{max} &= \|(I_n - \Theta)W(t+1)M(t) + \Theta\|_{max}\\ &\leq \|(I_n - \Theta)\frac{1}{n}\mathbb{1}_n\mathbb{1}_n^\top\mathbb{1}_n\mathbb{1}_n^\top + \Theta\|_{max}\\ &= \|(I_n - \Theta)\mathbb{1}_n\mathbb{1}_n^\top + \Theta\|_{max} = 1. \end{aligned}$$

(i) ⇒ (ii) From (5) we easily deduce that

$$\begin{aligned} M(t+1) - M(t) &= (I_n - \Theta)W(t+1)M(t)\\ &- (I_n - \Theta)W(t)M(t-1),\\ &= (I_n - \Theta)[W(t+1) - W(t)]M(t)\\ &+ (I_n - \Theta)W(t)[M(t) - M(t-1)]. \end{aligned}$$

This implies

$$\begin{aligned} \|M(t+1) - M(t)\|_{max} &\geq\\ &\geq \|(I_n - \Theta)[W(t+1) - W(t)]M(t)\|_{max} \quad (9)\\ &- \|(I_n - \Theta)W(t)[M(t) - M(t-1)]\|_{max}. \end{aligned}$$

We notice that

$$\begin{aligned} \|(I_n - \Theta)\ W(t)[M(t) - M(t-1)]\|_{max}&\\ &\leq n\|(I_n - \Theta)W(t)\|_{max}\|M(t) - M(t-1)\|_{max}\\ &< \|M(t) - M(t-1)\|_{max}, \quad (10) \end{aligned}$$

where we used the fact that $\|AB\|_{max} \leq n\|A\|_{max}\|B\|_{max}$ and that $\|(I_n - \Theta)W(t)\|_{max} \leq \max_i \frac{1-\theta_i}{n} < \frac{1}{n}$. So, (9) and (10) together lead to

$$\begin{aligned} \|M(t+1) - M(t)\|_{max} + \|M(t) - M(t-1)\|_{max}&\\ > \|(I_n - \Theta)[W(t+1) - W(t)]M(t)\|_{max}. \quad (11) \end{aligned}$$

From equation (5) we deduce

$$\begin{aligned} M(t) &= (I_n - \Theta)W(t)M(t-1) + \Theta\\ &= (I_n - \Theta)W(t)M(t)\\ &+ (I_n - \Theta)W(t)[M(t-1) - M(t)] + \Theta, \end{aligned}$$

that leads to

$$\begin{aligned} M(t) &= [I_n - (I_n - \Theta)W(t)]^{-1}[(I_n - \Theta)W(t)\, \cdot\\ &\cdot\ (M(t-1) - M(t)) + \Theta], \quad (12) \end{aligned}$$

where we used the fact that $(I_n - \Theta)W(t)$ is Schur[2] and hence $I_n - (I_n - \Theta)W(t)$ is nonsingular.
Assume, now, that there exists $\lim_{t\to+\infty} M(t)$. This means that for every $\varepsilon > 0$ there exists $T \in \mathbb{Z}_+$ such that for every $t \geq T$ we have $\|M(t) - M(t-1)\|_{max} < \varepsilon$. If ε is sufficiently small, this also ensures that, for every $t \geq T$, the matrix $M(t)-M(t+1)$ is infinitesimal, and hence the matrix $(I_n - \Theta)W(t)(M(t-1) - M(t)) + \Theta \approx \Theta$ is nonsingular.
As a consequence, (12) leads to

$$\begin{aligned} M(t)^{-1} &= [(I_n - \Theta)W(t)(M(t-1) - M(t)) + \Theta]^{-1}\, \cdot\\ &\cdot\ [I_n - (I_n - \Theta)W(t)]. \end{aligned}$$

To summarize, for every $t \geq T$ the matrix $M(t)^{-1}$ exists and is a bounded matrix (being the product of two bounded matrices). This guarantees that there exists $b > 0$, such that

$$\|M(t)^{-1}\|_{max} < b, \quad \forall t \geq T. \quad (13)$$

Now, we use the fact that

$$\begin{aligned} &\|(I_n - \Theta)[W(t+1) - W(t)]\|_{max}\\ &= \|(I_n - \Theta)[W(t+1) - W(t)]M(t)M(t)^{-1}\|_{max}\\ &\leq n\|(I_n - \Theta)[W(t+1) - W(t)]M(t)\|_{max}\|M(t)^{-1}\|_{max} \end{aligned}$$

[1] Note that $W(t+1), t \in \mathbb{Z}_+$, is always bounded, since it takes values in $\{-1/n, 0, 1/n\}$.

