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Circular Formation of General Linear Agents using Output Feedback

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Abstract— **This work is concern with achieving circular formation in a Multi-agent system (MAS) where each agent has the same linear dynamics. The approach taken here have several desirable features: (i) it requires at the minimum only one agent to know its own position while the rest need only relative position information from their neighbors; (ii) the approach is based on the output regulation methodology but uses the output feedback controller instead of the popular error feedback controller. The output feedback scheme has the advantage of a simpler design as the parameters are decoupled. With minimal modifications, the approach provided can be used to achieve MAS with user-defined center of the circle with different radii and elliptical formation. These features are illustrated using several examples.**

I. Introduction

Achieving formation of a multi-agent system (MAS) has been an active research area in recent years. One choice that is of great interest is the circular formation where agents move in a circle. Past works in this area are restricted to agents ([1], [2], [3], [4], [5], [6], [7], [8]) having special models, like unicycle [1], [2], [3], [4], [5], [6] or single integrator [7], [8]. There has been limited works on agents having general linear dynamics. Those that consider general linear dynamics typically use the approach of output regulation [9], [10], [11], [12].

Several considerations are important in the study of circular formation MAS. One is the amount of sensory information available to each agent. Obviously, an approach with fewer sensory feedback is desirable. In this regard, past works in the literature for general linear dynamics typically assumes all agents can measure its own position. The exception being the case of [5] where the minimum of one agent needs to know its own location while the rest require only relative position from its neighbors. Another consideration is the ability to specify the center of the circle and have a controller that realizes such a formation. Unfortunately, very few works deal with this aspect despite it being a useful requirement for practical implementations.

This work proposes a distributed controller, based on the output regulation approach, for circular formation of a MAS having agents with general linear dynamics. It differs from past approaches in the following ways. First, the proposed approach achieves circular formation with a user-specified radius and center of the circle. Second, the approach does not require all agents to know their own absolute positions; it requires that at least one agent knows its own position and assumes that the rest can measure the position of its neighbor relative to their own. This requirement is similar to that of [5] except that ours is for general linear dynamics while [5] is for unicycle models. Third, our approach uses a different controller from the standard controller used for output regulation. Specifically, standard output regulation approach uses the error feedback controller while our approach uses the output feedback controller. The use of output feedback is an important element in achieving distributed controllers and has the advantage of a decoupling design. Since the radius of the circle is bounded in value, an output feedback controller is feasible for circular formation while error feedback controller is particularly useful in tracking unbounded references like a ramp signal. Lastly, the approach can easily be adapted to tracking of an elliptical formation.

The notations used in this paper are standard. The sets of real numbers, n-dimensional real vectors and n by m real matrices are $\mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n\times m}$ respectively. The transpose of matrix A is A^T. Given $A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{q\times r}$, $A \otimes B \in \mathbb{R}^{nq\times mr}$ is the Kronecker product of A and B. $\mathrm{vec}(A)$ is a vector in $\mathbb{R}^{nm}$ and it is obtained by rearranging the columns of A in ascending order. e_i is column vector of all zeros with the i^{th} element being 1. For a square matrix Q, $Q \succ (\succeq) 0$ means Q is positive definite (semi-definite) and $\sigma(Q)$ denotes the set of all eigenvalues of Q. A directed graph is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $\mathcal{V} = \{1, \ldots, N\}$ being the nodes of the graph and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ being the set of edges. Hence, j is a neighbor of i if $(j,i) \in \mathcal{E}$ and $\mathcal{N}_i$ denotes the neighbors of i in $\mathcal{V}$. Associated with each $(j,i) \in \mathcal{E}$ is the weight α_{ij}. Its value is such that $\alpha_{ii} = 0$, $\alpha_{ij} > 0$ when $(j,i) \in \mathcal{E}$ and $\alpha_{ij} = 0$ otherwise. The Laplacian of $\mathcal{G}$ is $\mathcal{L} = [l_{ij}] \in R^{N\times N}$, where $l_{ii} = \sum_{j=1}^{N} \alpha_{ij}$ and $l_{ij} = -\alpha_{ij}$ if $i \neq j$.