[2] The result can be easily proved by resorting to Gershgorin Circles Theorem [10].

and hence

$$\|(I_n-\Theta)[W(t+1)-W(t)]M(t)\|_{max} \geq \frac{\|(I_n-\Theta)[W(t+1)-W(t)]\|_{max}}{n\|M(t)^{-1}\|_{max}}. \quad (14)$$

By replacing (14) in (11), and keeping into account (13), we obtain

$$\begin{aligned}\|M(t+1)- & M(t)\|_{max}+\|M(t)-M(t-1)\|_{max} \\ & > \frac{1}{nb}\|(I_n-\Theta)[W(t+1)-W(t)]\|_{max}. \quad (15)\end{aligned}$$

This guarantees that for every $t \geq T$

$$2\varepsilon > \frac{1}{nb}\|(I_n-\Theta)[W(t+1)-W(t)]\|_{max}.$$

Since the matrix W takes values in a finite set, and hence the nonzero entries of $W(t+1)-W(t)$ cannot be arbitrarily small, this ensures that $W(t+1)=W(t)$, $\forall t \geq T$.

(ii) ⇒ *(i)* If condition *(ii)* holds, then for every $t \geq T$ we have

$$\begin{aligned} M(t) &= (I_n-\Theta)W(t)M(t-1)+\Theta \\ &= (I_n-\Theta)W_\infty M(t-1)+\Theta, \end{aligned}$$

and hence

$$M(t+1)-M(t)=(I_n-\Theta)W_\infty[M(t)-M(t-1)].$$

This implies that for every $t \geq T$

$$\begin{aligned}\|M(t+1) - M(t)\|_{max} &= \|(I_n-\Theta)W_\infty[M(t)-M(t-1)]\|_{max} \\ &\leq n\|(I_n-\Theta)W_\infty\|_{max}\|M(t)-M(t-1)\|_{max} \\ &\leq \alpha\|M(t)-M(t-1)\|_{max} \\ &\leq \alpha^{t-T+1}\|M(T)-M(T-1)\|_{max},\end{aligned}$$

where we used the fact that $\|AB\|_{max} \leq n\|A\|_{max}\|B\|_{max}$ and that $\|(I_n-\Theta)W_\infty\|_{max} \leq \max_i \frac{1-\theta_i}{n} = \frac{\alpha}{n}$, $\alpha := \max_{i\in[1,n]}(1-\theta_i) < 1$. This ensures that

$$\lim_{t\to+\infty}\|M(t+1)-M(t)\|_{max}=0$$

and hence there exists $M_\infty = \lim_{t\to+\infty} M(t)$.

Condition (8) follows immediately. □

The main consequence of Theorem 2 is that, for every $Y(0) \in \mathbb{R}^{n\times m}$ for which $\exists M_\infty = \lim_{t\to+\infty} M(t)$, we can also ensure that $\exists Y_\infty := \lim_{t\to+\infty} Y(t)$ and $Y_\infty = M_\infty Y(0)$. Moreover, starting from some $T \in \mathbb{Z}_+, T \geq 1$, we have $W_\infty = W(T)$, and hence

$$W_\infty = \frac{1}{n}\text{sgn}\left(M(T)S_0M(T)^\top\right) \in \left\{-\frac{1}{n}, 0, \frac{1}{n}\right\}^{n\times n}. \quad (16)$$

We now explore some interesting properties of M_∞.

Proposition 3. *Given $Y(0) \in \mathbb{R}^{n\times m}$, if there exists $M_\infty = \lim_{t\to+\infty} M(t)$, then M_∞ is nonsingular and $\forall i \in [1,n]$*

(i) $\|M_\infty \mathbb{e}_i\|_\infty := \max_{j\in[1,n]} |[M_\infty]_{ji}| = |[M_\infty]_{ii}|$;

(ii) $[M_\infty]_{ii} > 0$.

Proof. We first prove that M_∞ is nonsingular. Suppose, by contradiction, that $v \in \mathbb{R}^n, v \neq \mathbb{0}_n$, belongs to the kernel of M_∞, i.e., $M_\infty v = \mathbb{0}_n$. Then, by making use of (8), we obtain

$$\mathbb{0}_n = M_\infty v = (I-\Theta)W_\infty M_\infty v + \Theta v \ \Rightarrow\ \Theta v = \mathbb{0}_n,$$

which is not possible as each $\theta_i \in (0,1)$, by Assumption 1.

(i) Let i be any index in $[1,n]$. Then

$$M_\infty \mathbb{e}_i = (I-\Theta)W_\infty M_\infty \mathbb{e}_i + \Theta \mathbb{e}_i.$$

If we permute the entries of $M_\infty \mathbb{e}_i$, using an $n\times n$ permutation matrix P, in such a way that $\tilde{v} := P^\top M_\infty \mathbb{e}_i = \begin{bmatrix}\tilde{v}_1 & \dots & \tilde{v}_n\end{bmatrix}^\top$, with $|\tilde{v}_1| \geq |\tilde{v}_2| \geq \dots \geq |\tilde{v}_n|$, we obtain

$$\begin{aligned}\tilde{v} &= P^\top M_\infty \mathbb{e}_i = P^\top(I-\Theta)PP^\top W_\infty PP^\top M_\infty \mathbb{e}_i + \\ &\quad + P^\top\Theta PP^\top \mathbb{e}_i = (I-\tilde{\Theta})\tilde{W}_\infty\tilde{v} + \tilde{\Theta}\mathbb{e}_j,\ \exists j \in [1,n],\end{aligned}$$

where $\tilde{\Theta} = P^\top\Theta P = \text{diag}\{\tilde{\theta}_1,\dots,\tilde{\theta}_n\}$ and $\tilde{W}_\infty = P^\top W_\infty P$. The first component of $\tilde{v}$, i.e., $\tilde{v}_1$, satisfies

$$\tilde{v}_1 = (1-\tilde{\theta}_1)\mathbb{e}_1^\top \tilde{W}_\infty \begin{bmatrix}\tilde{v}_1 & \dots & \tilde{v}_n\end{bmatrix}^\top + \tilde{\theta}_1 \mathbb{e}_1^\top \mathbb{e}_j,$$

which implies that

$$|\tilde{v}_1| \leq (1-\tilde{\theta}_1)\sum_{i=1}^n \frac{|\tilde{v}_i|}{n} + \tilde{\theta}_1 \mathbb{e}_1^\top \mathbb{e}_j. \quad (17)$$

Therefore, if $j \neq 1$, the right-hand side of (17) would be

$$(1-\tilde{\theta}_1)\sum_{i=1}^n \frac{|\tilde{v}_i|}{n} < \sum_{i=1}^n \frac{|\tilde{v}_i|}{n} \leq |\tilde{v}_1|,$$

a contradiction. Thus, it must be $j = 1$ and $\tilde{\theta}_1 = \theta_i$. So, we have $\tilde{v}_1 = [M_\infty \mathbb{e}_i]_i = [M_\infty]_{ii}$. This means that $\max_{j\in[1,n]} |[M_\infty]_{ji}| = |[M_\infty]_{ii}|$. Clearly, this is true for every index $i \in [1,n]$, namely for every column of M_∞.