II. Preliminaries and Problem Statement

All agents are assumed to have the same dynamics,

$$\dot{x}_i = Ax_i + Bu_i \quad i \in \mathcal{V} \tag{1}$$

with $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$. The outputs available to the agents depend on whether i is a leader or a follower agent. For this purpose, let $\mathcal{V} = \mathcal{V}_L \cup \mathcal{V}_F$ with $\mathcal{V}_L \cap \mathcal{V}_F = \emptyset$ and $\mathcal{V}_L \subset \mathcal{V} (\mathcal{V}_F \subset \mathcal{V})$ containing the indices of the leader (follower) agents. A leader agent is assumed to have

$$y_i = Cx_i \quad i \in \mathcal{V}_L \tag{2}$$

where $y_i \in \mathbb{R}^2$ refers to the 2-dimensional positional vector of agent i. Follower agents do not have positional information at their outputs. Instead, they are equipped with sensors that measure positions of its neighbors relative to its own position, explicitly represented by $y_{ij} (= y_i - y_j)$ as an entity for $i \in \mathcal{V}_F$ and $j \in \mathcal{N}_i$. Leader agents can also

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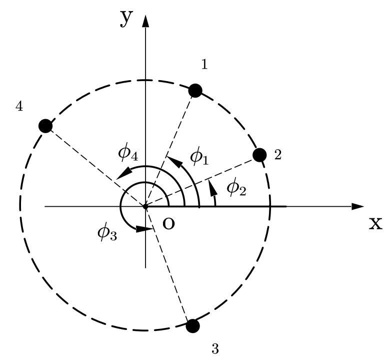

Fig. 1: A circular formation for 4 agents when $t = 0$

measure the relative positions y_{ij} and, since y_i is measurable independently, means that y_i and y_j are available separately to $i \in \mathcal{V}_L$. More information on the graph structure is given in section IV.

The objective of this work is to construct a distributed controller such that the MAS system achieves circular formation that is defined below.

Definition 1: (*Circular Formation*) Given a center $\bar{c} \in \mathbb{R}^2$, angular velocity $\omega \neq 0$, radius $r > 0$, and a set of phases $\{\phi_1, \phi_2, \dots, \phi_N\}$ where $\phi_i \in [0, 2\pi)$ for all $i \in \mathcal{V}$. Denote the reference trajectories by

$$y_i^r(t) := \bar{c} + (r\cos(\omega t + \phi_i), r\sin(\omega t + \phi_i))^T \tag{3}$$

If $y_i(t) \to y_i^r(t)$ for $i \in \mathcal{V}$, the agents are said to have achieved a circular formation.
An example of circular formation with 4 agents for the case where $\bar{c} = 0$ is shown in Fig.1.

The following assumptions are needed for our purpose.
(A1) The pair (A, B) is stabilizable.
(A2) (C, A) is detectable.

Clearly, these assumptions are mild requirement. Additional assumption on network connectivity is given in section IV. The next two sections contain discussions for the case with $\bar{c} = 0$ in (3). The case of non-zero $\bar{c}$ is discussed in subsection III-B.

III. Circular Motion for a Single Agent

This section discusses the design of a controller for one single agent to follow the prescribed circular path. Since only one agent is involved, subscript i is dropped from the expressions of (1), (2) and (3). Our approach follows the output regulation approach but uses an output feedback controller instead of the standard error feedback controller [13]. Our choice of controller is motivated by two considerations: (i) output feedback controller provides greater flexibility as the conditions needed to ensure tracking are decoupled, see Remark 1 in the sequel; (ii) the output feedback controller structure facilitates the tracking of circular formation in a multi-agent setting. These points will be made specific after the necessary exposition.

A. The case of $\bar{c} = 0$

Like standard output regulation [13], a model that specifies the reference trajectories is given by

$$\dot{v} = Sv = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix} v, \tag{4}$$

$$y^r = Qv \tag{5}$$

Clearly, $v(t) = \exp(St)v(0) = \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix} v(0) = \begin{pmatrix} r\cos(\omega t + \phi) \\ r\sin(\omega t + \phi) \end{pmatrix}$ when $v(0) = (r\cos\phi, r\sin\phi)^T$ and $y^r(t) = v(t)$ when $Q = I_2$. The proposed output regulation controller is a dynamic compensator given by

$$\begin{cases} \dot{\xi} = F\xi + Gy + L_2 v \\ u = K\xi + L_1 v \end{cases} \tag{6}$$

where $\xi \in \mathbb{R}^l$ is the state of the controller, L_1, L_2, F, G and K are the controller parameters. The combined system of (1), (4) and (6) is

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} A & BK & BL_1 \\ GC & F & L_2 \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} x \\ \xi \\ v \end{pmatrix} \tag{7}$$

The following properties of this system is known and is given next.

Lemma 1: (Output regulation via Output Feedback) Consider the system of (7) with (A1), (A2) holding. Then, (i) there exists $\{K, G, F\}$ such that

$$A_f := \begin{pmatrix} A & BK \\ GC & F \end{pmatrix} \tag{8}$$

is stable. (ii) Suppose $\{K, G, F\}$ has been found such that A_f is stable. The system of (7) achieves

$$\lim_{t\to\infty} (y(t) - Qv(t)) = 0 \tag{9}$$

for any $\{x(0), \xi(0), v(0)\}$ if and only if there are $\{L_1, L_2\}$ such that

$$\begin{cases} \Pi S = A\Pi + BK\Sigma + BL_1 \\ 0 = C\Pi - Q \\ \Sigma S = GC\Pi + F\Sigma + L_2 \end{cases} \tag{10}$$

has a solution $\{\Pi, \Sigma\}$.
Proof: Given in the appendix.

In the standard literature of output regulation, two controller schemes [13] are typically used: full-state feedback and error feedback. It is also known [13], [14] that feasibility of the full-state feedback conditions will lead to the feasibility conditions of the error feedback condition under (A1) and (A2) (Theorem 1.4.1 at page 19 of [13]). This result also holds for the case of the proposed controller under (A1) and (A2) and is shown next. Specifically, the conditions needed for a standard full-state feedback output regulation condition is that the set of matrix equation

$$\begin{cases} \Pi S = A\Pi + B\Gamma \\ 0 = C\Pi - Q \end{cases} \tag{11}$$

has a solution $\{\Pi, \Gamma\}$. While the full-state feedback controller is different from our proposed controller, the solvability of (10) is similar to that of (11). This connection is now made.

Lemma 2: Suppose $\{K, G, F\}$ are chosen such that A_f is stable. Then, there exists some $\{L_1, L_2\}$ such that (10) has a solution $\{\Pi, \Sigma\}$ if and only if (11) has a solution $\{\Pi, \Gamma\}$.

Proof: ($\Rightarrow$) Suppose a solution $\{\Pi, \Gamma\}$ of (11) is known. Let $L_1 = \Gamma - K\Sigma$ and $L_2 = \Sigma S - GC\Pi - F\Sigma$ for any arbitrary Σ and they satisfy (10). ($\Leftarrow$) Let $\Gamma = K\Sigma + L_1$ and then the first two equations of (10) becomes (11). ■

Remark 1: In the standard error feedback controller [13], the controller is of the form

$$\begin{cases} u = K\xi \\ \dot{\xi} = F\xi + G(y - Qv) \end{cases}$$

(where $y - Qv$ is the error term) and the conditions to be satisfied by $\{K, G.F\}$ are (i) A_f of (8) is stable and (ii) the set of equations

$$\begin{cases} \Pi S = A\Pi + BK\Sigma \\ 0 = C\Pi - Q \\ \Sigma S = F\Sigma \end{cases}$$

has a solution $\{\Pi, \Sigma\}$. Hence, the choice of $\{K, G.F\}$ has to simultaneously satisfy conditions (i) and (ii). This is a harder design problem than that given in Lemma 1. More exactly, the proposed approach allows the design of $\{K, G, F, L_1, L_2\}$ to be decoupled into two separate design problems: (i) the design of $\{K, G, F\}$ such that A_f is stable, and (ii) the design of $\{L_1, L_2\}$ such that (10) admits a solution $\{\Pi, \Sigma\}$ for the choice of $\{K, G, F\}$ obtained from (i).

B. The case of non-zero $\bar{c}$

The preceding section assumes that $y^r(t) = (r\cos(\omega t + \phi), r\sin(\omega t + \phi))^T$. In this section, the design of a controller that tracks $y^r(t) = \bar{c}$ is first discussed. Consider the reference model of

$$\dot{\bar{v}} = \bar{S}\bar{v} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\bar{v}, \quad \bar{v}(0) = \bar{c} \tag{12}$$

Clearly, the solution of the above is $\bar{v}(t) = \bar{c}$ for all t. Suppose $\{K, G, F\}$ have been chosen based on the procedure given in section III-A. Using the same procedure as in section III-A but with $Q = I_2$ and S replaced by $\bar{S}$ of (12), it is possible to obtain $\bar{L}_1$ and $\bar{L}_2$ from (10). With L_1, L_2 and $\bar{L}_1, \bar{L}_2$ available, the controller that can track $y^r(t) = \bar{c} + (r\cos(\omega t + \phi), r\sin(\omega t + \phi))^T$ is given by

$$\begin{cases} u = K\xi + L_1 v + \bar{L}_1\bar{v} \\ \dot{\xi} = F\xi + GCx + L_2 v + \bar{L}_2\bar{v} \end{cases} \tag{13}$$

This result holds since the combined system of (13) and (1) is

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & BK \\ GC & F \end{pmatrix}\begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} BL_1 \\ L_2 \end{pmatrix} v + \begin{pmatrix} B\bar{L}_1 \\ \bar{L}_2 \end{pmatrix}\bar{v} \tag{14}$$

$$y = Cx \tag{15}$$

and that the output is given by $y(t) = y_1(t) + y_2(t)$ where $y_1(t)$ is the output of the single agent with $u(t)$ given by (6) with L_1, L_2 obtained from (10) with S being that of (4) and $y_2(t)$ is the output of the same system with $u(t)$ given by (6) with $\bar{L}_1, \bar{L}_2$ obtained from (10) using $\bar{S}$ of (12).