(ii) We want to prove that $[M_\infty]_{ii} > 0, \forall i \in [1,n]$, which is equivalent to showing that $\tilde{v}_1 > 0$, by referring to the notation adopted in part *(i)*. Suppose, by contradiction, that $\tilde{v}_1 \leq 0$. Then, we get

$$\begin{aligned}\tilde{v}_1 &= (1-\tilde{\theta}_1)\mathbb{e}_1^\top \tilde{W}_\infty \begin{bmatrix}\tilde{v}_1 & \dots & \tilde{v}_n\end{bmatrix}^\top + \tilde{\theta}_1 \\ &= (1-\tilde{\theta}_1)[\tilde{W}_\infty]_{11}\tilde{v}_1 + (1-\tilde{\theta}_1)\sum_{j\neq 1}[\tilde{W}_\infty]_{1j}\tilde{v}_j + \tilde{\theta}_1.\end{aligned}$$

Consequently,

$$\left(1-(1-\tilde{\theta}_1)[\tilde{W}_\infty]_{11}\right)\tilde{v}_1 - \tilde{\theta}_1 = (1-\tilde{\theta}_1)\sum_{j\neq 1}[\tilde{W}_\infty]_{1j}\tilde{v}_j.$$

Note that since

$$\begin{aligned}[\tilde{W}_\infty]_{11} &= \left[\frac{1}{n}\text{sgn}(P^\top Y(T)Y(T)^\top P)\right]_{11} \\ &= \frac{1}{n}\text{sgn}(\mathbb{e}_1^\top P^\top Y(T)Y(T)^\top P\mathbb{e}_1) \\ &= \frac{1}{n}\text{sgn}(\|Y(T)^\top P\mathbb{e}_1\|_2^2)\end{aligned}$$

for some $T \in \mathbb{Z}_+$, $[\tilde{W}_\infty]_{11}$ belongs to $\{0, \frac{1}{n}\}$, and $1-(1-\tilde{\theta}_1)[\tilde{W}_\infty]_{11} > 0$. If $\tilde{v}_1 \le 0$, we get

$$\begin{aligned}&\left(1-(1-\tilde{\theta}_1)[\tilde{W}_\infty]_{11}\right)|\tilde{v}_1| + \tilde{\theta}_1\\ &= \left|\left(1-(1-\tilde{\theta}_1)[\tilde{W}_\infty]_{11}\right)\tilde{v}_1 - \tilde{\theta}_1\right|\\ &= (1-\tilde{\theta}_1)\left|\sum_{j\neq 1}[\tilde{W}_\infty]_{1j}\tilde{v}_j\right| \le (1-\tilde{\theta}_1)\tfrac{n-1}{n}|\tilde{v}_1|,\end{aligned}$$

which implies that

$$\left[1-(1-\tilde{\theta}_1)\left([\tilde{W}_\infty]_{11} + \frac{n-1}{n}\right)\right]|\tilde{v}_1| + \tilde{\theta}_1 \le 0. \quad (18)$$

Since $[\tilde{W}_\infty]_{11}$ belongs to $\{0, \frac{1}{n}\}$, then $1-(1-\tilde{\theta}_1)\left([\tilde{W}_\infty]_{11} + \frac{n-1}{n}\right) > 0$. So, all quantities on the left-hand side of (18) are nonnegative and, in particular, $\tilde{\theta}_1$ is positive. This contradicts inequality (18). Therefore, $\tilde{v}_1$ must be positive, which is equivalent to saying that $[M_\infty]_{ii} > 0, \forall i \in [1, n]$. □

The previous result means that, in case of convergence to a constant solution, the initial opinion of each agent impacts more on his/her own final opinion than on the final opinions of the other agents. In other words, the agent that weights more agent i's initial opinion is agent i himself/herself. Moreover, (and not unexpectedly!) such impact is always positive.

IV. Single-topic case

We now address the case where $m = 1$, namely there is only one discussion topic. When so, the opinion matrix is a column vector, that we now denote by $y(t) \in \mathbb{R}^n$, containing the opinions of the agents on the topic. It is easy to see that if we define $v(t) := \text{sgn}(y(t))$, then $\text{sgn}\left(y(t)y(t)^\top\right) = v(t)v(t)^\top$, and model (1)-(2) becomes:

$$\begin{aligned} y(t+1) &= (I-\Theta)W(t+1)y(t) + \Theta y(0) \quad (19)\\ W(t+1) &= \frac{1}{n}\text{sgn}\left(y(t)y(t)^\top\right) = \frac{1}{n}v(t)v(t)^\top, \quad (20)\end{aligned}$$

leading to the difference equation:

$$y(t+1) = \frac{1}{n}(I-\Theta)v(t)v(t)^\top y(t) + \Theta y(0). \quad (21)$$

We also note that in this context Assumption 2 amounts to imposing that $y(0)$ is devoid of zero entries. In fact, condition $y_i(0) = 0$ would lead the i-th agent to remain isolated and stick to the zero opinion.