IV. Circular Formation for the Multi-agent System

This section deals with the controller design for circular formation for the multi-agent system (1) consisting of $|\mathcal{V}_L|$ leaders and $|\mathcal{V}_F|$ followers. Like the discussion in section III, the controller consists of two parts: one for the case of $\bar{c} = 0$ and the other when $\bar{c} \neq 0$. Since the case of $\bar{c} \neq 0$ is relatively simple, the focus here is on the case of $\bar{c} = 0$. Without loss of generality, let $Q = I_2$ in this section. To distinguish leaders and followers in $\mathcal{G}$, an additional virtual node, node 0, is introduced. The resulting graph is denoted by $\overline{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ with $\bar{\mathcal{V}} = \mathcal{V} \cup \{0\}$ and $\bar{\mathcal{E}}$ includes all edges to the virtual node in addition to the original $\mathcal{E}$. As usual, α_{ij} is the weights associated with the edge from node j to node i and for the virtual node,

$$\begin{cases} \alpha_{i0} & > 0 \text{ when } i \in \mathcal{V}_L, \\ \alpha_{i0} & = 0 \text{ when } i \in \mathcal{V}_F. \end{cases}$$

Note that $\overline{\mathcal{G}}$ is the graph representation among agents for both relative position sensing and general communications. Both modes of information exchanges are needed in implementing the controller, as shown in the following paragraphs.

Consider the reference system for agent i as given in (4). The reference model for the combined N agents is

$$\dot{v}_c = S_c v_c = (I_N \otimes S)v_c \tag{16}$$

$$v_c = \begin{pmatrix} v_1^T & v_2^T & \cdots & v_N^T \end{pmatrix}^T \tag{17}$$

$$y_c^r(t) = ((y_1^r)^T, \cdots, (y_N^r)^T)^T = (I_N \otimes I_2)v_c \tag{18}$$

with S of (4) and $v_i(0) = (r\cos\phi_i, r\sin\phi_i)^T$ for all $i \in \mathcal{V}$. The controller for the i^{th} agent is given by

$$\begin{cases} u_i = K\xi_i + L_{1ci}v_c \\ \dot{\xi}_i = D_1\xi_i + D_2\eta_i + Ez_i + L_{2ci}v_c \end{cases} \tag{19}$$

where $\{D_1, D_2, E, K\}$ and $\{L_{1ci}, L_{2ci}\}$ are the parameters to be designed. Here $z_i := \sum_{j\in\mathcal{N}_i} \alpha_{ij}y_{ij} + \alpha_{i0}y_i$ and $\eta_i := \sum_{j\in\mathcal{N}_i} \alpha_{ij}(\xi_i - \xi_j) + \alpha_{i0}\xi_i$. Note that z_i can be evaluated by all i including the follower since only relative states y_{ij} are needed and $\alpha_{i0} = 0$ when $i \in \mathcal{V}_F$. Similarly, η_i can be evaluated as ξ_j for $j \in \mathcal{N}_i$ is obtained from neighboring agent j via communication. For (19) to be a truly distributed controller, additional conditions on L_{1ci} and L_{2ci} are needed so that only v_j with $j \in \mathcal{N}_i \cup \{i\}$ are used. Specifically, L_{1ci} and L_{2ci} of (19) are to be chosen such that

$$\begin{cases} L_{1ci} v_c = L_1 v_i \\ L_{2ci} v_c = L_{21} v_i + L_{22} \zeta_i \end{cases} \tag{20}$$

where $\zeta_i := \sum_{j \in \mathcal{N}_i} \alpha_{ij} (v_i - v_j) + \alpha_{i0} v_i$. Similar to η_i, ζ_i can also be evaluated by agent i since v_j for $j \in \mathcal{N}_i$ is available via communication. L_1, L_{21}, L_{22} are parameters to be designed. In this form, (19) is a distributed controller of the form

$$\begin{cases} u_i = K\xi_i + L_1 v_i \\ \dot{\xi}_i = D_1 \xi_i + L_{21} v_i + D_2 \eta_i + E z_i + L_{22} \zeta_i \end{cases} \tag{21}$$

For subsequent discussion, the following assumption is needed.