Under the previous hypotheses, we can derive the following results.

Lemma 4. *For $m = 1$,*

$$v(t) = \text{sgn}(y(t)) = \text{sgn}(y(0)) = v(0), \qquad \forall\, t \in \mathbb{Z}_+.$$

Consequently,

$$W(t+1) = W(1) = \frac{1}{n}v(0)v(0)^\top, \quad \forall t \ge 1,$$

namely the influence matrix remains constant.

Proof. By induction on t. For $t = 1$, we have $v(1) = \text{sgn}(y(1)) = \text{sgn}\Big[(I-\Theta)\frac{1}{n}v(0)v(0)^\top y(0) + \Theta y(0)\Big] = \text{sgn}(y(0)) = v(0)$, where we used the fact that $v(0)^\top y(0) = \text{sgn}(y(0))^\top y(0) = \sum_{i=1}^n |y_i(0)| > 0$ (Assumption 2 rules out the case $y(0) = \mathbb{0}_n$). Suppose, now, that the result holds for $t < \bar{t}$. For $t = \bar{t}$:

$$\begin{aligned} v(\bar{t}+1) &= \text{sgn}(y(\bar{t}+1))\\ &= \text{sgn}\Big[(I-\Theta)\frac{1}{n}\overbrace{v(\bar{t})}^{=v(0)}\underbrace{v(\bar{t})^\top y(\bar{t})}_{\sum_{i=1}^n |y_i(\bar{t})|>0} + \Theta y(0)\Big]\\ &= v(0),\end{aligned}$$

where we used the inductive assumption $v(\bar{t}) = \text{sgn}(y(\bar{t})) = \text{sgn}(y(0)) = v(0)$ that ensures, in particular, $y(\bar{t}) \neq \mathbb{0}_n$. The second part immediately follows. □

As a consequence of the previous lemma, for $m = 1$ the model in (19)-(20) becomes time-invariant and the dynamics of $y(t)$ can be expressed as:

$$y(t+1) = \frac{1}{n}(I-\Theta)v(0)v(0)^\top y(t) + \Theta y(0). \quad (22)$$

Lemma 4 implies that the whole opinion dynamics evolves at each time step with an influence matrix that corresponds to a situation of structural balance [2], [18], by this meaning that $\mathcal{G}(W(t+1))$ is structurally balanced for every $t \ge 0$. We can now derive the following result.

Theorem 5. *For $m = 1$, the matrix sequence $\{M(t)\}_{t\in\mathbb{Z}_+}$ always converges to a constant limit matrix M_∞ and*

$$\begin{aligned} M_\infty &= \left[I_n + \frac{1}{\sum_{i=1}^n \theta_i}(I_n-\Theta)v(0)v(0)^\top\right]\Theta, \quad (23)\\ W_\infty &= \frac{1}{n}v(0)v(0)^\top,\end{aligned}$$

Proof. Lemma 4 ensures that $W(t) = \frac{1}{n}v(0)v(0)^\top$ for every $t \ge 1$. So, by Theorem 2, we can claim that $\exists M_\infty = \lim_{t\to\infty} M(t)$ and that $W_\infty = \frac{1}{n}v(0)v(0)^\top$. Moreover, from (8) we get $[I_n - (I_n-\Theta)W_\infty]M_\infty = \Theta$. By Gershgorin Circles Theorem [10] and Assumption 1, we can claim that $(I_n-\Theta)W_\infty$ is Schur stable and hence $I_n - (I_n-\Theta)W_\infty$ is invertible. Consequently,

$$M_\infty = [I_n - (I_n-\Theta)W_\infty]^{-1}\Theta.$$

Finally,

$$\begin{aligned}&[I_n - (I_n-\Theta)W_\infty]^{-1} = I_n + \sum_{k=1}^{+\infty}[(I_n-\Theta)\tfrac{1}{n}v(0)v(0)^\top]^k\\ &= I_n + [(I_n-\Theta)\tfrac{1}{n}v(0)v(0)^\top]\sum_{k=1}^{+\infty}\left(\frac{\sum_{i=1}^n(1-\theta_i)}{n}\right)^{k-1}\\ &= I_n + \frac{1}{\sum_{i=1}^n \theta_i}(I_n-\Theta)v(0)v(0)^\top.\end{aligned}$$