(A3) The interaction graph $\overline{\mathcal{G}}$ has a directed spanning graph with the node 0 as its root.

(A3) is made for connectivity requirement. It also implies that there is at least one agent in $\mathcal{V}_L$, which is a minimum requirement for our approach although the general case is when there are multiple leaders. Combining the model of agents (1) and the controller (21), the overall system becomes

$$\begin{aligned} \begin{pmatrix} \dot{x}_c \\ \dot{\xi}_c \end{pmatrix} =& \begin{pmatrix} A_c & B_c K_c \\ H \otimes EC & I_N \otimes D_1 + H \otimes D_2 \end{pmatrix} \begin{pmatrix} x_c \\ \xi_c \end{pmatrix} \\ &+ \begin{pmatrix} B_c L_{1c} \\ L_{2c} \end{pmatrix} v_c \\ :=& A_{fc} \begin{pmatrix} x_c \\ \xi_c \end{pmatrix} + \begin{pmatrix} B_c L_{1c} \\ L_{2c} \end{pmatrix} v_c \end{aligned} \tag{22}$$

$$y_c = C_c x_c \tag{23}$$

where $H := \mathcal{L} + \mathrm{diag}\begin{pmatrix} \alpha_{10} & \dots & \alpha_{N0} \end{pmatrix}$, $A_c := I_N \otimes A, B_c := I_N \otimes B, K_c := I_N \otimes K$, $C_c := I_N \otimes C$ $x_c := \begin{pmatrix} x_1^T \; x_2^T \cdots \; x_N^T \end{pmatrix}^T$, $\xi_c := \begin{pmatrix} \xi_1^T \; \xi_2^T \cdots \; \xi_N^T \end{pmatrix}^T$, $L_{1c} := \begin{pmatrix} L_{1c1}^T \; L_{1c2}^T \cdots \; L_{1cN}^T \end{pmatrix}^T$, and L_{2c} is similarly defined.

The rest of the discussion here follows that of section III - to first design $\{D_1, D_2, E, K\}$ to stabilize A_{fc} to be followed by the result that $y_i(t) \to y_i^r(t)$ for all $i \in \mathcal{V}$. One such design procedure is given in [15] and [16] under assumptions (A1), (A2) and (A3) where there exists $\{K, E\}$ with $D_1 = A + BK$, $D_2 = -EC$ such that A_{fc} is stable. Specifically, K is chosen such that $A + BK$ is stable and $E = (\min_{\lambda \in \sigma(H)} \mathrm{Re}(\lambda))^{-1} P^{-1} C^{\mathrm{T}}$ where P is the solution of the equation $A^{\mathrm{T}} P + PA - 2C^{\mathrm{T}} C \prec 0$.

The next theorem shows the procedure to determine that values of L_{1ci} and L_{2ci} of (20).

Theorem 1: Suppose A_{fc} of (22) is stable. Then $y_c(t) \to y_c^r(t)$ with the distributed controller (21) if (11) with $Q = I_2$ has a solution $\{\Pi, \Gamma\}$.

Proof: Suppose (11) with $Q = I_2$ has a solution $\{\Pi, \Gamma\}$. Let $\Pi_c = I_N \otimes \Pi$, $\Gamma_c = I_N \otimes \Gamma$ and $I_c := I_N \otimes I_2$. Then it follows from (11) that

$$\begin{cases} \Pi_c S_c = A_c \Pi_c + B_c \Gamma_c \\ \quad 0 = C_c \Pi_c - I_c \end{cases} \tag{24}$$

Now let $L_1 = \Gamma - K\Sigma$ for an arbitrary $\Sigma \in \mathbb{R}^{l \times 2}$ which results in

$$L_{1c} = I_N \otimes (\Gamma - K\Sigma) = \Gamma_c - K_c \Sigma_c \tag{25}$$

where $\Sigma_c = I_N \otimes \Sigma$. Using this choice of Γ_c in (24) results in

$$\Pi_c S_c = A_c \Pi_c + B_c K_c \Sigma_c + B_c L_{1c} \tag{26}$$

Now let $L_{21} = \Sigma S - D_1 \Sigma$ and $L_{22} = -(EC\Pi + D_2 \Sigma)$ resulting in

$$\begin{aligned} L_{2c} =& I_N \otimes (\Sigma S - D_1 \Sigma) - H \otimes (EC\Pi + D_2 \Sigma) \\ =& (\Sigma_c S_c - (I_N \otimes D_1) \Sigma_c) \\ & - ((H \otimes EC) \Pi_c + (H \otimes D_2) \Sigma_c) \\ =& \Sigma_c S_c - (H \otimes EC) \Pi_c \\ & - (I_N \otimes D_1 + H \otimes D_2) \Sigma_c \\ =& \Sigma_c S_c - G_c C_c \Pi_c - F_c \Sigma_c \end{aligned} \tag{27}$$

where $G_c := H \otimes EC$ and $F_c := I_N \otimes D_1 + H \otimes D_2$. Combining the results of (24), (26) and (27), yields

$$\begin{cases} \Pi_c S_c = A_c \Pi_c + B_c K_c \Sigma_c + B_c L_{1c} \\ 0 = C_c \Pi_c - I_c \\ \Sigma_c S_c = G_c C_c \Pi_c + F_c \Sigma_c + L_{2c} \end{cases} \tag{28}$$

These equations are the equivalent of (10) for the single combined system given by (22) and (23) for the reference system of (16). Hence, using the result of Lemma 1, $y_c(t) \to y_c^r(t)$.

■

Remark 2: The discussion thus far is for achieving circular formation where all agents move along a circle of the same radius. The procedure above can be easily modified to achieve circular formation where each agent is on a circle of different radius, r_i. This is done by choosing $v_i(0) = (r_i \cos\phi_i, r_i \sin\phi_i)^T$. The procedure can be further generalized to track elliptical path of different sizes by choosing $y_i^r(t) = Q v_i(t)$ where Q is a positive definite matrix and $v_i(0) = (r_i \cos\phi_i, r_i \sin\phi_i)^T$.

V. Numerical Example

Example 1 has 4 agents having dynamics of (1) with $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T$, $C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. $x_1(0) = \begin{pmatrix} -10 & 0 & -14 & 0 \end{pmatrix}^T$, $x_2(0) = \begin{pmatrix} 6 & 0 & -12 & 0 \end{pmatrix}^T$, $x_3(0) = \begin{pmatrix} -14 & 0 & 10 & 0 \end{pmatrix}^T$, $x_4(0) = \begin{pmatrix} -7 & 0 & 6 & 0 \end{pmatrix}^T$. Also, $\xi_c(0) = 0$ and the communication graph is as shown in Fig. 2 where agent 3 is the leader. The y_i^r of (3) is given by $\omega = 1$, $\bar{c} = (2\ 7)^T$ with $\phi_i = \{0, \frac{2}{3}\pi, \frac{4}{3}\pi, \pi\}$ and $r = 7$.

The values of $\{D_1, D_2, E, K\}$ in (21) are $D_1 = A + BK$, $D_2 = -EC$, $K = \begin{pmatrix} -1.47 & -2.43 & -0.71 & -0.14 \\ -0.91 & -0.14 & -0.39 & -2.30 \end{pmatrix}$ and $E = \begin{pmatrix} 6.30 & 3.77 & 0.34 & 0.94 \\ 0.34 & -3.96 & 3.02 & 4.11 \end{pmatrix}^T$. Note that when $\omega =$

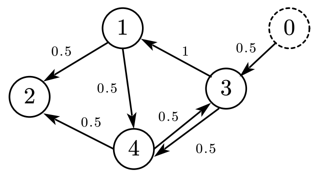

Fig. 2: The interaction graph $\overline{\mathcal{G}}$

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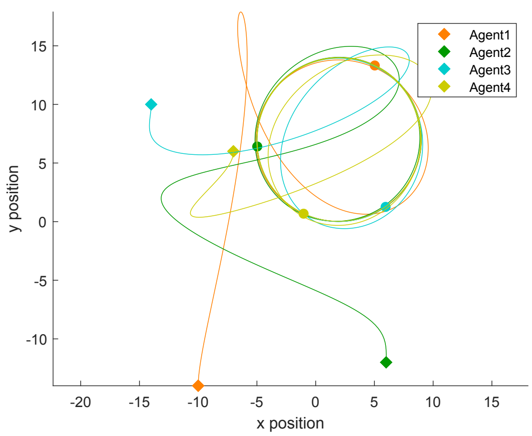

Fig. 3: The trajectories of 4 agents in the circular formation

1, (11) with $Q = I_2$ has a solution which implies, following Theorem 1, that

$$L_1 = \begin{pmatrix} 0.95 & 4.98 \\ 4.03 & 0.20 \end{pmatrix}$$

$$L_{21} = \begin{pmatrix} -0.27 & 3.41 & 0.16 & 3.07 \\ -1.80 & 3.22 & 0.20 & 0.28 \end{pmatrix}^T$$

$$L_{22} = \begin{pmatrix} -1.12 & -2.42 & 1.20 & 1.54 \\ 4.86 & 4.33 & -0.76 & -0.66 \end{pmatrix}^T$$