Thus, M_∞ is expressed as in (24). □

To conclude, we can provide an explicit expression for the agents' asymptotic opinions $y_\infty := \lim_{t\to+\infty} y(t)$, namely

$$y_\infty = \left[I_n + \frac{1}{\sum_{i=1}^n \theta_i}(I_n-\Theta)v(0)v(0)^\top\right]\Theta y(0).$$

Note, finally, that

$$W_\infty = \frac{1}{n}\text{sgn}(y_\infty y_\infty^\top) = \frac{1}{n}\text{sgn}(M_\infty y(0)y(0)^\top M_\infty^\top).$$

V. Example

Example. We consider a group of $n = 6$ agents discussing $m = 6$ topics. We assume that $\theta_1 = \theta_6 = \frac{2}{3}, \theta_2 = \theta_5 = \frac{1}{2}, \theta_3 = \theta_4 = \frac{1}{3}$ and that $Y(0)$ is:

$$Y(0) = \begin{bmatrix} -0.1317 & 1.7035 & -0.2350 & 0.0802 & 0.7824 & -0.6380 \\ 0.2968 & -0.6272 & 0.9015 & -0.4425 & -0.1206 & -0.7040 \\ -0.6075 & -0.3453 & 0.3935 & -0.9496 & 0.5671 & -0.3654 \\ 0.5217 & -0.2691 & -0.2884 & -0.1193 & -0.3721 & -1.1914 \\ 0.0244 & -0.2168 & -0.2278 & 1.1211 & -0.3104 & -0.7398 \\ -0.3392 & 0.7993 & 0.1429 & -0.9816 & -1.4906 & 0.2002 \end{bmatrix}.$$

The evolutions of the opinions on the 6 topics as well as the evolution of the influence matrix (that becomes stationary at time $t = 2$) and its graph of W_∞ are illustrated in Fig. 1.

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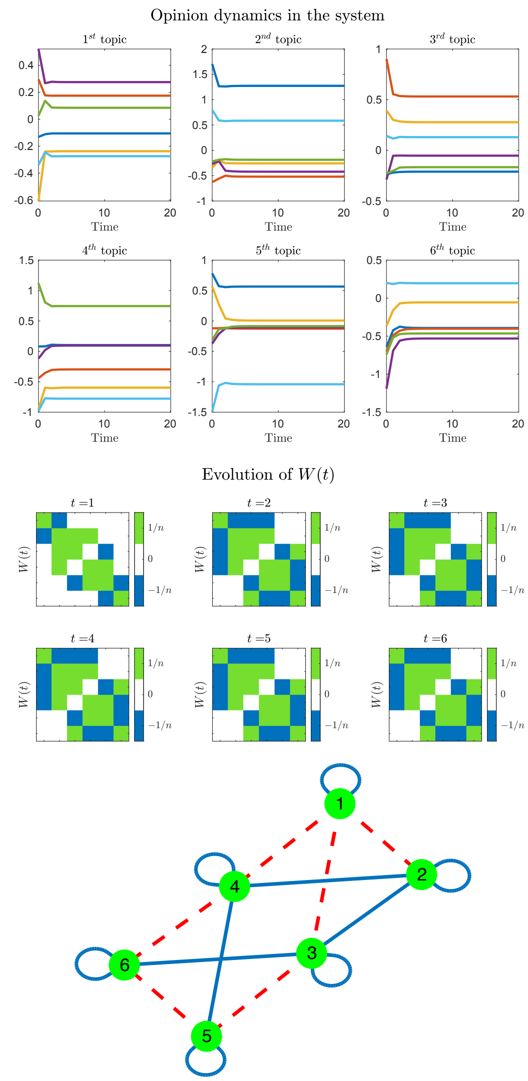

Fig. 1: Evolutions of the opinions on the 6 topics (top); Evolution of the influence matrix (middle); Graph associated with W_∞ (bottom).