The controller parameters needed to shift the center of the circle are $\bar{L}_1 = \begin{pmatrix} 1.47 & -1.58 \\ 0.91 & 2.55 \end{pmatrix}$, $\bar{L}_{21} = B\bar{L}_1, \bar{L}_{22} = 0$. The simulation results are shown in Fig. 3, 4 and 5. It is clear from Fig. 3 and Fig. 4 that the agents achieved the specified circular formation with the center at $\bar{c}$. Since $v_i(t)$ and $\bar{v}_i(t)$ are bounded for the circular formation, the input u_i is also bounded, as seen in Fig. 5.

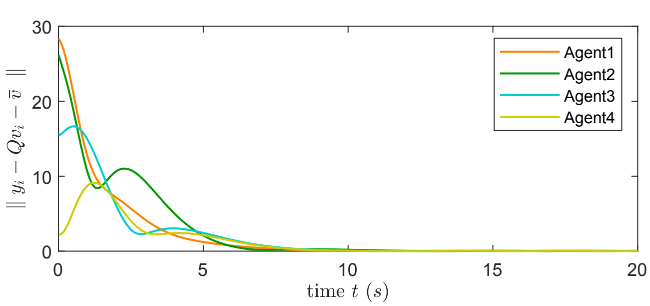

Fig. 4: Norm of $y_i(t) - Qv_i(t) - \bar{v}_i(t)$ for the case in Fig. 3

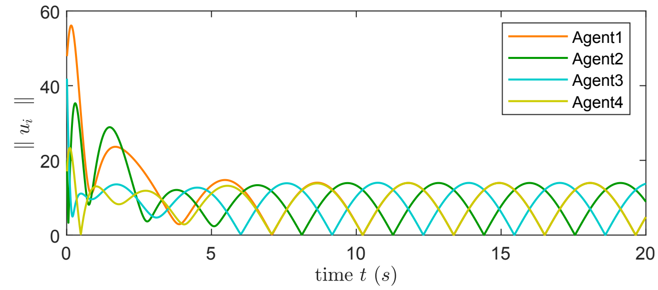

Fig. 5: Norm of u_i for the case in Fig. 3

The next example is similar to Example 1 in all aspects except for the values of Q as mentioned in Remark 2 and $v_i(0)$. Specifically, $Q = \begin{pmatrix} 1.5 & 0 \\ 0 & 1 \end{pmatrix}$ and $v_i(0) = (r_i \cos\phi_i, r_i \sin\phi_i)^T$ where $r_i = \{4, 4, 7, 7\}$ while ϕ_i remains unchanged from those in Example 1. With $\omega = 1$ and $Q = \begin{pmatrix} 1.5 & 0 \\ 0 & 1 \end{pmatrix}$, (11) has a solution and following Theorem 1,

$$L_1 = \begin{pmatrix} 0.45 & 4.98 \\ 4.03 & 0.20 \end{pmatrix}$$

$$L_{22} = \begin{pmatrix} -4.27 & -4.30 & 1.03 & 1.07 \\ 4.86 & 4.33 & -0.76 & -0.66 \end{pmatrix}^T$$

while L_{21} is same as in Example 1. The simulation results are shown in Fig. 6, 7 and 8.

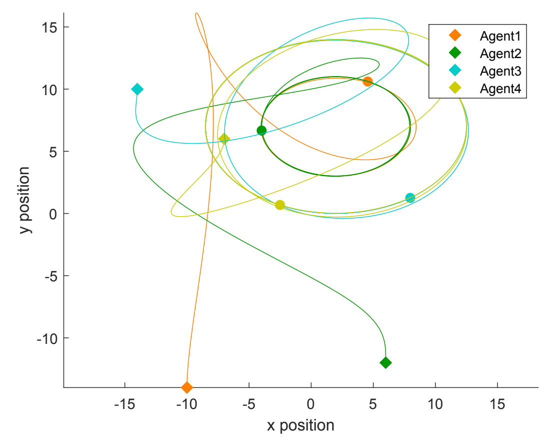

Fig. 6: The trajectories of 4 agents in the elliptical formation

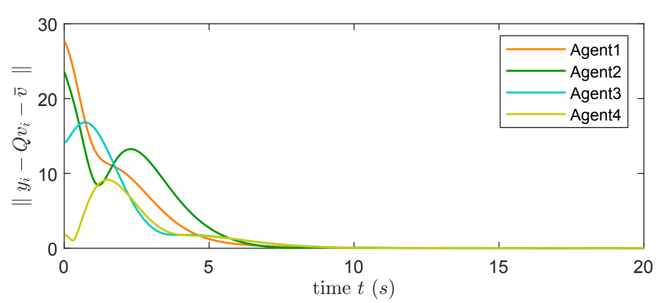

Fig. 7: Norm of $y_i(t) - Qv_i(t) - \bar{v}_i(t)$ for the case in Fig. 6

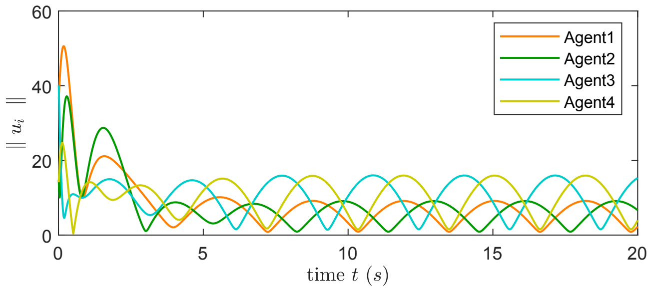

Fig. 8: Norm of u_i for the case in Fig. 6

VI. Conclusions

This work provides a design procedure that achieves circular formation for a Multi-Agent System using a distributed control law. The controller design is based on the output regulation approach but uses output feedback instead of the typical error feedback. Using the output feedback control scheme has the advantage of a decoupled design and is an important element in having the properties needed for circular formation. The other advantages of the approach include the ability to achieving circular formation with arbitrary radius and center, only one of the agents needs to know its own location and easy extension to achieving elliptical formation.

VII. Appendix

Proof: (i) It is well-known ([17] Chapter 3.5) that a stabilizable and detectable system can be stabilized using a dynamic compensator. In this case, A_f can be stabilized by letting $F = A + BK - GC$ since both $A + BK$ and $A - GC$ are stable under (A1) and (A2).

(ii) The proof follows similar reasoning as Lemma 1.4.1 in [13] except for the use of L_1 and L_2.

($\Rightarrow$) If $\{L_1, L_2\}$ are such that (10) has a solution $\{\Pi, \Sigma\}$, then by the first and the third equation in (10) and using the coordinate transformation,

$$\widetilde{x} = x - \Pi v, \quad \widetilde{\xi} = \xi - \Sigma v \tag{29}$$

(7) could be rewritten as

$$\begin{pmatrix} \dot{\widetilde{x}} \\ \dot{\widetilde{\xi}} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} A & BK & 0 \\ GC & F & 0 \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} \widetilde{x} \\ \widetilde{\xi} \\ v \end{pmatrix} \tag{30}$$

Since A_f is stable and the system is decoupled, it follows that $\widetilde{x} \to 0$ and $\widetilde{\xi} \to 0$ for all $\{x(0), \xi(0), v(0)\}$. The tracking error becomes

$$\begin{aligned} y - Qv =& Cx - Qv = C\left(\widetilde{x} + \Pi v\right) - Qv \\ =& C\widetilde{x} + \left(C\Pi - Q\right) v. \end{aligned} \tag{31}$$

Since $\widetilde{x} \to 0$ and $C\Pi - Q = 0$ in (10), $(Cx(t) - Qv(t)) \to 0$ for all $\{x(0), \xi(0), v(0)\}$.

($\Leftarrow$) The following fact is first established. Since S is unstable and A_f is stable, there is no common eigenvalue between S and A_f. It then implies that the following Sylvester equation has a unique solution $\{\Pi, \Sigma\}$ for any $\{L_1, L_2\}$,

$$\begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} S = \begin{pmatrix} A & BK \\ GC & F \end{pmatrix} \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} + \begin{pmatrix} BL_1 \\ L_2 \end{pmatrix}. \tag{32}$$

Define $\widetilde{x}$ and $\widetilde{\xi}$ as in (29) and perform the coordinate transformation from (x, ξ, v) to $(\widetilde{x}, \widetilde{\xi}, v)$ leading to (30). The only if part is now shown. Since $e = y - Qv \to 0$ for any $\{x(0), \xi(0), v(0)\}$, this implies $C\widetilde{x} + (C\Pi - Q)e^{St}v(0) = 0$. Since A_f is stable, $\widetilde{x} \to 0$ which implies that $C\Pi - Q = 0$. Combining $C\Pi - Q = 0$ and (32) result in (10). ■

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